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Research Article

A New Instability Result to Nonlinear Vector Differential Equations of Fifth Order

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By constructing a Lyapunov function, a new instability result is established, which guarantees that the trivial solution of a certain nonlinear vector differential equation of the fifth order is unstable. An example is also given to illustrate the importance of the result obtained. By this way, our findings improve an instability result related to a scalar differential equation in the literature to instability of the trivial solution to the afore-mentioned differential equation.

1. Introduction

In 1990, Li and Yu [1] investigated the instability of trivial solution to fifth-order nonlinear scalar differential equation

$$x^{(5)} + ax^{(4)} + b\dddot{x} + \varphi\left(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}\right)\ddot{x} + g(x)\dot{x} + f(x) = 0 \tag{1.1}$$

by introducting a Lyapunov function, where a and b are some positive constants.

In this paper, based on the result of Li and Yu [1], we are concerned with the instability of trivial solution to fifth-order nonlinear vector differential equation described by

$$X^{(5)} + AX^{(4)} + B\dddot{X} + \Psi\left(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}\right)\ddot{X} + G(X)\dot{X} + F(X)X = 0 \tag{1.2}$$

in the real Euclidean space $\mathfrak{R}^n$ (with the usual norm denoted in what follows by $\|\cdot\|$), where $X \in \mathfrak{R}^n$; A and B are constant $n \times n$-symmetric matrices; Ψ, G, and F are $n \times n$-symmetric continuous matrix functions depending, in each case, on the arguments shown. Throughout this paper, we consider, instead of (1.2), the equivalent differential system

$$\begin{gathered} \dot{X} = Y, \qquad \dot{Y} = Z, \qquad \dot{Z} = W, \qquad \dot{W} = U, \\ \dot{U} = -AU - BW - \Psi(X, Y, Z, W, U)Z - G(X)Y - F(X)X, \end{gathered} \tag{1.3}$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$, $X^{(4)} = U$ from (1.2). For the sake of brevity, we assume that the symbol $J_G(X)$ denotes the Jacobian matrix

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right) \quad (i, j = 1, 2, \ldots, n), \tag{1.4}$$

where $(x_1, x_2, \ldots, x_n)$ and $(g_1, g_2, \ldots, g_n)$ are components of X and G, respectively. In addition, it is assumed, as basic throughout the paper, that the Jacobian matrix $J_G(X)$ exists and is continuous and symmetric. The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\mathfrak{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)$, $(A = (a_{ij}))$ $(i, j = 1, 2, \ldots, n)$ are the eigenvalues of the $n \times n$-symmetric matrix A, and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathfrak{R}^n$ and X^T denotes the transpose of X.

At the same time, up to now, we should also recognize that some significant theoretical results related to instability of trivial solution of some nonlinear scalar and vector differential equations of fifth order have been achieved in the literature, see, for example, the papers of Ezeilo [2, 3], Tunç [4, 5], and the references registered in these papers. However, it should be noted that nearly all of the papers have been published on the subject without including any example related to the topic. The equation considered that the assumptions and Lyapunov [6] function that will be established here are completely different than those mentioned in the literature.

2. Main result

Our main result is the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed on A, B, Ψ, G, and F appearing in* (1.2)*, we assume there are constants a, b and a positive constant k_1 such that the following conditions hold:*

(i) $\lambda_i(A) \geq a$, $\lambda_i(B) \geq b$, $b \operatorname{sgn} a > 0$;

(ii) $\lambda_i(F(X))\operatorname{sgn} a - (1/4|a|)(\lambda_i(\Psi(X, Y, Z, W, U))^2 > k_1$ $(i = 1, 2, \ldots, n)$.

Then the trivial solution $X = 0$ of (1.2) *is unstable.*

Now, in order to prove our main result, we give a well-known lemma which plays an essential role throughout the proof of theorem.

Remark 2.2. It should be noted that there is no restriction on symmetric matrix G appearing in (1.2).

Lemma 2.3. *Let A be a real symmetric $n \times n$-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, \ldots, n), \tag{2.1}$$

where a', a are constants.

Then

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle, \qquad a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle. \tag{2.2}$$

Proof. See [7]. □

Proof of Theorem 2.1. To achieve the proof of theorem, we define the following Lyapunov function $V := V(X,Y,Z,W,U)$:

$$V = V_0(X,Y,Z,W,U)\operatorname{sgn} a, \tag{2.3}$$

where

$$\begin{aligned} V_0 = {} & \langle Y,W\rangle + \langle Y,AZ\rangle - \langle X,U\rangle - \langle X,AW\rangle - \langle X,BZ\rangle \\ & - \frac{1}{2}\langle Z,Z\rangle + \frac{1}{2}\langle BY,Y\rangle - \int_0^1 \langle \sigma G(\sigma X)X,X\rangle d\sigma. \end{aligned} \tag{2.4}$$

Now, under the assumptions of the theorem, it will be shown that the Lyapunov function $V = V(X,Y,Z,W,U)$ satisfies the entire Krasovskiĭ [8] criteria.

(K_1) In every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi,\eta,\zeta,\mu,\rho)$ such that $V(\xi,\eta,\zeta,\mu,\rho) > 0$.

(K_2) The time derivative $\dot{V} = (d/dt)V(X,Y,Z,W,U)$ along solution paths of system (1.3) is positive semidefinite.

(K_3) The only solution $(X,Y,Z,W,U) = (X(t),Y(t),Z(t),W(t),U(t))$ of system (1.3) which satisfies $\dot{V} = 0$ $(t \geq 0)$ is the trivial solution $(0,0,0,0,0)$. These properties guarantee that the trivial solution of (1.2) is unstable.

First, it is easy to see from (2.3) and (2.4) that

$$V(0,0,0,0,0) = V_0(0,0,0,0,0)\operatorname{sgn} a = 0. \tag{2.5}$$

Next, evidently, one can easily get

$$V(0,\varepsilon\operatorname{sgn} a,0,\varepsilon,0) = \langle \varepsilon,\varepsilon\rangle + \frac{1}{2}b\operatorname{sgn} a\langle \varepsilon,\varepsilon\rangle = \|\varepsilon\|^2 + \frac{1}{2}b\operatorname{sgn} a\|\varepsilon\|^2 > 0 \tag{2.6}$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \Re^n$.

Finally, let $(X,Y,Z,W,U) = (X(t),Y(t),Z(t),W(t),U(t))$ be an arbitrary solution of system (1.3). Differentiating (2.4) with respect to t, along this solution, calculations give that

$$\begin{aligned} \dot{V}_0 = \frac{d}{dt}V_0(X,Y,Z,W,U) = {} & \langle AZ,Z\rangle + \langle F(X)X,X\rangle + \langle X,\Psi(X,Y,Z,W,U)Z\rangle \\ & + \langle G(X)Y,X\rangle - \frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X,X\rangle d\sigma. \end{aligned} \tag{2.7}$$

Now, recall that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X,X\rangle d\sigma &= \int_0^1 \langle \sigma G(\sigma X)Y,X\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma G(\sigma X)Y,X\rangle d\sigma \\ &= \sigma^2\langle G(\sigma X)Y,X\rangle\big|_0^1 = \langle G(X)Y,X\rangle. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$\dot{V}_0 = \langle AZ,Z\rangle + \langle F(X)X,X\rangle + \langle X,\Psi(X,Y,Z,W,U)Z\rangle. \tag{2.9}$$

Now, it follows from (2.3) and (2.9) that

$$\begin{aligned}
\dot{V} &= \operatorname{sgn} a\langle Z, AZ\rangle + \operatorname{sgn} a\langle X, F(X)X\rangle + \operatorname{sgn} a\langle X, \Psi(X,Y,Z,W,U)Z\rangle \\
&\geq |a|\langle Z, Z\rangle + \operatorname{sgn} a\langle X, F(X)X\rangle + \operatorname{sgn} a\langle X, \Psi(X,Y,Z,W,U)Z\rangle \\
&= |a|\left\| Z + \frac{1}{2|a|}\Psi(X,Y,Z,W,W,U)X \operatorname{sgn} a\right\|^2 \\
&\quad + \left[\operatorname{sgn} a\langle X, F(X)X\rangle - \frac{1}{4|a|}\langle \Psi(X,Y,Z,W,U)X, \Psi(X,Y,Z,W,U)X\rangle\right] \\
&\geq \left[\operatorname{sgn} a\langle X, F(X)X\rangle - \frac{1}{4|a|}\langle \Psi(X,Y,Z,W,U)X, \Psi(X,Y,Z,W,U)X\rangle\right] \\
&\geq k_1\|X\|^2 \geq 0,
\end{aligned} \tag{2.10}$$

by (i) and (ii).

Thus, the assumptions of theorem show that $\dot{V}(t) \geq 0$ for all $t \geq 0$, that is, $\dot{V}$ is positive semidefinite. Furthermore, $\dot{V} = 0$ $(t \geq 0)$ necessarily implies that $X = 0$ for all $t \geq 0$, and $Z = \dot{Y} = 0$, $W = \ddot{Y} = 0$, $\dot{W} = \dddot{Y} = 0$ for all $t \geq 0$. Thus, it follows the estimates

$$X = Y = Z = W = U = 0. \tag{2.11}$$

Therefore, the function V has the entire criteria of Krasovskiĭ [8] if the assumptions of theorem hold. Thus, the basic properties of the function $V(X,Y,Z,W,U)$, which are proved above, verify that the trivial solution of system (1.3) is unstable. The system of (1.3) is equivalent to the differential equation (1.2).

Hence, this fact completes the proof of theorem. □

Example 2.4. As special cases of system (1.3), let us choose for $n = 2$, A, B, Ψ, and F:

$$\begin{gathered}
A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \\
\Psi(X,Y,Z,W,U) = \begin{bmatrix} 2 + \left(1 + x^2 + y^2 + z^2 + w^2 + u^2\right)^{-1} & 1 \\ 1 & 2 + \left(1 + x^2 + y^2 + z^2 + w^2 + u^2\right)^{-1} \end{bmatrix}, \\
F(X) = \begin{bmatrix} 20 + \dfrac{1}{1+x^2} & 1 \\ 1 & 20 + \dfrac{1}{1+x^2} \end{bmatrix}.
\end{gathered} \tag{2.12}$$

Then, respectively, we get

$$\begin{gathered}
\lambda_1(A) = 2, \qquad \lambda_2(A) = 4, \qquad \lambda_1(B) = 2, \qquad \lambda_2(B) = 6, \\
\lambda_1\big(\Psi(X,Y,Z,W,U)\big) = 1 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}, \\
\lambda_2\big(\Psi(X,Y,Z,W,U)\big) = 3 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}, \\
\lambda_i\big(\Psi(X,Y,Z,W,U)\big) \geq 1, \\
\lambda_1\big(F(X)\big) = 19 + \frac{1}{1+x^2}, \qquad \lambda_2\big(F(X)\big) = 21 + \frac{1}{1+x^2}.
\end{gathered} \tag{2.13}$$

Now, taking into account this facts, it can be easily seen that

$$\begin{gathered}
\lambda_i(A) \geq 2 = a, \qquad \lambda_i(B) \geq 2 = b, \qquad b \operatorname{sgn} a > 0, \qquad \lambda_i\big(F(X)\big) \operatorname{sgn} \geq 19 \quad (i = 1,2), \\
\left(19 + \frac{1}{1+x^2}\right) \operatorname{sgn} 2 - \frac{1}{8}\left(1 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}\right)^2 > 0, \\
\left(19 + \frac{1}{1+x^2}\right) \operatorname{sgn} 2 - \frac{1}{8}\left(3 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}\right)^2 > 0, \\
\left(21 + \frac{1}{1+x^2}\right) \operatorname{sgn} 2 - \frac{1}{8}\left(1 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}\right)^2 > 0, \\
\left(21 + \frac{1}{1+x^2}\right) \operatorname{sgn} 2 - \frac{1}{8}\left(3 + \frac{1}{1+x^2+y^2+z^2+w^2+u^2}\right)^2 > 0.
\end{gathered} \tag{2.14}$$

Clearly, these last four expressions imply that

$$\lambda_i\big(F(X)\big) \operatorname{sgn} a - \frac{1}{4|a|}\big(\lambda_i\big(\Psi(X,Y,Z,W,U)\big)\big)^2 > 0 \quad (i = 1,2). \tag{2.15}$$

Thus, it is shown that all the assumptions of the theorem hold.

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