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Research Article

Existence Results for Second-Order Impulsive Neutral Functional Differential Equations with Nonlocal Conditions

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The existence of mild solutions for second-order impulsive semilinear neutral functional differential equations with nonlocal conditions in Banach spaces is investigated. The results are obtained by using fractional power of operators and Sadovskii's fixed point theorem.

1. Introduction

The study of impulsive functional differential equations is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place "instantaneously" in the form of impulses. The theory of impulsive differential and functional differential equations has been extensively developed; see the monographs of Bainov and Simeonov [1], Lakshmikantham et al. [2], and Samoilenko and Perestyuk [3], where numerous properties of their solutions are studied, and detailed bibliographies are given.

This paper is devoted to extending existing results to second-order differential equations. To be precise, in [4], the authors used Sadovsii's fixed point theorem for a condensing map to establish existence results for first-order impulsive semilinear neutral functional differential inclusions with nonlocal conditions. Here, we obtain existence results for second-order semilinear impulsive differential equations with nonlocal conditions of the form

$$\frac{d}{dt}\left[x'(t) - F(t, x(h_1(t)))\right] = Ax(t) + G(t, x(h_2(t))), \quad t \in J = [0, b],\ t \neq t_k,$$
$$\Delta x|_{t=t_k} = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, \ldots, m,$$

$$\Delta x'\big|_{t=t_k} = \overline{I}_k\left(x\left(t_k^-\right)\right), \quad k = 1, \ldots, m,$$
$$x(0) + g(x) = x_0, \quad x'(0) = \eta, \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in R$, of bounded linear operators in X. Also, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$. Finally, $F, G, g, I_k, \overline{I}_k$ $(k = 1, \ldots, m)$ and h_1, h_2 are given functions to be specified later.

Other results on second order functional differential equations with and without impulsive effect can be founded in the monographs [5–8].

This paper is organized as follows. In Section 2, we recall briefly some basic definitions and lemmas. The existence theorem for (1.1) and its proof are arranged in Section 3. Our approaches are based on Sadovskii's fixed point theorem, and the theory of strongly continuous cosine families.

2. Preliminaries

Definition 2.1 (see [9]). A one-parameter family $C(t),\ t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in R$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is strongly continuous in t on R for each fixed $x \in X$.

We define the associated sine family $S(t),\ t \in R$, by

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X,\ t \in R. \tag{2.1}$$

We make the following assumption on A:

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in R$, of bounded linear operators from X into itself.

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in R$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x\bigg|_{t=0}, \quad x \in D(A), \tag{2.2}$$

where

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}. \tag{2.3}$$

We define

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}. \tag{2.4}$$

Lemma 2.2 (see [9]). *If $C(t)$, $t \in R$, be a strongly continuous cosine family in X, then*

(i) *there exist constants $K \geq 1$ and $\omega \geq 0$ so that $\|C(t)\| \leq Ke^{\omega|t|}$, for all $t \in R$, and*

$$\|S(t_1) - S(t_2)\| \leq K\left|\int_{t_1}^{t_2} e^{\omega|s|}ds\right|, \quad \forall t_1, t_2 \in R; \tag{2.5}$$

(ii) *if $x \in E$, then $S(t)x \in D(A)$ and $(d/dt)C(t)x = AS(t)x$.*

It is proved in [10] that for $0 \leq \alpha \leq 1$, the fractional powers $(-A)^\alpha$ exist as close linear operator in X, $D((-A)^\alpha) \subset D((-A)^\beta)$, for $0 \leq \beta \leq \alpha \leq 1$, and $(-A)^\alpha(-A)^\beta = (-A)^{\alpha+\beta}$ for $0 \leq \alpha + \beta \leq 1$.

We assume in addition the following assumption:

(H_2) for $0 \leq \alpha \leq 1$, $(-A)^\alpha$ maps onto X and is $1-1$, so that $D((-A)^\alpha)$ is a Banach space when endowed with the form $\|x\|_\alpha = \|(-A)^\alpha x\|$, $x \in D((-A)^\alpha)$. We denote this Banach space by X_α.

Denote $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. We define the following classes of functions:

$PC(J, X_\alpha) = \{x : J \to X_\alpha : x_k \in C(J_k, X_\alpha),\ k = 0, 1, \ldots, m$ and there exist $x(t_k^+)$, $x(t_k^-)$, $k = 1, \ldots, m$ with $x(t_k) = x(t_k^-)\}$:

$PC^1(J, X_\alpha) = \{x \in PC(J, X_\alpha) : x'_k \in C(J_k, X_\alpha),\ k = 0, 1, \ldots, m$ and there exist $x'(t_k^+)$, $x'(t_k^-)$, $k = 1, \ldots, m$ with $x'(t_k) = x'(t_k^-)\}$, where x_k and x'_k represent the restriction of x and x' to J_k, respectively, $(k = 0, \ldots, m)$, and $\|x_k\|_{J_k} = \sup_{s \in J_k}\|x_k(s)\|_\alpha$.

Obviously, $PC(J, X_\alpha)$ is a Banach space with the norm $\|x\|_{PC} = \max\{\|x_k\|_{J_k}, k = 0, \ldots, m\}$, and $PC^1(J, X_\alpha)$ is also a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Definition 2.3. A function $x(\cdot) \in PC^1(J, X_\alpha)$ is said to be a mild solution of (1.1) if

(i) $x(0) + g(x) = x_0$, $x'(0) = \eta$;

(ii) $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, \ldots, m$;

(iii) $\Delta x'|_{t=t_k} = \overline{I}_k(x(t_k^-))$, $k = 1, \ldots, m$;

(iv) the restriction of $x(\cdot)$ to the interval J_k $(k = 0, \ldots, m)$ is continuous and the following integral equation is verified:

$$\begin{aligned} x(t) &= C(t)\left[x_0 - g(x)\right] + S(t)\left[\eta - F(0, x(h_1(0)))\right] + \int_0^t C(t-s)F(s, x(h_1(s)))ds \\ &\quad + \int_0^t S(t-s)G(s, x(h_2(s)))ds + \sum_{0<t_k<t} C(t-t_k)I_k\left(x\left(t_k^-\right)\right) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k\left(x\left(t_k^-\right)\right), \quad t \in J. \end{aligned} \tag{2.6}$$

For (1.1), we assume that the following hypotheses are satisfied: for some $\alpha \in (0,1)$,

(H_3) there exists a constant $\beta \in (0,1)$ such that $F : J \times X_\alpha \to X_\beta$ is a continuous function, and $(-A)^\beta F : J \times X_\alpha \to X_\alpha$ satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that

$$\left\|(-A)^\beta F(t_1, x_1) - (-A)^\beta F(t_2, x_2)\right\|_\alpha \le L(|t_1 - t_2| + \|x_1 - x_2\|_\alpha), \tag{2.7}$$

for any $0 \le t_1,\ t_2 \le b,\ x_1, x_2 \in X_\alpha$. Moreover, there exists a constant $L_1 > 0$ such that the inequality

$$\left\|(-A)^\beta F(t, x)\right\|_\alpha \le L_1(\|x\|_\alpha + 1) \tag{2.8}$$

holds for any $x \in X_\alpha$;

(H_4) the function $G : J \times X_\alpha \to X$ satisfies the following conditions:

(i) for each $t \in J$, the function $G(t, \cdot) : X_\alpha \to X$ is continuous, and for each $x \in X_\alpha$, the function $G(\cdot, x) : J \to X$ is strongly measurable,

(ii) for each positive number $l \in N$, there is a positive function $w_l \in L^1(J)$ such that

$$\sup_{\|x\|_\alpha \le l} \|G(t,x)\| \le w_l(t) \quad \text{a.e. on } J, \qquad \liminf_{l \to \infty} \frac{1}{l}\int_0^b w_l(s)ds = \gamma < \infty, \tag{2.9}$$

where

$$\|x\|_\alpha = \sup_{0 \le s \le b} \|x(s)\|_\alpha; \tag{2.10}$$

(H_5) $h_i \in C(J, J),\ i = 1, 2.$ $g : PC^1(J, X_\alpha) \to X_\alpha$ is continuous and satisfies that

(i) there exist positive constants L_2 and L_2' such that

$$\|g(u)\|_\alpha \le L_2\|u\|_{PC^1} + L_2' \quad \forall u \in PC^1(J, X_\alpha), \tag{2.11}$$

(ii) g is a completely continuous map;

(H_6) $I_k, \overline{I}_k \in C(X_\alpha, X_\alpha)$, $k = 1, \ldots, m$ are all bounded, that is, there exist constants $d_k, \overline{d}_k, k = 1, \ldots, m$, such that $\|I_k(x)\|_\alpha \le d_k$, $\|\overline{I}_k(x)\|_\alpha \le \overline{d}_k$, for $x \in X_\alpha$;

(H_7) $C(t)$, $t \in J$, is completely continuous.

3. Main Result

Theorem 3.1. *Let $x_0 \in X_\alpha$. If the hypotheses (H_1)–(H_7) are satisfied, then* (1.1) *has a mild solution provided that*

$$L_0 := 2M_0LMb < 1, \tag{3.1}$$

$$M(L_2 + 2M_0bL_1 + b\gamma) < 1, \tag{3.2}$$

where

$$M = \sup\{\|C(t)\| : t \in J\}, \qquad M' = \sup\{\|C'(t)\| : t \in J\}, \qquad M_0 = \left\|(-A)^{-\beta}\right\|. \tag{3.3}$$

Proof. Consider the space $B = PC^1(J, X_\alpha)$ with morm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. We should now show that the operator P defined by

$$\begin{aligned}(Px)(t) &= C(t)[x_0 - g(x)] + S(t)[\eta - F(0, x(h_1(0)))] + \int_0^t C(t-s)F(s, x(h_1(s)))ds \\ &\quad + \int_0^t S(t-s)G(s, x(h_2(s)))ds + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k^-)) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(x(t_k^-))\end{aligned} \tag{3.4}$$

has a fixed point. This fixed point is then a solution of (2.6).

For each positive number l, let $B_l = \{x \in B : \|x(t)\|_\alpha \le l, t \in J\}$. Then for each l, B_l is clearly a bounded close convex set in B. We claim that there exists a positive integer l such that $PB_l \subseteq B_l$. If it is not true, then for each positive integer l, there is a function $x_l(\cdot) \in B_l$, but

$(Px_l)(\cdot) \notin B_l$, that is, $\|(Px_l)(t)\|_\alpha > l$ for some $t(l) \in J$, where $t(l)$ denotes t is dependent on l. However, on the other hand, we have

$$\begin{aligned}
l < \|(Px_l)(t)\|_\alpha &= \Big\| C(t)\left[x_0 - g(x_l)\right] + S(t)\left[\eta - F(0, x_l(h_1(0)))\right] \\
&\quad + \int_0^t C(t-s)F(s, x_l(h_1(s)))ds + \int_0^t S(t-s)G(s, x_l(h_2(s)))ds \\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k\left(x_l(t_k^-)\right) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k\left(x_l(t_k^-)\right)\Big\|_\alpha \\
&\le \|C(t)\left[x_0 - g(x_l)\right]\|_\alpha + \left\|S(t)\left[\eta - (-A)^{-\beta}(-A)^{\beta}F(0, x_l(h_1(0)))\right]\right\|_\alpha \\
&\quad + \left\|\int_0^t C(t-s)(-A)^{-\beta}(-A)^{\beta}F(s, x_l(h_1(s)))ds\right\|_\alpha \\
&\quad + \left\|\int_0^t S(t-s)G(s, x_l(h_2(s)))ds\right\|_\alpha \\
&\quad + \sum_{0<t_k<t} \|C(t-t_k)I_k\left(x_l(t_k^-)\right)\|_\alpha + \sum_{0<t_k<t} \left\|S(t-t_k)\bar{I}_k\left(x_l(t_k^-)\right)\right\|_\alpha \\
&\le M\left[\|x_0\|_\alpha + L_2 l + L_2'\right] + Mb\left[\|\eta\|_\alpha + M_0 L_1(l+1)\right] \\
&\quad + MM_0 bL_1(l+1) + Mb\int_0^b w_l(s)ds + M\sum_{k=1}^m d_k + M\sum_{k=1}^m (b-t_k)\bar{d}_k.
\end{aligned} \tag{3.5}$$

Dividing on both sides by l and taking the lower limits as $l \to +\infty$, we get $M(L_2 + 2M_0bL_1 + b\gamma) \ge 1$. This is a contradiction with the formula (3.2). Hence for some positive integer l, $PB_l \subseteq B_l$.

Next we will show that the operator P has a fixed point on B_l, which implies that (1.1) has a mild solution. For this purpose, we decompose P as $P = P_1 + P_2$, where the operators P_1, P_2 are defined on B_l, respectively, by

$$\begin{aligned}
(P_1x)(t) &= \int_0^t C(t-s)F(s, x(h_1(s)))ds - S(t)F(0, x(h_1(0))), \\
(P_2x)(t) &= C(t)\left[x_0 - g(x)\right] + S(t)\eta + \int_0^t S(t-s)G(s, x(h_2(s)))ds \\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k\left(x(t_k^-)\right) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k\left(x(t_k^-)\right),
\end{aligned} \tag{3.6}$$

for $t \in J$, and we will verify that P_1 is a contraction while P_2 is a completely continuous operator.

To prove that P_1 is a contraction, we take $x_1, x_2 \in B_l$ arbitrarily. Then for each $t \in J$ and by condition (H_3), we have that

$$\begin{aligned}
\|(P_1x_1)(t) - (P_1x_2)(t)\|_\alpha &\leq \left\| \int_0^t C(t-s)[F(s, x_1(h_1(s))) - F(s, x_2(h_1(s)))]ds \right\|_\alpha \\
&\quad + \|S(t)[F(0, x_1(h_1(0))) - F(0, x_2(h_1(0)))]\|_\alpha \\
&= \left\| \int_0^t C(t-s)(-A)^{-\beta}(-A)^{\beta}[F(s, x_1(h_1(s))) - F(s, x_2(h_1(s)))]ds \right\|_\alpha \\
&\quad + \left\| S(t)(-A)^{-\beta}\left\{(-A)^{\beta}[F(0, x_1(h_1(0))) - F(0, x_2(h_1(0)))]\right\} \right\|_\alpha \\
&\leq 2M_0LMb \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\|_\alpha \\
&= L_0 \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\|_\alpha.
\end{aligned} \tag{3.7}$$

Thus $\|P_1x_1 - P_1x_2\|_\alpha \leq L_0\|x_1 - x_2\|_\alpha$. Therefore, by assumption $0 < L_0 < 1$ (see (3.1)), we see that P_1 is a contraction.

To prove that P_2 is completely continuous, firstly we prove that P_2 is continuous on B_l. Let $x_n \to x_*, x_n \in B_l$, then by (H_4)(i), we have $G(s, x_n(h_2(s))) \to G(s, x_*(h_2(s))),\ n \to \infty$. Since $\|G(s, x_n(h_2(s))) - G(s, x_*(h_2(s)))\| \leq 2w_l(s)$, by the dominated convergence theorem, we have

$$\begin{aligned}
&\|P_2x_n - P_2x_*\|_{PC} \\
&\quad = \sup_{t\in J} \Bigg\| C(t)\left[g(x_*) - g(x_n)\right] + \int_0^t S(t-s)[G(s, x_n(h_2(s))) - G(s, x_*(h_2(s)))]ds \\
&\qquad + \sum_{0<t_k<t} C(t-t_k)\left[I_k\left(x_n(t_k^-)\right) - I_k\left(x_*(t_k^-)\right)\right] + \sum_{0<t_k<t} S(t-t_k)\left[\overline{I}_k\left(x_n(t_k^-)\right) - \overline{I}_k\left(x_*(t_k^-)\right)\right] \Bigg\|_\alpha \\
&\quad \leq M\left\|g(x_*) - g(x_n)\right\|_\alpha + \int_0^b \|S(t-s)[G(s, x_n(h_2(s))) - G(s, x_*(h_2(s)))]\|_\alpha ds \\
&\qquad + \sum_{k=1}^m M\left\|I_k(x_n(t_k^-)) - I_k\left(x_*(t_k^-)\right)\right\|_\alpha
\end{aligned}$$

$$+\sum_{k=1}^{m} M(b-t_k)\left\|\overline{I}_k\left(x_n\left(t_k^-\right)\right)-\overline{I}_k\left(x_*\left(t_k^-\right)\right)\right\|_\alpha \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty,$$

$$\begin{aligned}
&\left\|(P_2x_n)'-(P_2x_*)'\right\|_{PC}\\
&\quad=\sup_{t\in J}\Bigg\|C'(t)\left[g(x_*)-g(x_n)\right]+\int_0^t C(t-s)[G(s,x_n(h_2(s)))-G(s,x_*(h_2(s)))]ds\\
&\qquad+\sum_{0<t_k<t} C'(t-t_k)\left[I_k\left(x_n\left(t_k^-\right)\right)-I_k\left(x_*\left(t_k^-\right)\right)\right]\\
&\qquad+\sum_{0<t_k<t} C(t-t_k)\left[\overline{I}_k\left(x_n\left(t_k^-\right)\right)-\overline{I}_k\left(x_*\left(t_k^-\right)\right)\right]\Bigg\|_\alpha\\
&\quad\le M'\left\|g(x_*)-g(x_n)\right\|_\alpha+\int_0^b\left\|C(t-s)[G(s,x_n(h_2(s)))-G(s,x_*(h_2(s)))]\right\|_\alpha ds\\
&\qquad+\sum_{k=1}^{m} M'\left\|I_k\left(x_n\left(t_k^-\right)\right)-I_k\left(x_*\left(t_k^-\right)\right)\right\|_\alpha\\
&\qquad+\sum_{k=1}^{m} M\left\|\overline{I}_k\left(x_n\left(t_k^-\right)\right)-\overline{I}_k\left(x_*\left(t_k^-\right)\right)\right\|_\alpha \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{3.8}$$

Thus, P_2 is continuous.

Next, we prove that $\{P_2x : x\in B_l\}$ is a family of equicontinuous functions. Let $\tau_1,\tau_2\in J,\ \tau_1<\tau_2$. Then for each $t\in J$, we have

$$\begin{aligned}
&\left\|(P_2x)(\tau_2)-(P_2x)(\tau_1)\right\|_\alpha\\
&\quad\le\left\|[C(\tau_2)-C(\tau_1)](x_0-g(x))\right\|_\alpha+\left\|S(\tau_2)\eta-S(\tau_1)\eta\right\|_\alpha\\
&\qquad+\left\|\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]G(s,x(h_2(s)))ds\right\|_\alpha+\left\|\int_{\tau_1}^{\tau_2}S(\tau_2-s)G(s,x(h_2(s)))ds\right\|_\alpha\\
&\qquad+\left\|\sum_{0<t_k<\tau_1}[C(\tau_2-t_k)-C(\tau_1-t_k)]I_k\left(x\left(t_k^-\right)\right)\right\|_\alpha+\left\|\sum_{\tau_1\le t_k<\tau_2}C(\tau_2-t_k)I_k\left(x\left(t_k^-\right)\right)\right\|_\alpha\\
&\qquad+\left\|\sum_{0<t_k<\tau_1}[S(\tau_2-t_k)-S(\tau_1-t_k)]\overline{I}_k\left(x\left(t_k^-\right)\right)\right\|_\alpha+\left\|\sum_{\tau_1\le t_k<\tau_2}S(\tau_2-t_k)\overline{I}_k\left(x\left(t_k^-\right)\right)\right\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| [C(\tau_2) - C(\tau_1)](x_0 - g(x)) \right\|_\alpha + \left\| S(\tau_2)\eta - S(\tau_1)\eta \right\|_\alpha \\
&\quad + \int_0^{\tau_1} \| [S(\tau_2 - s) - S(\tau_1 - s)] G(s, x(h_2(s)))\|_\alpha ds + \left\| \int_{\tau_1}^{\tau_2} S(\tau_2 - s) G(s, x(h_2(s))) ds \right\|_\alpha \\
&\quad + \sum_{0<t_k<\tau_1} \| C(\tau_2 - t_k) - C(\tau_1 - t_k)\| d_k + \sum_{\tau_1 \leq t_k < \tau_2} \| C(\tau_2 - t_k)\| d_k \\
&\quad + \sum_{0<t_k<\tau_1} \| S(\tau_2 - t_k) - S(\tau_1 - t_k)\| \overline{d}_k + \sum_{\tau_1 \leq t_k < \tau_2} \| S(\tau_2 - t_k)\| \overline{d}_k,
\end{aligned}
\tag{3.9}
$$

and similarly

$$
\begin{aligned}
&\left\| (P_2 x)'(\tau_2) - (P_2 x)'(\tau_1) \right\|_\alpha \\
&\quad \leq \left\| \left[C'(\tau_2) - C'(\tau_1) \right] (x_0 - g(x)) \right\|_\alpha + \left\| \left[S'(\tau_2) - S'(\tau_1) \right] \eta \right\|_\alpha \\
&\quad + \left\| \int_0^{\tau_1} [C(\tau_2 - s) - C(\tau_1 - s)] G(s, x(h_2(s))) ds \right\|_\alpha + \left\| \int_{\tau_1}^{\tau_2} C(\tau_2 - s) G(s, x(h_2(s))) ds \right\|_\alpha \\
&\quad + \left\| \sum_{0<t_k<\tau_1} \left[C'(\tau_2 - t_k) - C'(\tau_1 - t_k) \right] I_k(x(t_k^-)) \right\|_\alpha + \left\| \sum_{\tau_1 \leq t_k < \tau_2} C'(\tau_2 - t_k) I_k(x(t_k^-)) \right\|_\alpha \\
&\quad + \left\| \sum_{0<t_k<\tau_1} \left[S'(\tau_2 - t_k) - S'(\tau_1 - t_k) \right] \overline{I}_k(x(t_k^-)) \right\|_\alpha + \left\| \sum_{\tau_1 \leq t_k < \tau_2} S'(\tau_2 - t_k) \overline{I}_k(x(t_k^-)) \right\|_\alpha \\
&\quad \leq \left\| \left[C'(\tau_2) - C'(\tau_1) \right] (x_0 - g(x)) \right\|_\alpha + \left\| \left[S'(\tau_2) - S'(\tau_1) \right] \eta \right\|_\alpha \\
&\quad + \int_0^{\tau_1} \| [C(\tau_2 - s) - C(\tau_1 - s)] G(s, x(h_2(s)))\|_\alpha ds + \left\| \int_{\tau_1}^{\tau_2} C(\tau_2 - s) G(s, x(h_2(s))) ds \right\|_\alpha \\
&\quad + \sum_{0<t_k<\tau_1} \left\| C'(\tau_2 - t_k) - C'(\tau_1 - t_k) \right\| d_k + \sum_{\tau_1 \leq t_k < \tau_2} \left\| C'(\tau_2 - t_k) \right\| d_k \\
&\quad + \sum_{0<t_k<\tau_1} \left\| S'(\tau_2 - t_k) - S'(\tau_1 - t_k) \right\| \overline{d}_k + \sum_{\tau_1 \leq t_k < \tau_2} \left\| S'(\tau_2 - t_k) \right\| \overline{d}_k.
\end{aligned}
\tag{3.10}
$$

The right-hand sides are independent of $x \in B_l$ and tend to zero as $\tau_2 - \tau_1 \to 0$, since $C(t), S(t), C'(t), S'(t)$ are uniformly continuous for $t \in J$ and the compactness of $C(t), S(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

The compactness of $S(t)$ follows from that of $C(t)$ and Lemma 2.2.

This shows that P_2 maps B_l into a family of equicontinuous functions.

It remains to prove that $V(t) = \{(P_2 x)(t) : x \in B_l\}$ is relatively compact in X.

Obviously, by condition (H_5)(ii), $V(0)$ is relatively compact in B. Let $0 < t \leq b$ be fixed and $0 < \epsilon < t$. For $x \in B_l$, we define

$$\begin{aligned} cc(P_{2,\epsilon}x)(t) &= C(t)\left[x_0 - g(x)\right] + S(t)\eta + \int_0^{t-\epsilon} S(t-s)G(s, x(h_2(s)))ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k\left(x\left(t_k^-\right)\right) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k\left(x\left(t_k^-\right)\right). \end{aligned} \tag{3.11}$$

Since $C(t), S(t)$ are compact operators, the set $V_\epsilon(t) = \{(P_{2,\epsilon}x)(t) : x \in B_l\}$ is relatively compact in B for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in \mathrm{B}_l$, we have

$$\begin{aligned} \|(P_2x)(t) - (P_{2,\epsilon}x)(t)\|_\alpha &= \left\| \int_{t-\epsilon}^t S(t-s)G(s, x(h_2(s)))ds \right\|_\alpha \leq \int_{t-\epsilon}^t \|S(t-s)\| w_l(s)ds, \\ \left\|(P_2x)'(t) - (P_{2,\epsilon}x)'(t)\right\|_\alpha &= \left\| \int_{t-\epsilon}^t C(t-s)G(s, x(h_2(s)))ds \right\|_\alpha \leq \int_{t-\epsilon}^t \|C(t-s)\| w_l(s)ds. \end{aligned} \tag{3.12}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence, the set $V(t)$ is relatively compact in B. □

Thus, by Arzela-Ascoli theorem, P_2 is a completely continuous operator. Those arguments enable us to conclude that $P = P_1 + P_2$ is a condensing map on B_l, and by the fixed point theorem of Sadovskii, there exists a fixed point $x(\cdot)$ for P on B_l. Therefore, the nonlocal Cauchy problem with impulsive effect (1.1) has a mild solution. The proof is completed.

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