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Research Article

Isometries of a Bergman-Privalov-Type Space on the Unit Ball

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We introduce a new space $AN_{\log,\alpha}(\mathbb{B})$ consisting of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ such that $\|f\|_{AN_{\log,\alpha}} := \int_{\mathbb{B}} \varphi_e(\ln(1+|f(z)|))dV_\alpha(z) < \infty$, where $\alpha > -1$, $dV_\alpha(z) = c_{\alpha,n}(1-|z|^2)^\alpha dV(z)$ ($dV(z)$ is the normalized Lebesgue volume measure on $\mathbb{B}$, and $c_{\alpha,n}$ is a normalization constant, that is, $V_\alpha(\mathbb{B}) = 1$), and $\varphi_e(t) = t\ln(e+t)$ for $t \in [0,\infty)$. Some basic properties of this space are presented. Among other results we proved that $AN_{\log,\alpha}(\mathbb{B})$ with the metric $d(f,g) = \|f-g\|_{AN_{\log,\alpha}}$ is an F-algebra with respect to pointwise addition and multiplication. We also prove that every linear isometry T of $AN_{\log,\alpha}(\mathbb{B})$ into itself has the form $Tf = c(f\circ\psi)$ for some $c \in \mathbb{C}$ such that $|c| = 1$ and some ψ which is a holomorphic self-map of $\mathbb{B}$ satisfying a measure-preserving property with respect to the measure dV_α. As a consequence of this result we obtain a complete characterization of all linear bijective isometries of $AN_{\log,\alpha}(\mathbb{B})$.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$, $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$, $dV(z)$ the normalized Lebesgue measure on $\mathbb{B}$, $d\sigma$ the normalized surface measure on the boundary $\partial\mathbb{B}$ of the unit ball, and $dV_\alpha(z) = c_{\alpha,n}(1-|z|^2)^\alpha dV(z)$, where $\alpha > -1$ and $c_{\alpha,n}$ is a normalization constant, that is, $V_\alpha(\mathbb{B}) = 1$. For each $\alpha > -1$ we define the holomorphic function space $AN_{\log,\alpha}(\mathbb{B}) = AN_{\log,\alpha}$ as follows:

$$AN_{\log,\alpha}(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) \mid \|f\|_{AN_{\log,\alpha}} := \int_{\mathbb{B}} \varphi_e\big(\ln\big(1+|f(z)|\big)\big)dV_\alpha(z) < \infty \right\}, \tag{1.1}$$

where $\varphi_e(t) = t\ln(e+t)$ for $t \in [0,\infty)$.

Since

$$\varphi_e'(t) = \ln(e+t) + \frac{t}{e+t} \geq 1, \quad \varphi_e''(t) = \frac{2e+t}{(e+t)^2} > 0, \quad t \in [0,\infty), \tag{1.2}$$

it follows that $\varphi_e(t)$ is an increasing convex function which is obviously nonnegative.

Let $s(t) = \varphi_e(\ln(1+t))$, then

$$s''(t) = \frac{\varphi_e''(\ln(1+t)) - \varphi_s'(\ln(1+t))}{(1+t)^2}. \tag{1.3}$$

Let $F(u) = \varphi_s'(u) - \varphi_e''(u)$, then $F(0) = 1 - (2/e)$, $F'(u) > 0$, so that $\varphi_e''(u) \leq \varphi_s'(u)$ for $u \geq 0$. Thus $s(t)$ is a nonnegative concave function on the interval $[0,\infty)$, so that

$$\frac{s(x+y) - s(x)}{y} \leq \frac{s(y) - s(0)}{y} \tag{1.4}$$

for $x \geq 0$ and $y > 0$, and consequently $s(x+y) \leq s(x) + s(y), x, y \geq 0$, which implies the following inequality:

$$\varphi_e(\ln(1+x+y)) \leq \varphi_e(\ln(1+x)) + \varphi_e(\ln(1+y)) \tag{1.5}$$

for all $x, y \in [0,\infty)$. It is easy to see that $\|cf\|_{AN_{\log,\alpha}}$ need not be equal to $|c|\|f\|_{AN_{\log,\alpha}}$, for every $c \in \mathbb{C}$ and $f \in AN_{\log,\alpha}(\mathbb{B})$. These facts imply that $\|\cdot\|_{AN_{\log,\alpha}}$ is not a norm on $AN_{\log,\alpha}(\mathbb{B})$ but satisfies the triangle inequality

$$\|f+g\|_{AN_{\log,\alpha}} \leq \|f\|_{AN_{\log,\alpha}} + \|g\|_{AN_{\log,\alpha}}. \tag{1.6}$$

Furthermore if we define $d(f,g) = \|f-g\|_{AN_{\log,\alpha}}$ for any $f, g \in AN_{\log,\alpha}(\mathbb{B})$, then we see that $AN_{\log,\alpha}(\mathbb{B})$ is a metric space with respect to $d(f,g)$.

Let X be a space of all holomorphic functions on some domain and T a linear isometry of X into X [1]. When X is the Hardy space $H^p(\mathbb{B})$ $(0 < p < \infty, p \neq 2)$, Forelli [2, 3] and Rudin [4] have determined the injective and/or surjective isometries of $H^p(\mathbb{B})$. For the case when X is the weighted Bergman spaces $A_\alpha^p(\mathbb{B})$ $(0 < p < \infty,\ p \neq 2)$, the isometries were completely characterized in a sequence of papers by Kolaski [5–7]. By these works we see that the isometries on these holomorphic function spaces are described as weighted composition operators, which is one of the reasons why these operators have been investigated so much recently in the settings of the unit ball or the unit polydisk (see, e.g., monograph [8], recent papers [9–19], and references therein). See also paper [20] for integral-type operators closely related with weighted composition operators. The case when X is not a Banach space has also been studied by many authors. The *Smirnov class* N^* and the *Privalov space* N^p $(1 < p < \infty)$ which are contained in the *Nevanlinna class* N are examples of such spaces. These type of spaces are F-spaces with respect to a suitable metric on them. For properties of these space, we can refer [21]. Stephenson [22], Iida and Mochizuki [23] and Subbotin [24–26] have studied linear isometries on these spaces. Their works showed that the injective isometries

are weighted composition operators induced by some inner functions and inner maps of $\mathbb{B}$ whose radial limit maps satisfy a measure-preserving property. Recently Matsugu and the second author of the present paper have studied the *weighted Bergman-Privalov space* $AN_\alpha^p(\mathbb{B})$ $(1 \le p < \infty)$ and characterized the isometries of this space in [27]. They showed that in this case, T has the form of a constant multiple composition operator which satisfies a measure-preserving property with respect to the measure dV_α.

Motivated by paper [28], in this paper, we investigate the space $AN_{\log,\alpha}(\mathbb{B})$. Some basic properties of the space $AN_{\log,\alpha}(\mathbb{B})$ are presented in Section 2; among other results it was proved that $AN_{\log,\alpha}(\mathbb{B})$ with the metric $d(f,g) = \|f-g\|_{AN_{\log,\alpha}}$ is an F-algebra with respect to pointwise addition and multiplication. Also an estimate for the point evaluation functional on $AN_{\log,\alpha}(\mathbb{B})$ is given. In Section 3, we will prove that every linear isometry T of $AN_{\log,\alpha}(\mathbb{B})$ has the form $Tf = c(f \circ \varphi)$, where $c \in \mathbb{C}$ such that $|c| = 1$ and φ is a holomorphic self-map of $\mathbb{B}$ satisfying a measure-preserving property with respect to the measure dV_α. As a consequence of this result we show that every surjective isometry T of $AN_{\log,\alpha}(\mathbb{B})$ is of the form $Tf = c(f \circ U)$ for any $f \in AN_{\log,\alpha}(\mathbb{B})$, where $c \in \mathbb{C}$ with $|c| = 1$ and U is a unitary operator on $\mathbb{C}^n$.

2. Basic Properties of $AN_{\log,\alpha}(\mathbb{B})$

This section is devoted to collecting fundamental results on $AN_{\log,\alpha}(\mathbb{B})$ which will be used in the proofs of the main results.

Recall that the *weighted Bergman space* $A_\alpha^1(\mathbb{B})$ is defined as follows:

$$A_\alpha^1(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) \mid \|f\|_{A_\alpha^1} := \int_{\mathbb{B}} |f(z)| dV_\alpha(z) < \infty \right\}. \tag{2.1}$$

First we prove an elementary inequality, which has the main role in determining the relationship between $AN_{\log,\alpha}(\mathbb{B})$ space and the weighted Bergman space $A_\alpha^1(\mathbb{B})$.

Lemma 2.1. *The following inequality holds:*

$$\varphi_e(\ln(1+x)) \le x, \quad x \in [0,\infty). \tag{2.2}$$

Proof. Note that inequality (2.2) is transformed into

$$e^t - 1 - \varphi_e(t) \ge 0, \quad t \in [0,\infty) \tag{2.3}$$

by the change $x = e^t - 1$.

Let $F(t) = e^t - 1 - \varphi_e(t)$. Then $F(0) = 0$ and

$$F'(t) = e^t - \ln(e+t) - \frac{t}{e+t}, \tag{2.4}$$

which implies $F'(0) = 0$ and

$$F''(t) = e^t - \frac{1}{e+t} - \frac{e}{(e+t)^2} \geq 1 - \frac{2}{e} > 0. \tag{2.5}$$

From all these relations it follows that inequality (2.3) holds and consequently inequality (2.2). □

From Lemma 2.1 we obtain the following corollary.

Corollary 2.2. *For each $\alpha > -1$, $A^1_\alpha(\mathbb{B}) \subset AN_{\log,\alpha}(\mathbb{B})$.*

Proof. From Lemma 2.1 it follows that

$$\varphi_e(\ln(1 + |f(z)|)) \leq |f(z)| \tag{2.6}$$

for every $f \in A^1_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$. Multiplying this inequality by $dV_\alpha(z)$, then integrating such obtained inequality over $\mathbb{B}$ we obtain

$$\|f\|_{AN_{\log,\alpha}} \leq \|f\|_{A^1_\alpha}, \tag{2.7}$$

from which the result follows. □

Remark 2.3. Note that from inequality (2.7) it follows that the inclusion $i : A^1_\alpha(\mathbb{B}) \to AN_{\log,\alpha}(\mathbb{B})$ has "norm" less than one, if we define operator norm as usual by

$$\|i\|_{A^1_\alpha \to AN_{\log,\alpha}} = \sup_{\|f\|_{A^1_\alpha} \leq 1} \|i(f)\|_{AN_{\log,\alpha}}. \tag{2.8}$$

Lemmas 2.5 and 2.7 are based on the following lemma.

Lemma 2.4. *Let $\Psi(x) = \ln(1+e^x)\ln\{e+\ln(1+e^x)\}$ $(x \in (-\infty, \infty))$. Then Ψ is a positive continuous, increasing, and convex function on $(-\infty, \infty)$.*

Proof. It is clear that Ψ is a positive and continuous function on the interval $(-\infty, \infty)$. Now we prove that it is increasing and convex on $(-\infty, \infty)$. Note that $\Psi(x) = \varphi_e(\ln(1 + e^x))$. Hence

$$\begin{aligned} \Psi'(x) &= \varphi'_e(\ln(1 + e^x))\frac{e^x}{1 + e^x}, \\ \Psi''(x) &= \varphi''_e(\ln(1 + e^x))\left(\frac{e^x}{1 + e^x}\right)^2 + \varphi'_e(\ln(1 + e^x))\frac{e^x}{(1 + e^x)^2}. \end{aligned} \tag{2.9}$$

From (1.2) and (2.9), and since $\ln(1+e^x) \in (0, \infty)$, when $x \in (-\infty, \infty)$, the lemma follows. □

Lemma 2.5. *Let $\alpha > -1$. If $f \in AN_{\log,\alpha}(\mathbb{B})$, then it holds that*

$$\lim_{r\to 1^-} \|f_r - f\|_{AN_{\log,\alpha}} = 0, \tag{2.10}$$

where $f_r(z) = f(rz)$ for each $r \in (0,1)$ and $z \in \mathbb{B}$.

Proof. Take an $f \in AN_{\log,\alpha}(\mathbb{B})$ and $\varepsilon > 0$. Then we can choose an $r_0 \in (0,1)$ such that

$$\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_e(\ln(1+|f(z)|))dV_\alpha(z) < \frac{\varepsilon}{3}, \tag{2.11}$$

where $r_0\overline{\mathbb{B}} = \{z \in \mathbb{B} : |z| \le r_0\}$. Since $\varphi_e(\ln(1+|f(z)|)) = \Psi(\ln|f(z)|)$, from Lemma 2.4 it follows that $\varphi_e(\ln(1+|f(z)|))$ is a positive plurisubharmonic function in $\mathbb{B}$. Hence we have

$$\int_{\partial\mathbb{B}} \varphi_e(\ln(1+|f(rt\zeta)|))d\sigma(\zeta) \le \int_{\partial\mathbb{B}} \varphi_e(\ln(1+|f(t\zeta)|))d\sigma(\zeta) \tag{2.12}$$

for any $r, t \in (0,1)$. This inequality implies that

$$\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_e(\ln(1+|f_r(z)|))dV_\alpha(z) \le \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \varphi_e(\ln(1+|f(z)|))dV_\alpha(z) < \frac{\varepsilon}{3} \tag{2.13}$$

for any $r \in (0,1)$.

Now we choose an $\varepsilon_0 > 0$ such that $\varphi_e(\ln(1+\varepsilon_0)) = \varepsilon/3$. By the continuity of f on the compact subset $r_0\overline{\mathbb{B}}$, we see that there exists a $\delta \in (0,1)$ such that if $z, w \in r_0\overline{\mathbb{B}}$ with $|z-w| < \delta$, then

$$|f(z) - f(w)| < \varepsilon_0. \tag{2.14}$$

Set $r_1 = 1-\delta$. If $r_1 < r < 1$, then

$$\int_{r_0\overline{\mathbb{B}}} \varphi_e(\ln(1+|f_r(z) - f(z)|))dV_\alpha(z) \le \varphi_e(\ln(1+\varepsilon_0)) = \frac{\varepsilon}{3}. \tag{2.15}$$

By (2.13), (2.15), and (1.5), we obtain

$$\begin{aligned}
\|f_r - f\|_{AN_{\log,\alpha}} &= \left(\int_{r_0\overline{\mathbb{B}}} + \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\right)\varphi_e(\ln(1+|f_r(z) - f(z)|))dV_\alpha(z) \\
&\le \int_{r_0\overline{\mathbb{B}}} \varphi_e(\ln(1+|f_r(z) - f(z)|))dV_\alpha(z) \\
&\quad + \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \{\varphi_e(\ln(1+|f_r(z)|)) + \varphi_e(\ln(1+|f(z)|))\}dV_\alpha(z) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{2.16}$$

This completes the proof. □

Corollary 2.6. *For each $\alpha > -1$ the space $AN_{\log,\alpha}(\mathbb{B})$ is separable.*

Proof. Since the dilated function f_r is approximated by the nth partial sum of its Taylor expansion uniformly on $\mathbb{B}$, Lemma 2.5 implies that polynomials are dense in $AN_{\log,\alpha}(\mathbb{B})$. Since the polynomials with rational coefficients approximate any polynomial on the close unit ball $\overline{\mathbb{B}}$, the corollary follows. □

Lemma 2.7. *Let $\alpha > -1$, $f \in H(\mathbb{B})$ and $z \in \mathbb{B}$. Then it holds that*

$$\ln(1+|f(z)|)\ln\{e+\ln(1+|f(z)|)\} \le \left(\frac{1+|z|}{1-|z|}\right)^{n+\alpha+1}\|f\|_{AN_{\log,\alpha}}. \tag{2.17}$$

Moreover, if $f \in AN_{\log,\alpha}(\mathbb{B})$, then f satisfies

$$\ln(1+|f(z)|)\ln\{e+\ln(1+|f(z)|)\} = o\left((1-|z|)^{-(n+\alpha+1)}\right) \qquad (\text{as } |z| \to 1^-). \tag{2.18}$$

Proof. Fix $f \in AN_{\log,\alpha}(\mathbb{B})$ and $z \in \mathbb{B}$. Let φ_z be the biholomorphic involution of $\mathbb{B}$ described in [29, page 25]. Since $\varphi_e(\ln(1+|f\circ\varphi_z|))$ is a positive plurisubharmonic function in $\mathbb{B}$ by Lemma 2.4, we have

$$\begin{aligned}
\varphi_e(\ln(1+|f(z)|)) &= \varphi_e(\ln(1+|f\circ\varphi_z(0)|)) \\
&= 2nc_{\alpha,n}\int_0^1 r^{2n-1}\left(1-r^2\right)^{\alpha}\varphi_e(\ln(1+|f(\varphi_z(0))|))dr \\
&\le 2nc_{\alpha,n}\int_0^1 r^{2n-1}\left(1-r^2\right)^{\alpha}dr\int_{\partial\mathbb{B}}\varphi_e(\ln(1+|f(\varphi_z(r\zeta))|))d\sigma(\zeta) \\
&= c_{\alpha,n}\int_{\mathbb{B}}\varphi_e(\ln(1+|f(\varphi_z(w))|))(1-|w|^2)^{\alpha}dV(w) \\
&= c_{\alpha,n}\int_{\mathbb{B}}\varphi_e(\ln(1+|f(w)|))(J_{\mathbb{R}}\varphi_z)(w)(1-|\varphi_z(w)|^2)^{\alpha}dV(w),
\end{aligned} \tag{2.19}$$

where $(J_{\mathbb{R}}\varphi_z)(w)$ denotes the real Jacobian of φ_z at w. By [29, Theorem 2.2.2 and 2.2.6], for $w \in \mathbb{B}$ we have

$$(J_{\mathbb{R}}\varphi_z)(w)(1-|\varphi_z(w)|^2)^{\alpha} \le \left(\frac{1+|z|}{1-|z|}\right)^{n+\alpha+1}(1-|w|^2)^{\alpha}. \tag{2.20}$$

By (2.19) and (2.20), we obtain

$$\varphi_e(\ln(1+|f(z)|)) \le \left(\frac{1+|z|}{1-|z|}\right)^{n+\alpha+1}\|f\|_{AN_{\log,\alpha}}, \tag{2.21}$$

which completes the proof of the first claim.

Next we prove the second claim. Fix an $\varepsilon > 0$. By Lemma 2.5, there exists an $r \in (0,1)$ such that $\|f_r - f\|_{AN_{\log,\alpha}} < \varepsilon$. From, (1.5) and (2.21) applied to the function $f_r - f$ we have that

$$\begin{aligned}\varphi_e(\ln(1+|f(z)|))(1-|z|)^{n+\alpha+1} &\le 2^{n+\alpha+1}\|f_r - f\|_{AN_{\log,\alpha}} + (1-|z|)^{n+\alpha+1}M_r \\ &< 2^{n+\alpha+1}\varepsilon + (1-|z|)^{n+\alpha+1}M_r,\end{aligned} \tag{2.22}$$

where $M_r = \max_{w\in r\overline{\mathbb{B}}}\varphi_e(\ln(1+|f(w)|)) < \infty$. Letting $|z| \to 1^-$ in (12) and using the fact that ε is an arbitrary positive number we obtain

$$\varphi_e(\ln(1+|f(z)|))(1-|z|)^{n+\alpha+1} \longrightarrow 0, \quad \text{as } |z| \to 1^-. \tag{2.23}$$

This completes the proof of this lemma. □

Lemma 2.8. *$AN_{\log,\alpha}(\mathbb{B})$ is a complete metric space.*

Proof. In the introduction we have seen that $AN_{\log,\alpha}(\mathbb{B})$ is a metric space. Since the convergence in $AN_{\log,\alpha}(\mathbb{B})$ implies the uniform convergence on compact subsets of $\mathbb{B}$ by Lemma 2.7, a usual normal family argument along with Fatou's lemma shows that every Cauchy sequence in $AN_{\log,\alpha}(\mathbb{B})$ converges to an element of $AN_{\log,\alpha}(\mathbb{B})$. Hence the space $AN_{\log,\alpha}(\mathbb{B})$ equipped with the metric $d(f,g) = \|f-g\|_{AN_{\log,\alpha}}$ is a complete metric space. □

Recall that a metric space (X,d) is called an *F-space* if it is a complete and

(i) $d(x,y) = d(x-y,0)$ for every $x,y \in X$;

(ii) for each sequence $\{x_k\} \subset X$ such that $d(x_k,0) \to 0$ as $k \to \infty$ it follows that $d(cx_k,0) \to 0$ for every $c \in \mathbb{C}$;

(iii) for each sequence $\{c_k\} \subset \mathbb{C}$ such that $c_k \to 0$ as $k \to \infty$, $d(c_k x,0) \to 0$ for each $x \in X$.

If in an F-space (X,d) is introduced an operation of pointwise multiplication such that X becomes an algebra and the operation of multiplication is continuous in the metric d, then the F-space (X,d) is called an *F-algebra*.

Before we prove our next result we will prove two technical lemmas.

Lemma 2.9. *The following inequality holds:*

$$\varphi_e(\ln(1+cx)) \le \max\left\{1, c^2\right\}\varphi_e(\ln(1+x)) \tag{2.24}$$

for $x, c \ge 0$.

Proof. It is enough to prove the case $c > 1$. Since φ_e is an increasing function and $\ln(1+cx) \le c\ln(1+x)$, we have

$$\varphi_e(\ln(1+cx)) \le \varphi_e(c\ln(1+x)) = c\ln(1+x)\ln(e + c\ln(1+x)). \tag{2.25}$$

Let

$$\eta(t) = (e+t)^c - (e+ct), \quad \text{for } t \in [0,\infty). \tag{2.26}$$

Then from

$$\eta'(t) = c\left((e+t)^{c-1} - 1\right), \tag{2.27}$$

we see that η is an increasing function on the interval $[0,\infty)$, so that $\eta(t) \geq \eta(0) = e^c - e > 0$. Hence $(e+ct) < (e+t)^c$. This implies that

$$\ln(e+ct) < c\ln(e+t). \tag{2.28}$$

So we obtain

$$\varphi_e(\ln(1+cx)) \leq c^2 \ln(1+x)\ln(e+\ln(1+x)) = c^2\varphi_e(\ln(1+x)), \tag{2.29}$$

completing the proof of the lemma. □

The next lemma improves inequality (19) in [28].

Lemma 2.10. *The following inequality holds:*

$$\varphi_e(x+y) \leq 2(\varphi_e(x) + \varphi_e(y)) \tag{2.30}$$

for $x, y \geq 0$.

Proof. First note that the function

$$f(x,y) = 2(\varphi_e(x) + \varphi_e(y)) - \varphi_e(x+y) \tag{2.31}$$

satisfies the following condition: $f(x,y) = f(y,x)$. Hence it is enough to prove inequality (2.30) for the case $x \geq y \geq 0$. Note that $f(0,y) = \varphi_e(y) \geq 0$ and

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= 2\left(\ln(e+x) + \frac{x}{e+x}\right) - \ln(e+x+y) - \frac{x+y}{e+x+y} \\
&\geq 2\left(\ln(e+x) + \frac{x}{e+x}\right) - \ln(e+2x) - \frac{2x}{e+2x} \\
&\geq \ln(e+x)^2 - \ln(e+2x) > 0.
\end{aligned} \tag{2.32}$$

Hence, for each fixed y we have that $f(x,y) \geq f(0,y) \geq 0$, from which inequality (2.30) follows. □

Theorem 2.11. *$AN_{\log,\alpha}(\mathbb{B})$ is an F-algebra with respect to pointwise addition and multiplication.*

Proof. By Lemma 2.8 $AN_{\log,\alpha}(\mathbb{B})$ is a complete metric space satisfying the condition $d(f,g) = d(f-g,0)$. From Lemma 2.9 we have the inequality

$$\varphi_e(\ln(1+st)) \le \max\left\{1, s^2\right\}\varphi_e(\ln(1+t)), \quad t,s \ge 0, \tag{2.33}$$

which implies that

$$\|cf\|_{AN_{\log,\alpha}} \le \max\left\{1, |c|^2\right\}\|f\|_{AN_{\log,\alpha}} \tag{2.34}$$

for each $c \in \mathbb{C}$, so that the operation of the multiplication by complex numbers is closed in $AN_{\log,\alpha}(\mathbb{B})$. On the other hand, by inequality (1.5) it follows that the pointwise addition is also closed in $AN_{\log,\alpha}(\mathbb{B})$. From (2.34) it follows that the condition $d(f_k,0) \to 0$ implies $d(cf_k,0) \to 0$ for every $c \in \mathbb{C}$, since

$$d(cf_k,0) = \|cf_k\|_{AN_{\log,\alpha}} \le \max\left\{1, |c|^2\right\}\|f_k\|_{AN_{\log,\alpha}} = \max\left\{1, |c|^2\right\}d(f_k,0). \tag{2.35}$$

Now assume that $\{c_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence such that $c_k \to 0$ as $k \to \infty$. Note that for each such sequence there is $k_0 \in \mathbb{N}$ such that $|c_k| \le 1$ for $k \ge k_0$. Hence we have that for each $f \in AN_{\log,\alpha}(\mathbb{B})$ and $z \in \mathbb{B}$

$$\varphi_e(\ln(1+|c_k f(z)|)) \le \varphi_e(\ln(1+|f(z)|)), \quad k \ge k_0. \tag{2.36}$$

From (2.36), since

$$\varphi_e(\ln(1+|c_k f(z)|)) \longrightarrow 0 \quad \text{as } k \to \infty \tag{2.37}$$

for each $z \in \mathbb{B}$, and by the Lebesgue dominated convergence theorem, we get $d(c_k f,0) = \|c_k f\|_{AN_{\log,\alpha}} \to 0$ as $k \to \infty$, which proves (iii) and that $AN_{\log,\alpha}(\mathbb{B})$ is an F-space.

Now we prove that $AN_{\log,\alpha}(\mathbb{B})$ is closed with respect to the pointwise multiplication. From Lemma 2.10 we have that

$$\varphi_e(t+s) \le 2(\varphi_e(t) + \varphi_e(s)), \quad t,s \ge 0, \tag{2.38}$$

from which it follows that

$$\|fg\|_{AN_{\log,\alpha}} \le 2\left(\|f\|_{AN_{\log,\alpha}} + \|g\|_{AN_{\log,\alpha}}\right), \tag{2.39}$$

so that $AN_{\log,\alpha}(\mathbb{B})$ is closed with respect to the pointwise multiplication.

Finally, we prove that the pointwise multiplication is continuous with respect to the metric d, that is, that from $d(f_k, f) \to 0$ as $k \to \infty$ and $d(g_l, g) \to 0$ as $l \to \infty$ it follows that $d(f_k g_l, fg) \to 0$ as $k, l \to \infty$. Note that from triangle inequality and by (2.39) it follows that

$$\begin{aligned}
&\|f_k g_l - fg\|_{AN_{\log,\alpha}} \\
&\quad \le \|(f_k - f)(g_l - g)\|_{AN_{\log,\alpha}} + \|(f_k - f)g\|_{AN_{\log,\alpha}} + \|f(g_l - g)\|_{AN_{\log,\alpha}} \\
&\quad \le 2\left(\|f_k - f\|_{AN_{\log,\alpha}} + \|g_l - g\|_{AN_{\log,\alpha}}\right) + \|(f_k - f)g\|_{AN_{\log,\alpha}} + \|f(g_l - g)\|_{AN_{\log,\alpha}}.
\end{aligned} \tag{2.40}$$

From (2.40) and the symmetry $\|fg\|_{AN_{\log,\alpha}} = \|gf\|_{AN_{\log,\alpha}}$ we see that it is enough to prove that from $d(f_k, f) \to 0$ as $k \to \infty$, for each $g \in AN_{\log,\alpha}(\mathbb{B})$, it follows that $d(f_k g, fg) \to 0$ as $k \to \infty$. Fix a $g \in AN_{\log,\alpha}(\mathbb{B})$ and put $E_s = \{z \in \mathbb{B} : |g(z)| \ge s\}$ for each $s \in [0, \infty)$. Hence we see that for every $\delta > 0$ there is an $s_0 > 0$ such that for every $s \ge s_0$

$$V_\alpha(E_s) \le \frac{1}{\varphi_e(\ln(1+s))} \int_{\mathbb{B}} \varphi_e(\ln(1 + |g(z)|)) dV_\alpha(z) < \delta. \tag{2.41}$$

Fix an $\varepsilon > 0$. Since $\varphi_e(\ln(1+|g|)) \in L^1(dV_\alpha)$, we have that for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that

$$\int_E \varphi_e(\ln(1 + |g(z)|)) dV_\alpha(z) < \varepsilon \tag{2.42}$$

for every set E such that $V_\alpha(E) < \delta_1$. From (2.41) with $\delta = \delta_1$ we have that $V_\alpha(E_s) < \delta$ for sufficiently large s. From this and inequalities (2.34) and (2.39) we have that

$$\begin{aligned}
\|g(f_k - f)\|_{AN_{\log,\alpha}} &= \int_{\mathbb{B}} \varphi_e(\ln(1 + |g(z)(f_k(z) - f(z))|)) dV_\alpha(z) \\
&= \left(\int_{\mathbb{B}\setminus E_s} + \int_{E_s}\right) \varphi_e(\ln(1 + |g(z)(f_k(z) - f(z))|)) dV_\alpha(z) \\
&\le 2\int_{E_s} \varphi_e(\ln(1 + |g(z)|)) dV_\alpha(z) \\
&\quad + 2\int_{E_s} \varphi_e(\ln(1 + |f_k(z) - f(z)|)) dV_\alpha(z) \\
&\quad + \int_{\mathbb{B}\setminus E_s} \varphi_e(\ln(1 + s|f_k(z) - f(z)|)) dV_\alpha(z) \\
&\le 2\varepsilon + \max\left\{3, 2 + s^2\right\} \|f_k - f\|_{AN_{\log,\alpha}}.
\end{aligned} \tag{2.43}$$

Letting $k \to \infty$ in (2.43) and since ε is an arbitrary positive number, it follows that $\|g(f_k - f)\|_{AN_{\log,\alpha}} \to 0$ as $k \to \infty$, finishing the proof of the theorem. □

3. Linear Isometries of $AN_{\log,\alpha}(\mathbb{B})$

In this section, we investigate linear isometries of $AN_{\log,\alpha}(\mathbb{B})$ into $AN_{\log,\alpha}(\mathbb{B})$, by modifying ideas from some earlier papers in this area (see, e.g., [23, 27, 28]).

The following two lemmas play an important role in the proofs of the main results in this section.

Lemma 3.1. *If T is a linear isometry of $AN_{\log,\alpha}(\mathbb{B})$ into itself, then the restriction of T to $A^1_\alpha(\mathbb{B})$ is also a linear isometry of $A^1_\alpha(\mathbb{B})$ into itself.*

Proof. Take an $f \in A^1_\alpha(\mathbb{B})$ and put $g = Tf$. For each $m \in \mathbb{N}$ we have $g/m = T(f/m)$, and so the assumption T which is an isometry of $AN_{\log,\alpha}(\mathbb{B})$ gives

$$\int_{\mathbb{B}} \varphi_e\left(\ln\left(1+\frac{|f(z)|}{m}\right)\right)dV_\alpha(z) = \int_{\mathbb{B}} \varphi_e\left(\ln\left(1+\frac{|g(z)|}{m}\right)\right)dV_\alpha(z). \tag{3.1}$$

By inequality (2.2), we have

$$m\varphi_e\left(\ln\left(1+\frac{|f(z)|}{m}\right)\right) \le |f(z)| \tag{3.2}$$

for any $m \in \mathbb{N}$ and $z \in \mathbb{B}$. Also, it is easy to see that

$$\lim_{m\to\infty} m\varphi_e\left(\ln\left(1+\frac{|f(z)|}{m}\right)\right) = |f(z)| \tag{3.3}$$

for each $z \in \mathbb{B}$. Hence the Lebesgue-dominated convergence theorem gives

$$\lim_{m\to\infty}\int_{\mathbb{B}} m\varphi_e\left(\ln\left(1+\frac{|f(z)|}{m}\right)\right)dV_\alpha(z) = \int_{\mathbb{B}} |f(z)|dV_\alpha(z). \tag{3.4}$$

On the other hand, Fatou's lemma (3.1) and (3.4) show that

$$\begin{aligned}\int_{\mathbb{B}} |g(z)|dV_\alpha(z) &\le \liminf_{m\to\infty}\int_{\mathbb{B}} m\varphi_e\left(\ln\left(1+\frac{|g(z)|}{m}\right)\right)dV_\alpha(z)\\ &= \liminf_{m\to\infty}\int_{\mathbb{B}} m\varphi_e\left(\ln\left(1+\frac{|f(z)|}{m}\right)\right)dV_\alpha(z)\\ &= \int_{\mathbb{B}} |f(z)|dV_\alpha(z),\end{aligned} \tag{3.5}$$

and so $g \in A^1_\alpha(\mathbb{B})$. By applying the Lebesgue-dominated convergence theorem once more again, we have

$$\lim_{m\to\infty}\int_{\mathbb{B}} m\varphi_e\left(\ln\left(1+\frac{|g(z)|}{m}\right)\right)dV_\alpha(z) = \int_{\mathbb{B}} |g(z)|dV_\alpha(z). \tag{3.6}$$

By (3.1), (3.4), and (3.6), we see that T is a linear isometry of $A^1_\alpha(\mathbb{B})$ into $A^1_\alpha(\mathbb{B})$. □

Lemma 3.2. *There exists a bounded continuous function $\theta(x)$ on $[0,\infty)$ such that*

$$\left(\frac{1}{2}-\frac{1}{e}\right)x^2 - x^3\theta(x) = x - \varphi_e(\ln(1+x)) \geq 0 \quad x\in[0,\infty). \tag{3.7}$$

Proof. Set

$$\theta(x) := \frac{\varphi_e(\ln(1+x)) - x - (-1/2+1/e)x^2}{x^3} \tag{3.8}$$

for $x \in (0,\infty)$. Since $\theta(x)$ is a continuous function on $(0,\infty)$ such that $\theta(x) \to 0$ as $x \to +\infty$, it is enough to prove that $\theta(x)$ has a finite limit as $x \to 0^+$. By the application of Taylor's theorem to $\varphi_e(\ln(1+x))$, we have

$$\varphi_e(\ln(1+x)) = x + \left(-\frac{1}{2}+\frac{1}{e}\right)x^2 + \left(\frac{1}{3}-\frac{1}{e}-\frac{1}{2e^2}\right)x^3 + R_4(x), \tag{3.9}$$

where $R_4(x)$ denotes the remainder term of order 4. Since $R_4(x)/x^3 \to 0$ as $x \to 0^+$, we obtain

$$\lim_{x\to 0^+}\theta(x) = \frac{1}{3}-\frac{1}{e}-\frac{1}{2e^2}, \tag{3.10}$$

which completes the proof. □

Theorem 3.3. *Every linear isometry T of $AN_{\log,\alpha}(\mathbb{B})$ into itself has the form $Tf = c(f\circ\psi)$ for any $f \in AN_{\log,\alpha}(\mathbb{B})$, where $c \in \mathbb{C}$ with $|c| = 1$ and $\psi : \mathbb{B} \to \mathbb{B}$ is a holomorphic map which satisfies the condition*

$$\int_{\mathbb{B}} h(\psi(z))dV_\alpha(z) = \int_{\mathbb{B}} h(z)dV_\alpha(z) \tag{3.11}$$

for every bounded or positive Borel function h in $\mathbb{B}$.

Proof. First suppose that $T : AN_{\log,\alpha}(\mathbb{B}) \to AN_{\log,\alpha}(\mathbb{B})$ is a linear isometry. By Lemma 3.1, the restriction of T to $A^1_\alpha(\mathbb{B})$ is a linear isometry of $A^1_\alpha(\mathbb{B})$ into $A^1_\alpha(\mathbb{B})$. Hence Kolaski's theorem [6, page 911, Theorem 2.11] implies that T has the following form:

$$Tf(z) = g(z)f(\psi(z)) \quad \left(f\in A^1_\alpha(\mathbb{B}), z\in\mathbb{B}\right), \tag{3.12}$$

where $g = T1$ and ψ is a holomorphic self-map of $\mathbb{B}$ such that

$$\int_{\mathbb{B}} h(\psi(z))|g(z)|dV_\alpha(z) = \int_{\mathbb{B}} h(z)dV_\alpha(z) \tag{3.13}$$

for every bounded or positive Borel function h in $\mathbb{B}$. Fix an $f \in AN_{\log,\alpha}(\mathbb{B})$. Since $\{f_r\}_{0<r<1} \subset A^1_\alpha(\mathbb{B})$, the representation (3.12) and Lemmas 2.5 and 2.7 give

$$Tf(z) = \lim_{r\to 1^-} Tf_r(z) = \lim_{r\to 1^-} g(z)f_r(\psi(z)) = g(z)f(\psi(z)) \tag{3.14}$$

for all $z \in \mathbb{B}$. Since $g = T1 \in A^1_\alpha(\mathbb{B})$ and $V_\alpha(\mathbb{B}) = 1$, Hölder's inequality gives

$$1 = \|1\|_{A^1_\alpha} = \|g\|_{A^1_\alpha} \le \|g\|_{A^2_\alpha}, \tag{3.15}$$

where $\|g\|^2_{A^2_\alpha} = \int_{\mathbb{B}} |g|^2 dV_\alpha$. On the other hand, from Lemma 3.2 it follows that

$$\begin{aligned}
&\int_{\mathbb{B}} \left\{ \left(\frac{1}{2} - \frac{1}{e}\right)|tg(z)|^2 - |tg(z)|^3\theta(|tg(z)|) \right\} dV_\alpha(z) \\
&\quad = \int_{\mathbb{B}} \{|tg(z)| - \varphi_e(\ln(1 + |tg(z)|))\} dV_\alpha(z) \\
&\quad = \int_{\mathbb{B}} |T(t)(z)| dV_\alpha(z) - \int_{\mathbb{B}} \varphi_e(\ln(1 + |T(t)(z)|)) dV_\alpha(z) \\
&\quad = \int_{\mathbb{B}} t dV_\alpha(z) - \int_{\mathbb{B}} \varphi_e(\ln(1 + |t|)) dV_\alpha(z) \\
&\quad = \left(\frac{1}{2} - \frac{1}{e}\right)t^2 - t^3\theta(t).
\end{aligned} \tag{3.16}$$

By Lemma 3.2 and Fatou's lemma we have

$$\begin{aligned}
\int_{\mathbb{B}} \left(\frac{1}{2} - \frac{1}{e}\right)|g(z)|^2 dV_\alpha(z) &\le \liminf_{t\to 0^+} \int_{\mathbb{B}} \left\{ \left(\frac{1}{2} - \frac{1}{e}\right)|g(z)|^2 - t|g(z)|^3\theta(|tg(z)|) \right\} dV_\alpha(z) \\
&= \liminf_{t\to 0^+} \left\{ \left(\frac{1}{2} - \frac{1}{e}\right) - t\theta(t) \right\} = \frac{1}{2} - \frac{1}{e},
\end{aligned} \tag{3.17}$$

and so $\|g\|_{A^2_\alpha} \le 1$. Combining this with (3.15), we obtain

$$\|g\|_{A^1_\alpha} = \|g\|_{A^2_\alpha} = 1. \tag{3.18}$$

This implies that $g \equiv c$ in $\mathbb{B}$ for some $c \in \mathbb{C}$ with $|c| = 1$. Hence (3.13) and (3.14) show that $Tf = c(f \circ \psi)$ for any $f \in AN_{\log,\alpha}(\mathbb{B})$ and ψ satisfies the condition in (3.11).

Conversely, for some complex number c with $|c| = 1$ and holomorphic self-map φ of $\mathbb{B}$ which satisfies (3.11) we define an operator T on $AN_{\log,\alpha}(\mathbb{B})$ by $Tf = c(f \circ \varphi)$ for $f \in AN_{\log,\alpha}(\mathbb{B})$. Since $\varphi_e(\ln(1+|f|))$ is a positive Borel function in $\mathbb{B}$, condition (3.11) implies that T is a linear isometry of $AN_{\log,\alpha}(\mathbb{B})$ into $AN_{\log,\alpha}(\mathbb{B})$. This completes the proof. □

Corollary 3.4. *Every surjective isometry T of $AN_{\log,\alpha}(\mathbb{B})$ is of the form $Tf = c(f \circ U)$ for any $f \in AN_{\log,\alpha}(\mathbb{B})$, where $c \in \mathbb{C}$ with $|c| = 1$ and U is a unitary operator on $\mathbb{C}^n$.*

Proof. Assume that $T : AN_{\log,\alpha}(\mathbb{B}) \to AN_{\log,\alpha}(\mathbb{B})$ is a surjective isometry. Then Theorem 3.3 implies that T is written in the form $Tf = c_T(f \circ \varphi_T)$ where $c_T \in \mathbb{C}$ with $|c_T| = 1$ and φ_T is a holomorphic self-map of $\mathbb{B}$ which satisfies (3.11). Since T^{-1} is also a surjective isometry of $AN_{\log,\alpha}(\mathbb{B})$, we see that

$$f = T^{-1}(T(f)) = T^{-1}(c_T(f \circ \varphi_T)) = c_{T^{-1}} c_T (f \circ \varphi_T \circ \varphi_{T^{-1}}), \tag{3.19}$$

so that $c_{T^{-1}} = c_T^{-1}$ and $\varphi_{T^{-1}} = \varphi_T^{-1}$, that is, φ_T is an automorphism of $\mathbb{B}$.

We prove that $\varphi := \varphi_T$ fixes the origin. Let φ_j $(1 \le j \le n)$ be the components of φ. For each $j \in \{1, \dots, n\}$ and an $r \in (0,1)$ we have

$$\int_{\partial\mathbb{B}} \varphi_j(r\zeta) d\sigma(\zeta) = \int_{\partial\mathbb{B}} d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_j\left(re^{i\theta}\zeta\right) d\theta. \tag{3.20}$$

Since the slice function $(\varphi_j)_\zeta$ of φ_j is holomorphic in the open unit disc of $\mathbb{C}$, the mean value theorem gives $(1/2\pi)\int_{-\pi}^{\pi} \varphi_j(re^{i\theta}\zeta) d\theta = \varphi_j(0)$ for each $\zeta \in \partial\mathbb{B}$ and $j \in \{1, \dots, n\}$. Using this fact in formula (3.20), multiplying such obtained equality by $2nc_{\alpha,n}(1-r^2)^\alpha r^{2n-1} dr$, then integrating it from 0 to 1 and using the polar coordinates on the unit ball we get

$$\int_{\mathbb{B}} \varphi_j(z) dV_\alpha(z) = \varphi_j(0) \tag{3.21}$$

for each $j \in \{1, \dots, n\}$. Now we fix a $j \in \{1, \dots, n\}$. By applying the condition (3.11) to a bounded Borel function $h(w) = \langle w, e_j \rangle$, $j \in \{1, \dots, n\}$, where e_j is the standard orthonormal base vector in $\mathbb{C}^n$, we have

$$\begin{aligned} \int_{\mathbb{B}} \varphi_j(z) dV_\alpha(z) &= \int_{\mathbb{B}} \langle \varphi(z), e_j \rangle dV_\alpha(z) \\ &= \int_{\mathbb{B}} \langle z, e_j \rangle dV_\alpha(z) \\ &= 2nc_{\alpha,n} \int_0^1 r^{2n} \left(1-r^2\right)^\alpha dr \int_{\partial\mathbb{B}} \langle \zeta, e_j \rangle d\sigma(\zeta). \end{aligned} \tag{3.22}$$

From [29, page 15, §1.4.5(2)] we have that

$$\int_{\partial\mathbb{B}} \langle \zeta, e_j \rangle d\sigma(\zeta) = \frac{n-1}{\pi} \int_{-\pi}^{\pi} \int_0^1 (1-t^2)^{n-2} t^2 e^{i\theta} dt d\theta = 0. \tag{3.23}$$

Hence (3.21), (3.22), and (3.23) show that $\psi_j(0) = 0$ for each $j \in \{1, \ldots, n\}$, and so ψ fixes the origin. By a well-known theorem [29] we see that ψ is a unitary operator on $\mathbb{C}^n$. This completes the proof. □

Remark 3.5. Let Isom $AN_{\log,\alpha}(\mathbb{B})$ denote the set of all surjective isometries of $AN_{\log,\alpha}(\mathbb{B})$. Since the transformations ψ in Corollary 3.4 form a group of unitary transformations the corollary implies the following isomorphism:

$$\text{Isom } AN_{\log,\alpha}(\mathbb{B}) \simeq \mathbb{T} \times U(n, \mathbb{C}). \tag{3.24}$$

Here $\mathbb{T}$ denotes the set of all complex numbers with $|c| = 1$ and $U(n, \mathbb{C})$ being the group of all unitary operators on $\mathbb{C}^n$. For the Smirnov class N^* and the Privalov space N^p $(p > 1)$, analogous results are obtained by Stephenson and Subbotin (see [22, 24–26]). So we see that every surjective isometry of $AN_{\log,\alpha}(\mathbb{B})$ has the same form as one of the Smirnov class or the Privalov space.

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