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Research Article

Multiple Positive Symmetric Solutions to p-Laplacian Dynamic Equations on Time Scales

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This paper makes a study on the existence of positive solution to p-Laplacian dynamic equations on time scales $\mathbb{T}$. Some new sufficient conditions are obtained for the existence of at least single or twin positive solutions by using Krasnosel'skii's fixed point theorem and new sufficient conditions are also obtained for the existence of at least triple or arbitrary odd number positive solutions by using generalized Avery-Henderson fixed point theorem and Avery-Peterson fixed point theorem. As applications, two examples are given to illustrate the main results and their differences. These results are even new for the special cases of continuous and discrete equations, as well as in the general time-scale setting.

1. Introduction

Initiated by Hilger in his Ph.D. thesis [1] in 1988, the theory of time scales has been improved ever since, especially in the unification of the theory of differential equations in the continuous case and the theory of difference equations in the discrete case. For the time being, it remains a field of vitality and attracts attention of many distinguished scholars worldwide. In particular, the theory is also widely applied to biology, heat transfer, stock market, wound healing, epidemic models [2–5], and so forth.

Recent research results indicate that considerable achievement has been made in the existence problems of positive solutions to dynamic equations on time scales. For details, please see [6–13] and the references therein. Symmetry and pseudosymmetry have been widely used in science and engineering [14]. The reason is that symmetry and pseudosymmetry are not only of its theoretical value in studying the metric manifolds [15] and symmetric graph [16, 17], and so forth, but also of its practical value, for example, we can

apply this characteristic to study graph structure [18, 19] and chemistry structure [20]. Yet, few literature resource [21, 22] is available concerning the characteristics of positive solutions to p-Laplacian dynamic equations on time scales.

Throughout this paper, we denote the p-Laplacian operator by $\varphi_p(u)$, that is, $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ with $(\varphi_p)^{-1} = \varphi_q$ and $1/p + 1/q = 1$.

For convenience, we think of the blanket as an assumption that a, b are points in $\mathbb{T}$, for an interval $(a, b)_{\mathbb{T}}$ we always mean $(a, b) \cap \mathbb{T}$. Other type of intervals is defined similarly.

We would like to mention the results of Sun and Li [11, 12]. In [12], Sun and Li considered the two-point BVP

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\Delta} + h(t)f(u^{\sigma}(t)) = 0, \quad t \in [a, b]_{\mathbb{T}}, \\ u(a) - B_0\left(u^{\Delta}(a)\right) = 0, \qquad u^{\Delta}(\sigma(b)) = 0,\end{gathered} \tag{1.1}$$

and established the existence theory for positive solutions of the above problem. They [11] also considered the m-point boundary value problem with p-Laplacian

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + h(t)f(t, u(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \\ u^{\Delta}(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} c_i u(\xi_i),\end{gathered} \tag{1.2}$$

and gave the existence of *single* or *multiple* positive solutions to the above problem. The main tools used in these two papers are some fixed-point theorems [23–25].

It is also noted that the researchers mentioned above [11, 12] only considered the existence of positive solutions. As a results, they failed to further provide characteristics of solutions, such as, symmetry. Naturally, it is quite necessary to consider the characteristics of solutions to p-Laplacian dynamic equations on time scales.

Let $\mathbb{T}$ be a symmetric time scale such that $0, T \in \mathbb{T}$. we consider the following p-Laplacian boundary value problem on time scales $\mathbb{T}$ of the form:

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + h(t)f(u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\ u(0) = u(T) = 0, \qquad u^{\Delta}(0) = -u^{\Delta}(T).\end{gathered} \tag{1.3}$$

By using symmetric technique, the Krasnosel'skii's fixed point theorem [24], the generalized Avery-Henderson fixed point theorem [26], and Avery-Peterson fixed point theorem [27], we obtain the existence of at least single, twin, triple, or arbitrary odd positive symmetric solutions of problem (1.3). As applications, two examples are given to illustrate the main results and their differences. These results are even new for the special cases of continuous and discrete equations as well as in the general time-scale setting.

The rest of the paper is organized as follows. In Section 2, we present several fixed point results. In Section 3, by using Krasnosel'skii's fixed point theorem, we obtain the existence of at least *single* or *twin* positive symmetric solutions to problem (1.3). In Section 4, the existence criteria for at least *triple* positive or *arbitrary odd* positive symmetric solutions to

problem (1.3) are established. In Section 5, we present two simple examples to illustrate our results.

For convenience, we now give some symmetric definitions.

Definition 1.1. The interval $[0,T]_{\mathbb{T}}$ is said to be symmetric if any given $t \in [0,T]_{\mathbb{T}}$, we have $T-t \in [0,T]_{\mathbb{T}}$.

We note that such a symmetric time scale $\mathbb{T}$ exists. For example, let

$$\mathbb{T} = \{0, 0.05, 0.1, 0.15\} \cup [0.22, 0.44] \cup \{0.5, 0.85, 0.9, 0.95, 1\} \cup [0.56, 0.78]. \tag{1.4}$$

It is obvious that $\mathbb{T}$ is a symmetric time scale.

Definition 1.2. A function $u : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is said to be symmetric if u is symmetric over the interval $[0,T]_{\mathbb{T}}$. That is, $u(t) = u(T-t)$, for any given $t \in [0,T]_{\mathbb{T}}$.

Definition 1.3. We say u is a symmetric solution to problem (1.3) on $[0,T]_{\mathbb{T}}$ provided that u is a solution to boundary value problem (1.3) and is symmetric over the interval $[0,T]_{\mathbb{T}}$.

Basic definitions on time scale can be found in [6, 7, 28]. Another excellent sources on dynamical systems on measure chains are the book in [29].

Throughout this paper, it is assumed that

(H1) $f : [0,\infty) \to [0,\infty)$ is continuous, and does not vanish identically;

(H2) $h \in C_{\mathrm{ld}}([0,T]_{\mathbb{T}}, [0,\infty))$ is symmetric over the interval $[0,T]_{\mathbb{T}}$ and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $C_{\mathrm{ld}}([0,T]_{\mathbb{T}}, [0,\infty))$ denotes the set of all left dense continuous functions from $[0,T]_{\mathbb{T}}$ to $[0,\infty)$.

2. Preliminaries

Let $E = C_{\mathrm{ld}}([0,T]_{\mathbb{T}}, \mathbb{R})$ and equip norm

$$\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|, \tag{2.1}$$

then E is a Banach space. Define a cone $P \subset E$ by

$$P = \{u \in E \mid u(0) = u(T) = 0, u \text{ is symmetric, nonnegative, and concave on the interval } [0,T]_{\mathbb{T}}\}. \tag{2.2}$$

Assume that $r, \eta \in (0, T/2)_{\mathbb{T}}$ with $\eta < r$. By using the symmetric and concave characters of $u \in P$ and $u(0) = u(T) = 0$, it is easy to obtain the following results.

Lemma 2.1. *Assume that $r, \eta \in (0, T/2)_{\mathbb{T}}$ with $\eta < r$. If $u \in P$, then*

(i) $u(\eta) \geq (\eta/r)u(r)$;

(ii) $(T/2)u(r) \geq ru(T/2)$.

From the previous lemma we know that $\|u\| = u(T/2)$ for $u \in P$.

The operator $A : P \to E$ is defined by

$$Au(t) = \begin{cases} \displaystyle\int_0^t \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s, & t \in \left[0, \dfrac{T}{2}\right]_{\mathbb{T}}, \\ \displaystyle\int_t^T \varphi_q\left(\int_{T/2}^s h(r)f(u(r))\nabla r\right)\Delta s, & t \in \left[\dfrac{T}{2}, T\right]_{\mathbb{T}}. \end{cases} \tag{2.3}$$

It is obvious that A is completely continuous operator and all the fixed points of A are the solutions to the boundary value problem (1.3).

In addition, it is easy to see that the operator A is symmetric. In fact, for $t \in [0, T/2]_{\mathbb{T}}$, we have $T - t \in [T/2, T]_{\mathbb{T}}$, by using the integral transform, we have

$$\begin{aligned} Au(T-t) &= \int_{T-t}^T \varphi_q\left(\int_{T/2}^s h(r)f(u(r))\nabla r\right)\Delta s \\ &\overset{s\to T-s}{=} \int_0^t \varphi_q\left(\int_{T/2}^{T-s} h(r)f(u(r))\nabla r\right)\Delta s \\ &\overset{r\to T-r}{=} \int_0^t \varphi_q\left(\int_s^{T/2} h(T-r)f(u(T-r))\nabla r\right)\Delta s \\ &= \int_0^t \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s = Au(t). \end{aligned} \tag{2.4}$$

Hence, A is symmetric.

Now, we provide some background material from the theory of cones in Banach spaces [24, 26, 27, 30], and then state several fixed point theorems needed later.

Firstly, we list the Krasnosel'skii's fixed point theorem [24].

Lemma 2.2 (see [24]). *Let P be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \le \|x\|$, *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|$, *for all* $x \in P \cap \partial\Omega_2$ *or*

(ii) $\|Ax\| \ge \|x\|$, *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|$, *for all* $x \in P \cap \partial\Omega_2$,

then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define, for each $d > 0$, the set $P(\gamma, d) = \{x \in P : \gamma(x) < d\}$.

Secondly, we state the generalized Avery-Henderson fixed point theorem [26].

Lemma 2.3 (see [26]). *Let P be a cone in a real Banach space E. Let α, β, and γ be increasing, nonnegative continuous functional on P such that for some $c > 0$ and $H > 0, \gamma(x) \leq \beta(x) \leq \alpha(x)$ and $\|x\| \leq H\gamma(x)$ for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ and $A : \overline{P(\gamma, c)} \to P$ is a completely continuous operator such that*

(i) *$\gamma(Ax) < c$ for all $x \in \partial P(\gamma, c)$;*

(ii) *$\beta(Ax) > b$ for all $x \in \partial P(\beta, b)$;*

(iii) *$P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) < a$ for $x \in \partial P(\alpha, a)$,*

then A has at least three fixed points x_1, x_2, and x_3 belonging to $\overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2) \quad \text{with } \beta(x_2) < b < \beta(x_3),\ \gamma(x_3) < c. \tag{2.5}$$

The following lemma can be found in [21].

Lemma 2.4 (see [21]). *Let P be a cone in a real Banach space E. Let α, β, and γ be increasing, nonnegative continuous functional on P such that for some $c > 0$ and $H > 0, \gamma(x) \leq \beta(x) \leq \alpha(x)$ and $\|x\| \leq H\gamma(x)$ for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ and $A : \overline{P(\gamma, c)} \to P$ is a completely continuous operator such that:*

(i) *$\gamma(Ax) > c$ for all $x \in \partial P(\gamma, c)$;*

(ii) *$\beta(Ax) < b$ for all $x \in \partial P(\beta, b)$;*

(iii) *$P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in \partial P(\alpha, a)$,*

then A has at least three fixed points x_1, x_2, and x_3 belonging to $\overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2) \quad \text{with } \beta(x_2) < b < \beta(x_3),\ \gamma(x_3) < c. \tag{2.6}$$

Let β and ϕ be nonnegative continuous convex functionals on P, λ is a nonnegative continuous concave functional on P, and φ is a nonnegative continuous functional, respectively on P. We define the following convex sets:

$$\begin{gathered} P(\phi, \lambda, b, d) = \{x \in P : b \leq \lambda(x), \phi(x) \leq d\}, \\ P(\phi, \beta, \lambda, b, c, d) = \{x \in P : b \leq \lambda(x), \beta(x) \leq c, \phi(x) \leq d\}, \end{gathered} \tag{2.7}$$

and a closed set $R(\phi, \varphi, a, d) = \{x \in P : a \leq \varphi(x), \phi(x) \leq d\}$.

Finally, we list the fixed point theorem due to Avery-Peterson [27].

Lemma 2.5 (see [27]). *Let P be a cone in a real Banach space E and $\beta, \phi, \lambda, \varphi$ defined as above, moreover, φ satisfies $\varphi(\lambda' x) \leq \lambda' \varphi(x)$ for $0 \leq \lambda' \leq 1$ such that, for some positive numbers h and d,*

$$\lambda(x) \leq \varphi(x), \quad \|x\| \leq h\phi(x) \tag{2.8}$$

for all $x \in \overline{P(\phi, d)}$. Suppose that $A : \overline{P(\phi, d)} \to \overline{P(\phi, d)}$ is completely continuous and there exist positive real numbers a, b, c, with $a < b$ such that

(i) *$\{x \in P(\phi, \beta, \lambda, b, c, d) : \lambda(x) > b\} \neq \emptyset$ and $\lambda(A(x)) > b$ for $x \in P(\phi, \beta, \lambda, b, c, d)$;*

(ii) *$\lambda(A(x)) > b$ for $x \in P(\phi, \lambda, b, d)$ with $\beta(A(x)) > c$;*

(iii) *$0 \notin R(\phi, \varphi, a, d)$ and $\lambda(A(x)) < a$ for all $x \in R(\phi, \varphi, a, d)$ with $\varphi(x) = a$,*

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\phi, d)}$ such that

$$\phi(x_i) \leq d \quad \text{for } i = 1, 2, 3, \qquad b < \lambda(x_1), \qquad a < \varphi(x_2), \qquad \lambda(x_2) < b \qquad \text{with } \varphi(x_3) < a. \tag{2.9}$$

3. Single or Twin Solutions

Let

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{\varphi_p(u)}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{\varphi_p(u)}. \tag{3.1}$$

We define i_0 = number of zeros in the set $\{f_0, f_\infty\}$ and i_∞ = number of infinities in the set $\{f_0, f_\infty\}$. Clearly, $i_0, i_\infty = 0, 1$, or 2 and there exist six possible cases: (i) $i_0 = 1$ and $i_\infty = 1$; (ii) $i_0 = 0$ and $i_\infty = 0$; (iii) $i_0 = 0$ and $i_\infty = 1$; (iv) $i_0 = 0$ and $i_\infty = 2$; (v) $i_0 = 1$ and $i_\infty = 0$; (vi) $i_0 = 2$ and $i_\infty = 0$. In the following, by using Krasnosel'skii's fixed point theorem in a cone, we study the existence of positive symmetric solutions to problem (1.3) under the above six possible cases.

3.1. For the Case $i_0 = 1$ and $i_\infty = 1$

In this subsection, we discuss the existence of single positive symmetric solution of the problem (1.3) under $i_0 = 1$ and $i_\infty = 1$.

Theorem 3.1. *Problem* (1.3) *has at least one positive symmetric solution in the case $i_0 = 1$ and $i_\infty = 1$.*

Proof. We divide the proof into two cases.

Case 1 ($f_0 = 0$ and $f_\infty = \infty$). In view of $f_0 = 0$, there exists an $H_1 > 0$ such that $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$ for $u \in (0, H_1]$, where ε arbitrary small and satisfies $0 < (\varepsilon T/2)\varphi_q(\int_0^{T/2} h(r)\nabla r) \leq 1$.

If $u \in P$ with $\|u\| = H_1$, then

$$\begin{aligned}
\|Au\| &= \sup_{t\in[0,T]_{\mathbb{T}}} |Au| \\
&= Au\left(\frac{T}{2}\right) \\
&= \int_0^{T/2} \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\le \varepsilon\|u\|\int_0^{T/2} \varphi_q\left(\int_0^{T/2} h(r)\nabla r\right)\Delta s \\
&\le \|u\|\frac{\varepsilon T}{2}\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right) \\
&\le \|u\|.
\end{aligned} \tag{3.2}$$

We let $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$, then $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_{H_1}$.

From $f_\infty = \infty$, there exists an $H_2' > 0$ such that $f(u) \ge \varphi_p(k)\varphi_p(u) = \varphi_p(ku)$ for $u \in [H_2', \infty)$, where $k > 0$, and satisfies the following inequality:

$$\frac{2k\eta^2}{T}\varphi_q\left(\int_\eta^{T/2} h(r)\nabla r\right) \ge 1. \tag{3.3}$$

Set

$$H_2 = \max\left\{2H_1, H_2'\frac{T}{2\eta}\right\}, \qquad \Omega_{H_2} = \{u \in E : \|u\| < H_2\}. \tag{3.4}$$

If $u \in P$ with $\|u\| = H_2$, then, by Lemma 2.1, one has

$$\min_{t\in[\eta,T/2]_{\mathbb{T}}} u(t) \ge \frac{2\eta}{T}u\left(\frac{T}{2}\right) \ge H_2'. \tag{3.5}$$

For $u \in P \cap \partial\Omega_{H_2}$, in terms of (3.3) and (3.5), we get

$$\begin{aligned}
\|Au\| &= \sup_{t\in[0,T]_{\mathbb{T}}} |Au| \\
&\geq Au(\eta) \\
&= \int_0^{\eta} \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\geq \int_0^{\eta} \varphi_q\left(\int_{\eta}^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\geq \int_0^{\eta} \varphi_q\left(\int_{\eta}^{T/2} h(r)\varphi_p(ku(r))\nabla r\right)\Delta s \\
&\geq \frac{2k\eta^2}{T}\|u\|\varphi_q\left(\int_{\eta}^{T/2} h(r)\nabla r\right) \\
&\geq \|u\|.
\end{aligned} \tag{3.6}$$

Thus, by (i) of Lemma 2.2, problem (1.3) has at least single positive symmetric solution u in $P \cap (\overline{\Omega}_{H_2} \setminus \Omega_{H_1})$ with $H_1 \leq \|u\| \leq H_2$.

Case 2 ($f_0 = \infty$ and $f_\infty = 0$). Since $f_0 = \infty$, there exists an $H_3 > 0$ such that $f(u) \geq \varphi_p(m)\varphi_p(u) = \varphi_p(mu)$ for $u \in (0, H_3]$, where m is such that

$$\frac{2m\eta^2}{T}\varphi_q\left(\int_{\eta}^{T/2} h(r)\nabla r\right) \geq 1. \tag{3.7}$$

If $u \in P$ with $\|u\| = H_3$, then, by (3.7), one has

$$\begin{aligned}
\|Au\| &= \sup_{t\in[0,T]_{\mathbb{T}}} |Au| \\
&\geq \int_0^{\eta} \varphi_q\left(\int_{\eta}^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\geq \eta\varphi_q\left(\int_{\eta}^{T/2} h(r)\varphi_p(mu(r))\nabla r\right) \\
&\geq \eta m\frac{2\eta}{T}\|u\|\varphi_q\left(\int_{\eta}^{T/2} h(r)\nabla r\right) \\
&\geq \|u\|.
\end{aligned} \tag{3.8}$$

If we let $\Omega_{H_3} = \{u \in E : \|u\| < H_3\}$, then $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{H_3}$.

Now, we consider $f_\infty = 0$. By definition, there exists $H_4' > 0$ such that

$$f(u) \le \varphi_p(\delta)\varphi_p(u) = \varphi_p(\delta u) \quad \text{for } u \in \left[H_4', \infty\right), \tag{3.9}$$

where $\delta > 0$ satisfies

$$\frac{\delta T}{2}\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right) \le 1. \tag{3.10}$$

Suppose that f is bounded, then $f(u) \le \varphi_p(K)$ for all $u \in [0,\infty)$ and some constant $K > 0$. Pick

$$H_4 = \max\left\{2H_3, \frac{KT}{2}\varphi_q\left(\int_0^{T/2} h(s)\Delta s\right)\right\}. \tag{3.11}$$

If $u \in P$ with $\|u\| = H_4$, then

$$\begin{aligned}
\|Au\| &= Au\left(\frac{T}{2}\right) \\
&\le \int_0^{T/2} \varphi_q\left(\int_0^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\le K\frac{T}{2}\varphi_q\left(\int_0^{T/2} h(s)\Delta s\right) \\
&\le H_4 \\
&= \|u\|.
\end{aligned} \tag{3.12}$$

Suppose that f is unbounded. From $f \in C([0,+\infty),[0,+\infty))$, we have $f(u) \le C_3$ for arbitrary $u \in [0, C_4]$, here C_3 and C_4 are arbitrary positive constants. This implies that $f(u) \to +\infty$ if $u \to +\infty$. Hence, it is easy to know that there exists $H_4 \ge \max\{2H_3, (T/2\eta)H_4'\}$ such

that $f(u) \leq f(H_4)$ for $u \in [0, H_4]$. If $u \in P$ with $\|u\| = H_4$, then by using (3.9) and (3.10), we have

$$\begin{aligned}
\|Au\| &= Au\left(\frac{T}{2}\right) \\
&= \int_0^{T/2} \varphi_q\left(\int_s^{T/2} h(r) f(u(r)) \nabla r\right) \Delta s \\
&\leq \int_0^{T/2} \varphi_q\left(\int_0^{T/2} h(r) f(H_4) \nabla r\right) \Delta s \\
&\leq \delta H_4 \frac{T}{2} \varphi_q\left(\int_0^{T/2} h(r) \nabla r\right) \\
&\leq \|u\|.
\end{aligned} \tag{3.13}$$

Consequently, in either case, if we take $\Omega_{H_4} = \{u \in E : \|u\| < H_4\}$, then, for $u \in P \cap \partial\Omega_{H_4}$, we have $\|Au\| \leq \|u\|$. Thus, condition (ii) of Lemma 2.2 is satisfied. Consequently, problem (1.3) has at least single positive symmetric solution u in $P \cap (\overline{\Omega}_{H_4} \setminus \Omega_{H_3})$ with $H_3 \leq \|u\| \leq H_4$. The proof is complete. □

3.2. For the Case $i_0 = 0$ and $i_\infty = 0$

In this subsection, we discuss the existence of positive symmetric solutions to problems (1.3) under $i_0 = 0$ and $i_\infty = 0$.

First, we will state and prove the following main result of problem (1.3).

Theorem 3.2. *Suppose that the following conditions hold:*

(i) *there exists constant $p' > 0$ such that $f(u) \leq \varphi_p(p'\Lambda_1)$ for $u \in [0, p']$, where $\Lambda_1 = \{(T/2)\varphi_q(\int_0^{T/2} h(r)\nabla r)\}^{-1}$;*

(ii) *there exists constant $q' > 0$ such that $f(u) \geq \varphi_p(q'\Lambda_2)$ for $u \in [(2\eta/T)q', q']$, where $\Lambda_2 = \{\eta\varphi_q(\int_\eta^{T/2} h(r)\nabla r)\}^{-1}$, furthermore, $p' \neq q'$,*

then problem (1.3) *has at least one positive symmetric solution u such that $\|u\|$ lies between p' and q'.*

Proof. Without loss of generality, we may assume that $p' < q'$.

Let $\Omega_{p'} = \{u \in E : \|u\| < p'\}$. For any $u \in P \cap \partial\Omega_{p'}$, in view of condition (i), we have

$$\begin{aligned}\|Au\| &= Au\left(\frac{T}{2}\right)\\ &= \int_0^{T/2} \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s\\ &\le p'\Lambda_1 \frac{T}{2}\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right)\\ &= p',\end{aligned} \tag{3.14}$$

which yields

$$\|Au\| \le \|u\| \quad \text{for } u \in P \cap \partial\Omega_{p'}. \tag{3.15}$$

Now, set $\Omega_{q'} = \{u \in E : \|u\| < q'\}$. For $u \in P \cap \partial\Omega_{q'}$, Lemma 2.1 implies that

$$\frac{2\eta}{T} q' \le u(t) \le q' \quad \text{for } t \in \left[\eta, \frac{T}{2}\right]_{\mathbb{T}}. \tag{3.16}$$

Hence, by condition (ii) we get

$$\begin{aligned}\|Au\| &= Au\left(\frac{T}{2}\right)\\ &\ge \int_0^{\eta} \varphi_q\left(\int_\eta^{T/2} h(r)f(u(r))\nabla r\right)\Delta s\\ &\ge q'\Lambda_2 \eta \varphi_q\left(\int_\eta^{T/2} h(r)\nabla r\right)\\ &= q'.\end{aligned} \tag{3.17}$$

So, if we take $\Omega_{q'} = \{u \in E : \|u\| < q'\}$, then

$$\|Au\| \ge \|u\|, \quad u \in P \cap \partial\Omega_{q'}. \tag{3.18}$$

Consequently, in view of $p' < q'$, (3.15) and (3.18), it follows from Lemma 2.2 that problem (1.3) has a positive symmetric solution u in $P \cap (\overline{\Omega}_{q'} \setminus \Omega_{p'})$. The proof is complete. □

3.3. *For the Case* $i_0 = 1$ *and* $i_\infty = 0$ *or* $i_0 = 0$ *and* $i_\infty = 1$

In this subsection, under the conditions $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$, we discuss the existence of positive symmetric solutions to problem (1.3).

Theorem 3.3. *Suppose that* $f_0 \in [0, \varphi_p(\Lambda_1))$ *and* $f_\infty \in (\varphi_p((T/2\eta)\Lambda_2), \infty)$ *hold. Then problem* (1.3) *has at least one positive symmetric solution.*

Proof. It is easy to see that under the assumptions, conditions (i) and (ii) in Theorem 3.2 are satisfied. So the proof is easy and we omit it here. □

Theorem 3.4. *Suppose that* $f_0 \in (\varphi_p((T/2\eta)\Lambda_2), \infty)$ *and* $f_\infty \in [0, \varphi_p(\Lambda_1))$ *hold, then problem* (1.3) *has at least one positive symmetric solution.*

Proof. Firstly, let $\varepsilon_1 = f_0 - \varphi_p((T/2\eta)\Lambda_2) > 0$, there exists a sufficiently small $q' > 0$ that satisfies

$$\frac{f(u)}{\varphi_p(u)} \geq f_0 - \varepsilon_1 = \varphi_p\left(\frac{T}{2\eta}\Lambda_2\right) \quad \text{for } u \in (0, q']. \tag{3.19}$$

Thus, $u \in [(2\eta/T)q', q']$, we have

$$f(u) \geq \varphi_p\left(\frac{T}{2\eta}\Lambda_2\right)\varphi_p(u) \geq \varphi_p(\Lambda_2 q'), \tag{3.20}$$

which implies that condition (ii) in Theorem 3.2 holds.

Nextly, for $\varepsilon_2 = \varphi_p(\Lambda_1) - f_\infty > 0$, there exists a sufficiently large p'' $(> q')$ such that

$$\frac{f(u)}{\varphi_p(u)} \leq f_\infty + \varepsilon_2 = \varphi_p(\Lambda_1) \quad \text{for } u \in [p'', \infty). \tag{3.21}$$

We consider two cases.

Case 1. Assume that f is bounded, that is,

$$f(u) \leq \varphi_p(K_1) \quad \text{for } u \in [0, \infty), \tag{3.22}$$

here $K_1 > 0$ some constant. If we take sufficiently large p' such that $p' \geq \max\{K_1/\Lambda_1, p''\}$, then

$$f(u) \leq \varphi_p(K_1) \leq \varphi_p(\Lambda_1 p') \quad \text{for } u \in [0, p']. \tag{3.23}$$

Consequently, from the above inequality, condition (i) of Theorem 3.2 is true.

Case 2. Assume that f is unbounded.

From $f \in C([0, \infty), [0, \infty))$, there exists $p' > p''$ such that

$$f(u) \leq f(p') \quad \text{for } u \in [0, p']. \tag{3.24}$$

Since $p' > p''$, by (3.21), we get $f(p') \leq \varphi_p(\Lambda_1 p')$, hence

$$f(u) \leq f(p') \leq \varphi_p(\Lambda_1 p') \quad \text{for } u \in [0, p']. \tag{3.25}$$

Thus, condition (i) of Theorem 3.2 is fulfilled.

Consequently, Theorem 3.2 implies that the conclusion of this theorem holds. □

From the proof of Theorems 3.1 and 3.2, respectively, we have the following two results.

Corollary 3.5. *Suppose that $f_0 = 0$ and condition (ii) in Theorem 3.2 hold, then problem* (1.3) *has at least one positive symmetric solution.*

Corollary 3.6. *Suppose that $f_\infty = 0$ and condition (ii) in Theorem 3.2 hold, then problem* (1.3) *has at least one positive symmetric solution.*

Theorem 3.7. *Suppose that $f_0 \in (0, \varphi_p(\Lambda_1))$ and $f_\infty = \infty$ hold, then problem* (1.3) *has at least one positive symmetric solution.*

Proof. First, in view of $f_\infty = \infty$, then by inequality (3.7), we have $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{H_2}$. Next, by $f_0 \in (0, \varphi_p(\Lambda_1))$, for $\varepsilon_3 = \varphi_p(\Lambda_1) - f_0 > 0$, there exists a sufficiently small $p' \in (0, H_2)$ such that

$$f(u) \leq (f_0 + \varepsilon_3)\varphi_p(u) = \varphi_p(\Lambda_1 u) \leq \varphi_p(\Lambda_1 p') \quad \text{for } u \in [0, p'], \tag{3.26}$$

which implies that (i) of Theorem 3.2 holds, that is, (3.14) is true. Hence, we obtain $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_{p'}$. The result is obtained and the proof is complete. □

Theorem 3.8. *Suppose that $f_0 = \infty$ and $f_\infty \in (0, \varphi_p(\Lambda_1))$ hold, then problem* (1.3) *has at least one positive symmetric solution.*

Proof. On one hand, since $f_0 = \infty$, by inequality (3.9), one gets $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_{H_3}$. On the other hand, since $f_\infty \in (0, \varphi_p(\Lambda_1))$, from the technique similar to the second part proof in Theorem 3.4, one obtains that condition (i) of Theorem 3.2 is satisfied, that is, inequality (3.14) holds, one has $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_{p'}$, where $p' > H_3$. Hence, problem (1.3) has at least one positive symmetric solution. The proof is complete. □

From Theorems 3.7 and 3.8, respectively, it is easy to obtain the following two corollaries.

Corollary 3.9. *Assume that $f_\infty = \infty$ and condition (i) in Theorem 3.2 hold, then problem* (1.3) *has at least one positive symmetric solution.*

Corollary 3.10. *Assume that $f_0 = \infty$ and condition (i) in Theorem 3.2 hold, then problem* (1.3) *has at least one positive symmetric solution.*

3.4. For the Case $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$

In this subsection, under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$, we study the existence of multiple positive solutions to problems (1.3).

Combining the proofs of Theorems 3.1 and 3.2, it is easy to prove the following two theorems.

Theorem 3.11. *Suppose that $i_0 = 0$ and $i_\infty = 2$ and condition (i) of Theorem 3.2 hold, then problem* (1.3) *has at least two positive solutions $u_1, u_2 \in P$ such that $0 < \|u_1\| < p' < \|u_2\|$.*

Theorem 3.12. *Suppose that $i_0 = 2$ and $i_\infty = 0$ and condition (ii) of Theorem 3.2 hold, then problem* (1.3) *has at least two positive solutions $u_1, u_2 \in P$ such that $0 < \|u_1\| < q' < \|u_2\|$.*

4. Triple Solutions

In the previous section, we have obtained some results on the existence of at least single or twin positive symmetric solutions to problem (1.3). In this section, we will further discuss the existence of positive symmetric solutions to problem (1.3) by using two different methods. And the conclusions we will arrive at are different with their own distinctive advantages.

Based on the obtained symmetric solution position and local properties, we can only get some local properties of solutions by using method one; however, the position of solutions is not determined. In contrast, by means of method two, we cannot only get some local properties of solutions but also give the position of all solutions, with regard to some subsets of the cone, which has to meet some conditions which are stronger than those of method one. Obviously, the local properties of obtained solutions are different by using the two different methods. Hence, it is convenient for us to comprehensively comprehend the solutions of the models by using the two different techniques.

In Section 5, two examples are given to illustrate the differences of the results obtained by the two different methods.

For the notational convenience, we denote

$$\begin{gathered} M_\xi = \eta\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right), \qquad N_\xi = \eta\varphi_q\left(\int_\eta^{T/2} h(r)\nabla r\right), \\ L_\xi = r\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right), \qquad L_\theta = r\varphi_q\left(\int_r^{T/2} h(r)\nabla r\right), \qquad W_\xi = \frac{T}{2}\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right). \end{gathered} \tag{4.1}$$

4.1. Result 1

In this subsection, in view of the generalized Avery-Henderson fixed-point theorem [26], the existence criteria for at least *triple* and arbitrary odd positive symmetric solutions to problems (1.3) are established.

For $u \in P$, we define the nonnegative, increasing, continuous functionals γ, β, and α by

$$\begin{gathered} \gamma(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta), \qquad \beta(u) = \min_{t\in[\eta,T/2]_{\mathbb{T}}} u(t) = u(\eta), \\ \alpha(u) = \max_{t\in[0,r]_{\mathbb{T}}} u(t) = u(r). \end{gathered} \tag{4.2}$$

It is obvious that $\gamma(u) \le \beta(u) \le \alpha(u)$ for each $u \in P$. By Lemma 2.1, one obtains $\|u\| \le C^*\gamma(u)$ for all $u \in P$, here $C^* = T/2\eta$.

We now present the results in this subsection.

Theorem 4.1. *If there are positive numbers a', b', c' such that $a' < (2r/T)b' < (2r/T)(c'N_\xi/M_\xi)$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u) < \varphi_p(c'/M_\xi)$ *for* $u \in [0, (T/2\eta)c']$;

(ii) $f(u) > \varphi_p(b'/N_\xi)$ *for* $u \in [b', (T/2\eta)b']$;

(iii) $f(u) < \varphi_p(a'/L_\xi)$ *for* $u \in [0, (T/2r)a']$.

Then problem (1.3) *has at least three positive symmetric solutions u_1, u_2, and u_3 such that*

$$\begin{gathered} 0 < \max_{t\in[0,r]_{\mathbb{T}}} u_1(t) < a' < \max_{t\in[0,r]_{\mathbb{T}}} u_2(t), \\ \min_{t\in[\eta,T/2]_{\mathbb{T}}} u_2(t) < b' < \min_{t\in[\eta,T/2]_{\mathbb{T}}} u_3(t), \qquad \max_{t\in[0,\eta]_{\mathbb{T}}} u_3(t) < c'. \end{gathered} \tag{4.3}$$

Proof. By the definition of completely continuous operator A and its properties, it has to be demonstrated that all the conditions of Lemma 2.3 hold with respect to A. It is easy to obtain that $A : \overline{P(\gamma, c)} \to P$.

Firstly, we verify that if $u \in \partial P(\gamma, c')$, then $\gamma(Au) < c'$.

If $u \in \partial P(\gamma, c')$, then

$$\gamma(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta) = c'. \tag{4.4}$$

Lemma 2.1 implies that

$$\|u\| \le \frac{T}{2\eta} u(\eta) = \frac{T}{2\eta} c', \tag{4.5}$$

we have

$$0 \le u(t) \le \frac{T}{2\eta} c', \quad t \in \left[0, \frac{T}{2}\right]_{\mathbb{T}}. \tag{4.6}$$

Thus, by condition (i), one has

$$\begin{aligned}
\gamma(Au) &= \max_{t\in[0,\eta]_{\mathbb{T}}} Au(t) \\
&= Au(\eta) \\
&= \int_0^{\eta} \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\le \int_0^{\eta} \varphi_q\left(\int_0^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&< \eta\varphi_q\left(\int_0^{T/2} h(r)\varphi_p\left(\frac{c'}{M_\xi}\right)\nabla r\right) \\
&= \frac{c'}{M_\xi}\eta\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right) \\
&= c'.
\end{aligned} \tag{4.7}$$

Secondly, we show that $\beta(Au) > b'$ for $u \in \partial P(\beta, b')$.

If we choose $u \in \partial P(\beta, b')$, then $\beta(u) = \min_{t\in[\eta,T/2]_{\mathbb{T}}} u(t) = b'$. In view of Lemma 2.1, we have

$$\|u\| \le \frac{T}{2\eta}u(\eta) = \frac{T}{2\eta}b'. \tag{4.8}$$

So

$$b' \le u(t) \le \frac{T}{2\eta}b', \quad t \in \left[\eta, \frac{T}{2}\right]_{\mathbb{T}}. \tag{4.9}$$

Using condition (ii), we get

$$\begin{aligned}
\beta(Au) &= Au(\eta) \\
&\ge \int_0^{\eta} \varphi_q\left(\int_\eta^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&> \eta\varphi_q\left(\int_\eta^{T/2} h(r)\varphi_p\left(\frac{b'}{N_\xi}\right)\nabla r\right) \\
&> \frac{b'}{N_\xi}\eta\varphi_q\left(\int_\eta^{T/2} h(r)\nabla r\right) \\
&= b'.
\end{aligned} \tag{4.10}$$

Finally, we prove that $P(\alpha, a') \neq \emptyset$ and $\alpha(Au) < a'$ for all $u \in \partial P(\alpha, a')$.

In fact, the constant function $a'/2 \in P(\alpha, a')$. Moreover, for $u \in \partial P(\alpha, a')$, we have $\alpha(u) = \max_{t\in[0,r]_{\mathbb{T}}} u(t) = a'$, which implies $0 \leq u(t) \leq a'$ for $t \in [0, r]_{\mathbb{T}}$. Hence, $\|u\| \leq (T/2r)u(r)$. Therefore

$$0 \leq u(t) \leq \frac{T}{2r}a', \quad t \in \left[0, \frac{T}{2}\right]_{\mathbb{T}}. \tag{4.11}$$

By using assumption (iii), one has

$$\begin{aligned}
\alpha(Au) &= (Au)(r) \\
&= \int_0^r \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\right)\nabla r\Delta s \\
&\leq \int_0^r \varphi_q\left(\int_0^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\leq \int_0^r \varphi_q\left(\int_0^{T/2} h(r)\varphi_p\left(\frac{a'}{L_\xi}\right)\nabla r\right)\Delta s \\
&= \frac{a'}{L_\xi} r\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right) \\
&= a'.
\end{aligned} \tag{4.12}$$

Thus, all the conditions in Lemma 2.3 are satisfied. From (H1) and (H2), we have that the solutions to problem (1.3) do not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$. Consequently, problem (1.3) has at least three positive symmetric solutions u_1, u_2, and u_3 belonging to $\overline{P(\gamma, c')}$, and satisfying (4.3). The proof is complete. □

From Theorem 4.1, we see that, when assumptions as (i), (ii), and (iii) are imposed appropriately on f, we can establish the existence of an arbitrary odd number of positive symmetric solutions to problem (1.3).

Theorem 4.2. *Let $l = 1, 2, \ldots, n$. Suppose that there exist positive numbers a'_{s_l}, b'_{s_l}, c'_{s_l} such that*

$$a'_{s_1} < \frac{2r}{T}b'_{s_1} < \frac{2r}{T}\frac{N_\xi}{M_\xi}c'_{s_1} < a'_{s_2} < \frac{2T}{r}b'_{s_2} < \frac{2r}{T}\frac{N_\xi}{M_\xi}c'_{s_2} < a'_{s_3} < \cdots < a'_{s_l}. \tag{4.13}$$

In addition, $f(u)$ satisfies the following conditions:

(i) $f(u) < \varphi_p(c'_{s_l}/M_\xi)$ *for* $u \in [0, (T/2\eta)c'_{s_l}]$;

(ii) $f(u) > \varphi_p(b'_{s_l}/N_\xi)$ *for* $u \in [b'_{s_l}, (T/2\eta)b'_{s_l}]$;

(iii) $f(u) < \varphi_p(a'_{s_l}/L_\xi)$ *for* $u \in [0, (T/2r)a'_{s_l}]$.

Then problem (1.3) *has at least* $2l + 1$ *positive symmetric solutions.*

Proof. When $l = 1$, it is clear that Theorem 4.1 holds. Then we can obtain at least three positive symmetric solutions u_1, u_2, and u_3 satisfying

$$\begin{gathered} 0 < \max_{t\in[0,r]_{\mathbb{T}}} u_1(t) < a_{s_1} < \max_{t\in[0,r]_{\mathbb{T}}} u_2(t), \\ \min_{t_i\in[\eta,T/2]_{\mathbb{T}}} u_2(t) < b_{s_1} < \min_{t_i\in[\eta,T/2]_{\mathbb{T}}} u_3(t), \qquad \max_{t\in[0,\eta]_{\mathbb{T}}} u_3(t) < c_{s_1}. \end{gathered} \tag{4.14}$$

Following this way, we finish the proof by induction. The proof is complete. □

Using Lemma 2.4, it is easy to have the following results.

Theorem 4.3. *Suppose that there are positive numbers a', b', c' such that $a' < (L_\theta/M_\xi)b' < (2\eta/T)(L_\theta/M_\xi)c'$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u) > \varphi_p(c'/N_\xi)$ *for* $u \in [c', (T/2\eta)c']$;

(ii) $f(u) < \varphi_p(b'/M_\xi)$ *for* $u \in [0, (T/2\eta)b']$;

(iii) $f(u) > \varphi_p(a'/L_\theta)$ *for* $u \in [a', (T/2r)a']$.

Then problem (1.3) *has at least three positive symmetric solutions u_1, u_2, and u_3 such that*

$$\begin{gathered} 0 < \max_{t\in[0,r]_{\mathbb{T}}} u_1(t) < a' < \max_{t\in[0,r]_{\mathbb{T}}} u_2(t), \\ \min_{t\in[\eta,T/2]_{\mathbb{T}}} u_2(t) < b' < \min_{t\in[\eta,T/2]_{\mathbb{T}}} u_3(t), \qquad \max_{t\in[0,\eta]_{\mathbb{T}}} u_3(t) < c'. \end{gathered} \tag{4.15}$$

From Theorem 4.3, we can obtain Theorem 4.4 and Corollary 4.5.

Theorem 4.4. *Let $l = 1, 2, \ldots, n$. Suppose that there existence positive numbers a'_{λ_l}, b'_{λ_l}, c'_{λ_l} such that*

$$a'_{\lambda_1} < \frac{L_\theta}{M_\xi} b'_{\lambda_1} < \frac{2\eta}{T}\frac{L_\theta c'_{\lambda_1}}{N_\xi} < a'_{\lambda_2} < \frac{L_\theta}{M_\xi} b'_{\lambda_2} < \frac{2\eta}{T}\frac{L_\theta c'_{\lambda_2}}{N_\xi} < a'_{\lambda_3} < \cdots < a'_{\lambda_l}. \tag{4.16}$$

In addition, $f(u)$ satisfies the following conditions:

(i) $f(u) > \varphi_p(c'_{\lambda_l}/N_\xi)$ *for* $u \in [c'_{\lambda_l}, (T/2\eta)c'_{\lambda_l}]$;

(ii) $f(u) < \varphi_p(b'_{\lambda_l}/M_\xi)$ *for* $u \in [0, (T/2\eta)b'_{\lambda_l}]$;

(iii) $f(u) > \varphi_p(a'_{\lambda_l}/L_\theta)$ *for* $u \in [a'_{\lambda_l}, (T/2r)a'_{\lambda_l}]$.

Then problem (1.3) *has at least $2l + 1$ positive symmetric solutions.*

Corollary 4.5. *Assume that f satisfies the following conditions:*

(i) $f_0 = \infty, f_\infty = \infty$,

(ii) *there exists $c_0 > 0$ such that $f(u) < \varphi_p((2\eta/T)(c_0/M_\xi))$ for $u \in [0, c_0]$,*

then problem (1.3) *has at least three positive symmetric solutions.*

Proof. First, by condition (ii), let $b' = (2\eta/T)c_0$, one gets

$$f(u) < \varphi_p\left(\frac{b'}{M_\xi}\right) \quad \text{for } u \in \left[0, \frac{T}{2\eta}b'\right], \tag{4.17}$$

which implies that (ii) of Theorem 4.3 holds.

Second, choose K_3 sufficiently large to satisfy

$$K_3 L_\theta = K_3 r \varphi_q\left(\int_r^{T/2} h(r)\nabla r\right) > 1. \tag{4.18}$$

Since $f_0 = \infty$, there exists $r_1' > 0$ sufficiently small such that

$$f(u) \geq \varphi_p(K_3)\varphi_p(u) = \varphi_p(K_3 u) \quad \text{for } u \in [0, r_1']. \tag{4.19}$$

Without loss of generality, suppose $r_1' \leq (L_\theta T/2rM_\xi)b'$. Choose $a' > 0$ such that $a' < (2r/T)r_1'$. For $a' \leq u \leq (T/2r)a'$, we have $u \leq r_1'$ and $a' < (L_\theta/M_\xi)b'$. Thus, by (4.18) and (4.19), we have

$$f(u) \geq \varphi_p(K_3 u) \geq \varphi_p(K_3 a') > \varphi_p\left(\frac{a'}{L_\theta}\right) \quad \text{for } u \in \left[a', \frac{T}{2r}a'\right], \tag{4.20}$$

this implies that (iii) of Theorem 4.3 is true.

Third, choose K_2 sufficiently large such that

$$K_2 N_\xi = K_2 \eta \varphi_q\left(\int_\eta^{T/2} h(r)\nabla r\right) > 1. \tag{4.21}$$

Since $f_\infty = \infty$, there exists $r_2' > 0$ sufficiently large such that

$$f(u) \geq \varphi_p(K_2)\varphi_p(u) = \varphi_p(K_2 u) \quad \text{for } u \geq r_2'. \tag{4.22}$$

Without loss of generality, suppose $r_2' > (T/2\eta)b'$. Choose $c' = r_2'$. Then

$$f(u) \geq \varphi_p(K_2 u) \geq \varphi_p(K_2 c') > \varphi_p\left(\frac{c'}{N_\xi}\right) \quad \text{for } u \in \left[c', \frac{T}{2\eta}c'\right], \tag{4.23}$$

which means that (i) of Theorem 4.3 holds.

From above analysis, we get

$$0 < a' < \frac{L_\theta}{M_\xi}b' < \frac{2\eta}{T}\frac{L_\theta c'}{M_\xi}, \tag{4.24}$$

then, all conditions in Theorem 4.3 are satisfied. Hence, problem (1.3) has at least three positive symmetric solutions. □

In terms of Theorem 4.1, we also have the following corollary.

Corollary 4.6. *Assume that f satisfies conditions*

(i) $f_0 = 0, f_\infty = 0$;

(ii) *there exists* $c_0 > 0$ *such that* $f(u) > \varphi_p((2\eta/T)(c_0/N_\xi))$ *for* $u \in [(2\eta/T)c_0, c_0]$,

then problem (1.3) *has at least three positive symmetric solutions.*

4.2. Result 2

In this subsection, the existence criteria for at least *triple* positive or arbitrary odd positive symmetric solutions to problems (1.3) are established by using the Avery-Peterson fixed point theorem [27].

Define the nonnegative continuous convex functionals ϕ and β, nonnegative continuous concave functional λ, and nonnegative continuous functional φ, respectively, on P by

$$\phi(u) = \max_{t\in[0,T/2]_{\mathbb{T}}} u(t) = u\left(\frac{T}{2}\right), \qquad \beta(u) = \max_{t\in[r,T/2]_{\mathbb{T}^\kappa}} \left|u^\Delta(t)\right| = \left|u^\Delta(r)\right|,$$
$$\lambda(u) = \varphi(u) = \min_{t\in[\eta,T/2]_{\mathbb{T}}} u(t) = u(\eta). \tag{4.25}$$

Now, we list and prove the results in this subsection.

Theorem 4.7. *Suppose that there exist constants a^*, b^*, d^* such that $0 < a^* < (2\eta/T)b^* < (2\eta/T)(N_\xi d^*/W_\xi)$. In addition, suppose that $W_\xi > \varphi_q(\int_\eta^{T/2} h(s)\nabla s)$ holds, f satisfies the following conditions:*

(i) $f(u) \le \varphi_p(d^*/W_\xi)$ *for* $u \in [0, d^*]$;

(ii) $f(u) > \varphi_p(b^*/N_\xi)$ *for* $u \in [b^*, d^*]$;

(iii) $f(u) < \varphi_p(a^*/M_\xi)$ *for* $u \in [0, (T/2\eta)a^*]$,

then problem (1.3) *has at least three positive symmetric solutions u_1, u_2, u_3 such that*

$$\|u_i\| \le d^*, \quad i = 1,2,3, \qquad b^* < u_1(\eta), \qquad u_2(\eta) < b^*, \qquad u_3(\eta) < a^*. \tag{4.26}$$

Proof. By the definition of completely continuous operator A and its properties, it suffices to show that all the conditions of Lemma 2.5 hold with respect to A.

For all $u \in P, \lambda(u) = \varphi(u)$ and $\|u\| = u(T/2) = \phi(u)$. Hence, condition (2.8) is satisfied.

Firstly, we show that $A : \overline{P(\phi, d^*)} \to \overline{P(\phi, d^*)}$.

For any $u \in \overline{P(\phi, d^*)}$, in view of $\phi(u) = \|u\| \le d^*$ and assumption (i), one has

$$\begin{aligned}
\|Au\| &= Au\left(\frac{T}{2}\right) \\
&= \int_0^{T/2} \varphi_q\left(\int_s^{T/2} h(r) f(u(r)) \nabla r\right) \Delta s \\
&\le \int_0^{T/2} \varphi_q\left(\int_0^{T/2} h(r) f(u(r)) \nabla r\right) \Delta s \\
&\le \frac{d^*}{W_\xi} \frac{T}{2} \varphi_q\left(\int_0^{T/2} h(r) \nabla r\right) \\
&= d^*.
\end{aligned} \tag{4.27}$$

From the above analysis, it remains to show that (i)–(iii) of Lemma 2.5 hold.

Secondly, we verify that condition (i) of Lemma 2.5 holds, let $u(t) \equiv kb^*$ with $k = W_\xi / N_\xi > 1$. From the definitions of N_ξ, W_ξ, and $\beta(u)$, respectively, it is easy to see that $u(t) = kb^* > b^*$ and $\beta(u) = 0 < kb^*$. In addition, in view of $b^* < (N_\xi / W_\xi) d^*$, we have $\phi(u) = kb^* < d^*$. Thus

$$\left\{u \in P\left(\phi, \beta, \lambda, b^*, kb^*, d^*\right) : \lambda(x) > b^*\right\} \ne \emptyset. \tag{4.28}$$

For any $u \in P(\phi, \beta, \lambda, b^*, kb^*, d^*)$, then we get $b^* \le u(t) \le d^*$ for all $t \in [\eta, T/2]_{\mathbb{T}}$. Hence, by assumption (ii), we have

$$\begin{aligned}
\lambda(Au) &= Au(\eta) \\
&= \int_0^{\eta} \varphi_q\left(\int_s^{T/2} h(r) f(u(r)) \nabla r\right) \Delta s \\
&\ge \int_0^{\eta} \varphi_q\left(\int_\eta^{T/2} h(r) f(u(r)) \nabla r\right) \Delta s \\
&> \frac{b^*}{N_\xi} \eta \varphi_q\left(\int_\eta^{T/2} h(r) \nabla r\right) \\
&= b^*.
\end{aligned} \tag{4.29}$$

Thirdly, we prove that condition (ii) of Lemma 2.5 holds. For any $u \in P(\phi, \lambda, b^*, d^*)$ with $\beta(Au) > kb^*$, that is,

$$\beta(Au) = \left|(Au)^{\Delta}(r)\right| = \varphi_q\left(\int_r^{T/2} h(s) f(u(s)) \nabla s\right) > kb^*. \tag{4.30}$$

So, in view of $k = W_\xi/N_\xi$, $W_\xi > \varphi_q(\int_\eta^{T/2} h(s)\nabla s)$ and (4.30), one has

$$\begin{aligned}
\lambda(Au) &= Au(\eta) \\
&= \int_0^\eta \varphi_q\left(\int_s^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&\geq \int_0^\eta \varphi_q\left(\int_\eta^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&> \int_0^\eta \varphi_q\left(\int_r^{T/2} h(r)f(u(r))\nabla r\right)\Delta s \\
&> \eta k b^* \\
&> b^*.
\end{aligned} \tag{4.31}$$

Finally, we check condition (iii) of Lemma 2.5. Clearly, since $\varphi(0) = 0 < a^*$, we have $0 \notin R(\phi, \varphi, a^*, d^*)$. If $u \in R(\phi, \varphi, a^*, d^*)$ with $\varphi(u) = \min_{t\in[\eta,T/2]_{\mathbb{T}}} u(t) = a^*$, then Lemma 2.1 implies that

$$\|u\| \leq \frac{T}{2\eta}u(\eta) = \frac{T}{2\eta}a^*. \tag{4.32}$$

This yields $0 \leq u(t) \leq (T/2\eta)a^*$ for all $t \in [0, T/2]_{\mathbb{T}}$. Hence, by assumption (iii), we have

$$\lambda(Au) = Au(\eta) \leq \int_0^\eta \varphi_q\left(\int_0^{T/2} h(r)f(u(r))\nabla r\right)\Delta s < \frac{a^*}{M_\xi}\eta\varphi_q\left(\int_0^{T/2} h(r)\nabla r\right) = a^*. \tag{4.33}$$

Consequently, all conditions of Lemma 2.5 are satisfied. The proof is completed. □

We remark that condition (i) in Theorem 4.7 can be replaced by the following condition (i′):

$$\lim_{u\to\infty} \frac{f(u)}{\varphi_p(u)} \leq \varphi_p\left(\frac{1}{W_\xi}\right), \tag{4.34}$$

which is a special case of (i).

Corollary 4.8. *If condition (i) in Theorem 4.7 is replaced by (i′), then the conclusion of Theorem 4.7 also holds.*

Proof. By Theorem 4.7, we only need to prove that (i′) implies that (i) holds, that is, if (i′) holds, then there is a number $d^* \geq \max\{(T/2\eta)a^*, (W_\xi/N_\xi)b^*\}$ such that $f(u) \leq \varphi_p(d^*/W_\xi)$ for $u \in [0, d^*]$.

Suppose on the contrary that for any $d^* \geq \max\{(T/2\eta)a^*, (W_\xi/N_\xi)b^*\}$, there exists $u_c \in [0, d^*]$ such that $f(u_c) > \varphi_p(d^*/W_\xi)$. Hence, if we choose

$$c_n' > \max\left\{\frac{T}{2\eta}a^*, \frac{W_\xi}{N_\xi}b^*\right\} \quad (n = 1, 2, \ldots) \tag{4.35}$$

with $c_n' \to \infty$, then there exist $u_n \in [0, c_n']$ such that

$$f(u_n) > \varphi_p\left(\frac{c_n'}{W_\xi}\right), \tag{4.36}$$

and so

$$\lim_{n\to\infty} f(u_n) = \infty. \tag{4.37}$$

Since condition (i′) holds, then there exists $\tau > 0$ such that

$$f(u) \leq \varphi_p\left(\frac{u}{W_\xi}\right) \quad \text{for } u \in (\tau, \infty). \tag{4.38}$$

Hence, we have $u_n \leq \tau$. Otherwise, if $u_n > \tau$, then it follows from (4.38) that

$$f(u_n) \leq \varphi_p\left(\frac{u_n}{W_\xi}\right) \leq \varphi_p\left(\frac{c_n'}{W_\xi}\right), \tag{4.39}$$

which contradicts (4.36).

Let $W = \max_{u\in[0,\tau]} f(u)$, then $f(u_n) \leq W$ $(n = 1, 2, \ldots)$, which also contradicts (4.37). The proof is complete. □

Theorem 4.9. *Let $l = 1, 2, \ldots, n$. Suppose that there exist constants a_l^*, b_l^*, d_l^* such that*

$$0 < a_1^* < \frac{2\eta}{T}b_1^* < \frac{2\eta}{T}\frac{d_1^* N_\xi}{W_\xi} < a_2^* < \frac{2\eta_i}{T}b_2^* < \frac{2\eta}{T}\frac{d_2^* N_\xi}{W_\xi} < a_3^* < \cdots < a_l^*. \tag{4.40}$$

In addition, suppose that $W_\xi > \varphi_q(\int_\eta^{T/2} h(s)\nabla s)$ holds, then f satisfies the following conditions:

(i) *$f(u) < \varphi_p(d_l^*/W_\xi)$ for $u \in [0, d_l^*]$;*

(ii) *$f(u) > \varphi_p(b_l^*/N_\xi)$ for $u \in [b_l^*, d_l^*]$;*

(iii) *$f(u) < \varphi_p(a_l^*/M_\xi)$ for $u \in [0, (T/2\eta)a_l^*]$,*

then problem (1.3) *has at least $2l + 1$ positive symmetric solutions.*

Proof. Similar to the proof of Theorem 4.2, we omit it here. □

5. Examples

In this section, we give two simple examples to illustrate that the conclusions we will arrive at are different with their own distinctive advantages.

Example 5.1. Let

$$\mathbb{T} = \{0, 0.05, 0.1, 0.15, 0.45, 0.48, 0.5, 0.55, 0.52, 0.85, 0.9, 0.95, 1\} \cup [0.22, 0.44] \cup [0.56, 0.78]. \tag{5.1}$$

Consider the following boundary value problem with $p = 3$:

$$\begin{gathered} \left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + h(t) f(u(t)) = 0, \quad t \in (0,1)_{\mathbb{T}}, \\ u(0) = u(1) = 0, \qquad u^{\Delta}(0) = -u^{\Delta}(1), \end{gathered} \tag{5.2}$$

where

$$\begin{gathered} h(t) = \begin{cases} t + \rho(t), & t \in [0, 0.5]_{\mathbb{T}}, \\ 1 - t + \rho(1-t), & t \in [0.5, 1]_{\mathbb{T}}, \end{cases} \\ f(u) = \begin{cases} 0.16, & u \in [0, 0.6], \\ 1049.6u - 629.6, & u \in [0.6, 1], \\ 420, & u \in [1, 5], \\ 2.8364u + 405.82, & u \in [5, 60], \\ 576, & u \in [60, \infty). \end{cases} \end{gathered} \tag{5.3}$$

Note that $T = 1$. If we choose $\eta = 0.1$, $r = 0.25$, then direct calculation shows that

$$M_{\xi} = \eta \varphi_q\left(\int_0^{1/2} h(r) \nabla r\right) = 0.05, \tag{5.4}$$

and $N_{\xi} \approx 0.0489$, $L_{\xi} = 0.125$, $W_{\xi} = 0.2500$. If we take $a' = 0.3$, $b' = 1$, $c' = 12$, then

$$a' = 0.3 < \frac{2r}{T} b' \approx 0.9714 < \frac{2r}{T} \frac{c' N_{\xi}}{M_{\xi}} \approx 9.173. \tag{5.5}$$

Furthermore,

$$\begin{aligned} f(u) &= 0.16 < 5.76 = \varphi_p\left(\frac{a'}{L_\xi}\right), \quad u \in [0, 0.6], \\ f(u) &= 420 > 418.2 \approx \varphi_p\left(\frac{b'}{N_\xi}\right), \quad u \in [1, 5], \\ f(u) &\leq 576 < 57600 = \varphi_p\left(\frac{c'}{M_\xi}\right), \quad u \in [0, 60]. \end{aligned} \tag{5.6}$$

By Theorem 4.1 we see that the boundary value problem (5.2) has at least three positive symmetric solutions u_1, u_2, and u_3 such that

$$0 < u_1(0.25) < 0.3 < u_2(0.25), \qquad u_2(0.1) < 1 < u_3(0.1), \qquad u_3(0.1) < 12. \tag{5.7}$$

Yet, $\varphi_q(\int_{0.1}^{1/2} h(s)\nabla s) \approx 0.489 > W_\xi \approx 0.25$, hence, the existence of positive solutions of boundary value problem (5.2) is not obtained by using Theorem 4.7.

Example 5.2. Let

$$\begin{aligned} \mathbb{T} = \{&0, 0.05, 0.1, 0.15, 0.45, 0.46, 0.48, 0.5, 0.52, 0.54, 0.55, 0.85, 0.9, 0.95, 1\} \\ &\cup [0.22, 0.44] \cup [0.56, 0.78]. \end{aligned} \tag{5.8}$$

Consider the following boundary value problem:

$$\begin{gathered} \left(\varphi_p\left(u^\Delta(t)\right)\right)^\nabla + h(t) f(u(t)) = 0, \quad t \in (0, 1)_{\mathbb{T}}, \\ u(0) = u(1) = 0, \qquad u^\Delta(0) = -u^\Delta(1), \end{gathered} \tag{5.9}$$

where

$$h(t) = \begin{cases} t + \rho(t), & t \in [0, 0.5]_{\mathbb{T}}, \\ 1 - t + \rho(1 - t), & t \in [0.5, 1]_{\mathbb{T}}. \end{cases} \tag{5.10}$$

Note that $T = 1$. If we choose $\eta = 0.45$, $r = 0.48$, then a direct calculation shows that

$$\begin{gathered} M_\xi = \eta\varphi_q\left(\int_0^{1/2} h(r)\nabla r\right) = 0.225, \\ N_\xi \approx 0.0981, \qquad L_\xi \approx 0.24, \qquad W_\xi \approx 0.25, \quad W_\xi \approx 0.25 > 0.218 \approx \varphi_q\left(\int_{0.45}^{0.5} h(r)\nabla r\right). \end{gathered} \tag{5.11}$$

Let ε be an arbitrary small positive number, a^*, b^*, and d^* are arbitrary positive numbers with

$$0 < a^* < 2\eta b^* < 2\eta \frac{N_\xi d^*}{W_\xi}. \tag{5.12}$$

That is

$$\begin{gathered} 0 < a^* < 0.9b^* < 0.3924d^*, \\ f(u) = \begin{cases} \varphi_p\left(\dfrac{a^*}{0.225}\right) - \varepsilon, & 0 \le u \le a^*, \\ l(u), & a^* \le u \le b^*, \\ \varphi_p\left(\dfrac{b^*}{0.0981}\right) + \varepsilon, & b^* \le u \le d^*, \end{cases} \end{gathered} \tag{5.13}$$

where $l(u)$ satisfy $l(a^*) = \varphi_p(a^*/0.225) - \varepsilon$, $l(b^*) = \varphi_p(b^*/0.0981) + \varepsilon$, $(l^\Delta(u))^\Delta = 0$, $u \in [a^*, b^*]$.

It is obvious that (i), (ii), and (iii) in Theorem 4.7 are satisfied. By Theorem 4.7, we see that the boundary value problem (5.9) has at least three positive solutions u_1, u_2, and u_3 such that

$$\|u_i\| \le d^*, \quad i = 1, 2, 3, \qquad b^* < u_1(0.45), \qquad a^* < u_2(0.45), \qquad u_3(0.45) < a^*. \tag{5.14}$$

However, for arbitrary positive numbers a^*, b^*, d^*, condition (iii) of Theorem 4.1 is not satisfied. Therefore, Theorem 4.1 is not fit to study the boundary value problem (5.9).

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