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Research Article

The Fixed Point Method for Fuzzy Approximation of a Functional Equation Associated with Inner Product Spaces

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Th. M. Rassias (1984) proved that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$, $\sum_{i=1}^{n} \|x_i - (1/n)\sum_{j=1}^{n} x_j\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\|(1/n)\sum_{i=1}^{n} x_i\|^2$ holds for all $x_1, \ldots, x_n \in X$. The aim of this paper is to extend the applications of the fixed point alternative method to provide a fuzzy stability for the functional equation $\sum_{i=1}^{n} f(x_i - (1/n)\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf((1/n)\sum_{i=1}^{n} x_i)$ which is said to be a functional equation associated with inner product spaces.

1. Introduction

Studies on fuzzy normed linear spaces are relatively recent in the field of fuzzy functional analysis. In 1984, Katsaras [1] first introduced the notion of fuzzy norm on a linear space and at the same year Wu and Fang [2] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [3–6].

Nowadays, fixed point and operator theory play an important role in different areas of mathematics, and its applications, particularly in mathematics, physics, differential equation, game theory and dynamic programming. Since fuzzy mathematics and fuzzy physics along with the classical ones are constantly developing, the fuzzy type of the fixed point and operator theory can also play an important role in the new fuzzy area and fuzzy mathematical physics. Many authors [4, 7–9] have also proved some different type of fixed point theorems in fuzzy (probabilistic) metric spaces and fuzzy normed linear spaces. In 2003, Bag and Samanta [10] modified the definition of Cheng and Mordeson [11] by removing a regular

condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed linear spaces [12].

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem concerning group homomorphisms was raised by Ulam [13] in 1940 and affirmatively solved by Hyers [14]. The result of Hyers was generalized by Aoki [15] for approximate additive function and by Th. M. Rassias [16] for approximate linear functions by allowing the difference Cauchy equation $\|f(x_1+x_2)-f(x_1)-f(x_2)\|$ to be controlled by $\varepsilon(\|x_1\|^p+\|x_2\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Th. M. Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th. M. Rassias is called the generalized Hyers-Ulam stability. In 1994, a generalization of Th. M. Rassias theorem was obtained by Găvruţa [17], who replaced $\varepsilon(\|x_1\|^p+\|x_2\|^p)$ by a general control function $\varphi(x_1,x_2)$.

On the other hand, J. M. Rassias [18–25] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruţa [26]. This stability phenomen on is called the Ulam-Găvruţa-Rassias stability (see also [27]).

Theorem 1.1 (J. M. Rassias [18]). *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\|x\|^p\|y\|^q, \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x)-L(x)\| \leq \frac{\theta}{|2^r-2|}\|x\|^r, \tag{1.2}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Very recently, K. Ravi [28] in the inequality (1.1) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{\|x\|^p\|y\|^p+(\|x\|^{2p}+\|y\|^{2p})\}$.

For more details about the results concerning such problems and mixed product-sum stability (J. M. Rassias Stability), the reader is referred to [29–41].

Quadratic functional equations were used to characterize inner product spaces [42–45]. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x_1+x_2\|^2+\|x_1-x_2\|^2 = 2\left(\|x_1\|^2+\|x_1\|^2\right). \tag{1.3}$$

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{1.4}$$

is related to a symmetric biadditive function [46, 47]. It is natural that this equation is called a quadratic functional equation, and every solution of the quadratic equation (1.4) is said to be a quadratic function.

It was shown by Th. M. Rassias [48] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$ as follows:

$$\sum_{i=1}^{n}\left\|x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2 - n\left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\|^2, \tag{1.5}$$

for all $x_1, \ldots, x_n \in X$. In [49], Park proved the generalized Hyers-Ulam stability of a functional equation associated with inner product spaces:

$$f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y) - 2f\left(\frac{x+y}{2}\right), \tag{1.6}$$

in fuzzy normed spaces.

The main objective of this paper is to prove the the generalized Hyers-Ulam stability of the following functional equation associated with inner product spaces

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right), \tag{1.7}$$

in fuzzy normed spaces, based on the fixed point method. Interesting new results concerning functional equations associated with inner product spaces have recently been obtained by Park et al. [50–52] and Najati and Th. M. Rassias [53] as well as for the fuzzy stability of a functional equation associated with inner product spaces by Park [49].

The stability of different functional equations in fuzzy normed spaces and random normed spaces has been studied in [20, 21, 54–77]. In this paper, we prove the generalized fuzzy stability of a functional equation associated with inner product spaces (1.7).

2. Preliminaries

We start our work with the following notion of fixed point theory. For the proof, refer to [78]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [79].

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

We recall the following theorem by Margolis and Diaz.

Theorem 2.1. *Suppose that one is given a complete generalized metric space (Ω, d) and a strictly contractive function $T : \Omega \to \Omega$ with Lipschitz constant L, then for each given $x \in \Omega$, either*

$$d\left(T^m x, T^{m+1} x\right) = \infty \quad \forall m \geq 0, \tag{2.1}$$

or other exists a natural number m_0 such that

(i) $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \geq m_0$;

(ii) *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;

(iii) y^* *is the unique fixed point of* T *in* $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;

(iv) $d(y, y^*) \leq (1/(1-L)) d(y, Ty)$ *for all* $y \in \Lambda$.

Next, we define the notion of a fuzzy normed linear space.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X [10] if and only if the following conditions are satisfied:

(N_1) $N(x,t) = 0$ for all $x \in X$ and $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N_3) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$ for all $x, y \in X$ and all $s, t \in \mathbb{R}$;

(N_5) $N(x, \cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$ for all $x \in X$.

In the following we will suppose that $N(x, \cdot\ )$ is left continuous for every x.

A fuzzy normed linear space is a pair (X, N), where X is a real linear space and N is a fuzzy norm on X.

Let $(X, \|\cdot\|)$ be a normed linear space, then

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0,\ x \in X, \\ 0, & t \leq 0,\ x \in X \end{cases} \tag{2.2}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N - \lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, \delta) > 1 - \epsilon$ $(m, n \geq n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

From now on, let X be a linear space, (Z, N') be a fuzzy normed space and (Y, N) be a fuzzy Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$\Delta f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) + n f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right), \tag{2.3}$$

for all $x_1, \ldots, x_n \in X$, where $n \geq 2$ is a fixed integer.

3. Fuzzy Approximation

In the following theorem, we prove the fuzzy stability of the functional equation (1.7) via fixed point method, for an even case.

Theorem 3.1. *Let $\phi : X \to (Z, N')$ be a function such that, $\phi(2x) = \alpha\phi(x)$ for some real number α with $|\alpha| < 4$. Suppose that an even function $f : X \to (Y, N)$ with $f(0) = 0$ satisfies the inequality*

$$N\left(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n\right) \geq \min\left\{N'\left(\phi(x_1), t_1\right), \ldots, N'\left(\phi(x_n), t_n\right)\right\} \tag{3.1}$$

for all $x_1, \ldots, x_n \in X$ and all $t_1, \ldots, t_n > 0$, then there exists a unique quadratic function $Q : X \to Y$ such that $Q(x) = N - \lim_{m\to\infty}(f(2^m x)/4^m)$ and

$$N\left(f(x) - Q(x), t\right) \geq M^e(x, (4-\alpha)t), \tag{3.2}$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned} M^e(x,t) = \min\Bigg\{ & N'\left(\phi(nx), \frac{2n-2}{2n^2+9n}t\right), N'\left(\phi((n-1)x), \frac{2n-2}{2n^2+9n}t\right), \\ & N'\left(\phi(x), \frac{2n-2}{2n^2+9n}t\right), N'\left(\phi(0), \frac{2n-2}{2n^2+9n}t\right)\Bigg\}. \end{aligned} \tag{3.3}$$

Proof. Letting $x_1 = nx_1$, $x_i = nx_2$ $(i = 2, \ldots, n)$, and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1) and using the evenness of f, we obtain

$$\begin{aligned} & N\left(nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2), nt\right) \\ & \quad \geq \min\left\{N'\left(\phi(nx_1), t\right), N'\left(\phi(nx_2), t\right)\right\}, \end{aligned} \tag{3.4}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Interchanging x_1 with x_2 in (3.4) and using the evenness of f, we obtain

$$\begin{aligned} & N\left(nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2), nt\right) \\ & \quad \geq \min\left\{N'\left(\phi(nx_1), t\right), N'\left(\phi(nx_2), t\right)\right\}, \end{aligned} \tag{3.5}$$

for all $x_1, x_2 \in X$ and all $t > 0$. It follows from (3.4) and (3.5) that

$$\begin{aligned} & N\left(nf((n-1)x_1 + x_2) + nf(x_1 + (n-1)x_2) + 2f((n-1)(x_1 - x_2)) + 2(n-1)f(x_1 - x_2)\right. \\ & \quad \left. -nf(nx_1) - nf(nx_2), 2nt\right) \geq \min\left\{N'\left(\phi(nx_1), t\right), N'\left(\phi(nx_2), t\right)\right\}, \end{aligned} \tag{3.6}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Setting $x_1 = nx_1$, $x_2 = -nx_2$, $x_i = 0$ $(i = 3, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1) and using the evenness of f, we obtain

$$\begin{aligned} &N\big(f((n-1)x_1 + x_2) + f(x_1 + (n-1)x_2) + 2(n-1)f(x_1 - x_2) - f(nx_1) - f(nx_2), nt\big) \\ &\quad \geq \min\{N'(\phi(nx_1), t), N'(\phi(-nx_2), t), N'(\phi(0), t)\}, \end{aligned} \tag{3.7}$$

for all $x_1, x_2 \in X$ and all $t > 0$. So we obtain from (3.6) and (3.7) that

$$\begin{aligned} &N\left(f((n-1)(x_1 - x_2)) - (n-1)^2 f(x_1 - x_2), \frac{n^2 + 2n}{2} t\right) \\ &\quad \geq \min\{N'(\phi(nx_1), t), N'(\phi(nx_2), t), N'(\phi(-nx_2), t), N'(\phi(0), t)\}, \end{aligned} \tag{3.8}$$

for all $x_1, x_2 \in X$ and all $t > 0$. So

$$N\left(f((n-1)x) - (n-1)^2 f(x), \frac{n^2 + 2n}{2} t\right) \geq \min\{N'(\phi(nx), t), N'(\phi(0), t)\}, \tag{3.9}$$

for all $x \in X$ and all $t > 0$. Putting $x_1 = nx$, $x_i = 0$ $(i = 2, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1), we get

$$N\big(f(nx) - f((n-1)x) - (2n-1)f(x), nt\big) \geq \min\{N'(\phi(nx), t), N'(\phi(0), t)\}, \tag{3.10}$$

for all $x \in X$ and all $t > 0$. It follows from (3.9) and (3.10) that

$$N\left(f(nx) - n^2 f(x), \frac{n^2 + 4n}{2} t\right) \geq \min\{N'(\phi(nx), t), N'(\phi(0), t)\}, \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Letting $x_2 = -(n-1)x_1$ in (3.7) and replacing x_1 by x/n in the obtained inequality, we get

$$\begin{aligned} &N\big(f((n-1)x) - f((n-2)x) - (2n-3)f(x), nt\big) \\ &\quad \geq \min\{N'(\phi(x), t), N'(\phi((n-1)x), t), N'(\phi(0), t)\}, \end{aligned} \tag{3.12}$$

for all $x \in X$ and all $t > 0$. It follows from (3.9), (3.10), (3.11) and (3.12) that

$$\begin{aligned} &N\left(f((n-2)x) - (n-2)^2 f(x), \frac{n^2 + 4n}{2} t\right) \\ &\quad \geq \min\{N'(\phi(nx), t), N'(\phi((n-1)x), t), N'(\phi(x), t), N'(\phi(0), t)\}, \end{aligned} \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Applying (3.11) and (3.13), we obtain

$$\begin{aligned} &N\Big(f(nx) - f((n-2)x) - 4(n-1)f(x), \big(n^2+4n\big)t\Big) \\ &\quad \geq \min\{N'(\phi(nx),t), N'(\phi((n-1)x),t), N'(\phi(x),t), N'(\phi(0),t)\}, \end{aligned} \tag{3.14}$$

for all $x \in X$ and all $t > 0$. Setting $x_1 = x_2 = nx$, $x_i = 0$ $(i = 3, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1), we obtain

$$N\Big(f((n-2)x) + (n-1)f(2x) - f(nx), \frac{n}{2}t\Big) \geq \min\{N'(\phi(nx),t), N'(\phi(0),t)\}, \tag{3.15}$$

for all $x \in X$ and all $t > 0$. It follows from (3.14) and (3.15) that

$$\begin{aligned} &N\left(f(2x) - 4f(x), \frac{2n^2+9n}{2n-2}t\right) \\ &\quad \geq \min\{N'(\phi(nx),t), N'(\phi((n-1)x),t), N'(\phi(x),t), N'(\phi(0),t)\}, \end{aligned} \tag{3.16}$$

for all $x \in X$ and all $t > 0$. Therefore

$$N(f(2x) - 4f(x), t) \geq M^e(x,t) \tag{3.17}$$

for all $x \in X$ and all $t > 0$, which implies that

$$N\left(\frac{f(2x)}{4} - f(x), t\right) \geq M^e(x,4t), \tag{3.18}$$

for all $x \in X$ and all $t > 0$. Let S be the set of all even functions $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h,k) = \inf\{u \in \mathbb{R}^+ : N(h(x) - k(x), ut) \geq M^e(x,t),\ \forall x \in X,\ \forall t > 0\}, \tag{3.19}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S,d) is a generalized complete metric space [80].

Without loss of generality, we consider $\alpha > 0$. Let us now consider the function $J : S \to S$ defined by $Jh(x) := h(2x)/4$ for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f,g) < \varepsilon$, then

$$N\Big(Jg(x) - Jf(x), \frac{\alpha u}{4}t\Big) = N(g(2x) - f(2x), \alpha ut) \geq M^e(2x, \alpha t) = M^e(x,t), \tag{3.20}$$

that is, if $d(f,g) < \varepsilon$ we have $d(Jf, Jg) < (\alpha/4)\varepsilon$. This means that $d(Jf, Jg) \leq (\alpha/4)d(f,g)$ for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\alpha/4$.

It follows from (3.18) that

$$N\left(Jf(x)-f(x),\frac{t}{4}\right)\geq M^{e}(x,t), \tag{3.21}$$

for all $x\in X$ and all $t>0$, which implies that $d(Jf,f)\leq 1/4$.

Due to Theorem 2.1, there exists a function $Q:X\to Y$ such that Q is a fixed point of J, that is, $Q(2x)=4Q(x)$ for all $x\in X$.

Also, $d(J^{m}g,Q)\to 0$ as $m\to\infty$, implies the equality $N-\lim_{m\to\infty}(f(2^{m}x)/4^{m})=Q(x)$ for all $x\in X$. Setting $x_{i}=2^{m}x_{i}$ $(i=1,\ldots,n)$ and $t_{i}=t/n$ $(i=1,\ldots,n)$ in (3.1), we obtain that

$$\begin{aligned}
&N\left(\frac{\Delta f(2^{m}x_{1},\ldots,2^{m}x_{n})}{4^{m}},t\right)\\
&\quad\geq\min\left\{N'\left(\phi(2^{m}x_{1}),4^{m}\frac{t}{n}\right),\ldots,N'\left(\phi(2^{m}x_{n}),4^{m}\frac{t}{n}\right)\right\}\\
&\quad=\min\left\{N'\left(\phi(2^{m}x_{1}),\alpha^{m}\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right),\ldots,N'\left(\phi(2^{m}x_{n}),\alpha^{m}\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right)\right\}\\
&\quad=\min\left\{N'\left(\alpha^{m}\phi(x_{1}),\alpha^{m}\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right),\ldots,N'\left(\alpha^{m}\phi(x_{n}),\alpha^{m}\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right)\right\}\\
&\quad=\min\left\{N'\left(\phi(x_{1}),\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right),\ldots,N'\left(\phi(x_{n}),\left(\frac{4}{\alpha}\right)^{m}\frac{t}{n}\right)\right\}
\end{aligned} \tag{3.22}$$

for all $x_{1},\ldots,x_{n}\in X$ and all $t>0$. By letting $m\to\infty$ in (3.22), we find that $N(\Delta Q(x_{1},\ldots,x_{n}),t)=1$ for all $t>0$, which implies $\Delta Q(x_{1},\ldots,x_{n})=0=0$. Thus Q satisfies (1.7). Hence the function $Q:X\to Y$ is quadratic (See Lemma 2.2 of [53]).

According to the fixed point alterative, since Q is the unique fixed point of J in the set $\Omega=\{g\in S:d(f,g)<\infty\}$, Q is the unique function such that

$$N\left(f(x)-Q(x),ut\right)\geq M^{e}(x,t), \tag{3.23}$$

for all $x_{1},\ldots,x_{n}\in X$ and all $t>0$. Again using the fixed point alterative, we get

$$d(f,Q)\leq\frac{1}{1-L}d(f,Jf)\leq\frac{1}{4(1-L)}=\frac{1}{4(1-\alpha/4)}, \tag{3.24}$$

which implies the inequality

$$N\left(f(x)-Q(x),\frac{t}{4-\alpha}\right)\geq M^{e}(x,t), \tag{3.25}$$

for all $x \in X$ and all $t > 0$. So

$$N\big(f(x) - Q(x), t\big) \geq M^e(x, (4-\alpha)t), \tag{3.26}$$

for all $x \in X$ and all $t > 0$. This completes the proof. □

In the following theorem, we prove the fuzzy stability of the functional equation (1.7) via fixed point method, for an odd case.

Theorem 3.2. *Let $\phi : X \to (Z, N')$ be a function such that $\phi(2x) = \alpha\phi(x)$ for some real number α with $|\alpha| < 2$. Suppose that an odd function $f : X \to (Y, N)$ satisfies the inequality* (3.1) *for all $x_1, \ldots, x_n \in X$ and all $t_1, \ldots, t_n > 0$, then there exists a unique additive function $A : X \to Y$ such that $A(x) = N - \lim_{m\to\infty}(f(2^m x)/2^m)$ and*

$$N\big(f(x) - A(x), t\big) \geq M^o(x, (2-\alpha)t) \tag{3.27}$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned} M^o(x,t) = \min\Bigg\{ & N'\left(\phi(2x), \frac{2}{n^2+4n}t\right), N'\left(\phi(x), \frac{2}{n^2+4n}t\right), \\ & N'\left(\phi(-x), \frac{2}{n^2+4n}t\right), N'\left(\phi(0), \frac{2}{n^2+4n}t\right)\Bigg\}. \end{aligned} \tag{3.28}$$

Proof. Letting $x_1 = nx_1$, $x_i = nx_1'$ $(i = 2, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1) and using the oddness of f, we obtain that

$$\begin{aligned} & N\big(nf(x_1 + (n-1)x_1') + f((n-1)(x_1 - x_1')) - (n-1)f(x_1 - x_1') \\ & \quad - f(nx_1) - (n-1)f(nx_1'), nt\big) \geq \min\{N'(\phi(nx_1), t), N'(\phi(nx_1'), t)\}, \end{aligned} \tag{3.29}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Interchanging x_1 with x_1' in (3.29) and using the oddness of f, we get

$$\begin{aligned} & N\big(nf((n-1)x_1 + x_1') - f((n-1)(x_1 - x_1')) + (n-1)f(x_1 - x_1') - (n-1)f(nx_1) - f(nx_1'), nt\big) \\ & \quad \geq \min\{N'(\phi(nx_1), t), N'(\phi(nx_1'), t)\}, \end{aligned} \tag{3.30}$$

for all $x_1, x_1' \in X$ and all $t > 0$. It follows from (3.29) and (3.30) that

$$\begin{aligned} & N\big(nf(x_1 + (n-1)x_1') - nf((n-1)x_1 + x_1') + 2f((n-1)(x_1 - x_1')) - 2(n-1)f(x_1 - x_1') \\ & \quad + (n-2)f(nx_1) - (n-2)f(nx_1'), 2nt\big) \geq \min\{N'(\phi(nx_1), t), N'(\phi(nx_1'), t)\}, \end{aligned} \tag{3.31}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Setting $x_1 = nx_1$, $x_2 = -nx_1'$, $x_i = 0$ $(i = 3, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1) and using the oddness of f, we get

$$\begin{aligned} &N\left(f\left((n-1)x_1 + x_1'\right) - f\left(x_1 + (n-1)x_1'\right) + 2f\left(x_1 - x_1'\right) - f(nx_1) + f\left(nx_1'\right), nt\right) \\ &\quad \geq \min\left\{N'\left(\phi(nx_1), t\right), N'\left(\phi\left(-nx_1'\right), t\right), N'\left(\phi(0), t\right)\right\}, \end{aligned} \tag{3.32}$$

for all $x_1, x_1' \in X$ and all $t > 0$. So we obtain from (3.31) and (3.32) that

$$\begin{aligned} &N\left(f\left((n-1)\left(x_1 - x_1'\right)\right) + f\left(x_1 - x_1'\right) - f(nx_1) + f\left(nx_1'\right), \frac{n^2+2n}{2}t\right) \\ &\quad \geq \min\left\{N'\left(\phi(nx_1), t\right), N'\left(\phi\left(nx_1'\right), t\right), N'\left(\phi\left(-nx_1'\right), t\right), N'\left(\phi(0), t\right)\right\}, \end{aligned} \tag{3.33}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Putting $x_1 = n(x_1 - x_1')$, $x_i = 0$ $(i = 2, \ldots, n)$ and $t_i = t$ $(i = 1, \ldots, n)$ in (3.1), we obtain

$$\begin{aligned} &N\left(f\left(n\left(x_1 - x_1'\right)\right) - f\left((n-1)\left(x_1 - x_1'\right)\right) - f\left(\left(x_1 - x_1'\right)\right), nt\right) \\ &\quad \geq \min\left\{N'\left(\phi\left(n\left(x_1 - x_1'\right)\right), t\right), N'\left(\phi(0), t\right)\right\}, \end{aligned} \tag{3.34}$$

for all $x_1, x_1' \in X$ and all $t > 0$. It follows from (3.33) and (3.34) that

$$\begin{aligned} &N\left(f\left(n\left(x_1 - x_1'\right)\right) - f(nx_1) + f\left(nx_1'\right), \frac{n^2+4n}{2}t\right) \\ &\quad \geq \min\left\{N'\left(\phi\left(n\left(x_1 - x_1'\right)\right), t\right), N'\left(\phi(nx_1), t\right), N'\left(\phi\left(nx_1'\right), t\right), N'\left(\phi\left(-nx_1'\right), t\right), N'\left(\phi(0), t\right)\right\}, \end{aligned} \tag{3.35}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Replacing x_1 and x_1' by x/n and $-x/n$ in (3.35), respectively, we obtain

$$N\left(f(2x) - 2f(x), \frac{n^2+4n}{2}t\right) \geq \min\left\{N'\left(\phi(2x), t\right), N'\left(\phi(x), t\right), N'\left(\phi(-x), t\right), N'\left(\phi(0), t\right)\right\}, \tag{3.36}$$

for all $x \in X$ and all $t > 0$. Therefore

$$N\left(f(2x) - 2f(x), t\right) \geq M^o(x, t), \tag{3.37}$$

for all $x \in X$ and all $t > 0$, which implies that

$$N\left(\frac{f(2x)}{2} - f(x), t\right) \geq M^o(x, 2t), \tag{3.38}$$

for all $x \in X$ and all $t > 0$. Let S be the set of all odd functions $h : X \to Y$ and introduce a generalized metric on S as follows:

$$d(h,k) = \inf\{u \in \mathbb{R}^+ : N(h(x) - k(x), ut) \geq M^o(x,t),\ \forall x \in X,\ \forall t > 0\}, \tag{3.39}$$

where, as usual, $\inf \emptyset = +\infty$. So (S,d) is a generalized complete metric space. We consider the function $J : S \to S$ defined by $Jh(x) := h(2x)/2$ for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f,g) < \varepsilon$, then

$$N\left(Jg(x) - Jf(x), \frac{\alpha u}{2}t\right) = N\left(g(2x) - f(2x), \alpha ut\right) \geq M^o(2x, \alpha t) = M^o(x,t), \tag{3.40}$$

that is, if $d(f,g) < \varepsilon$ we have $d(Jf, Jg) < (\alpha/2)\varepsilon$. This means that $d(Jf, Jg) \leq (\alpha/2)d(f,g)$ for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\alpha/2$.

It follows from (3.38) that

$$N\left(Jf(x) - f(x), \frac{t}{2}\right) \geq M^o(x,t), \tag{3.41}$$

for all $x \in X$ and all $t > 0$, which implies that $d(Jf, f) \leq 1/2$.

Due to Theorem 2.1, there exists a function $A : X \to Y$ such that A is a fixed point of J, that is, $A(2x) = 2A(x)$ for all $x \in X$.

Also, $d(J^m g, A) \to 0$ as $m \to \infty$, implies the equality $N\text{–}\lim_{m\to\infty}(f(2^m x)/2^m) = A(x)$ for all $x \in X$. Setting $x_i = 2^m x_i$ $(i = 1, \ldots, n)$ and $t_i = (t/n)$ $(i = 1, \ldots, n)$ in (3.1), we obtain that

$$\begin{aligned}
&N\left(\frac{\Delta f(2^m x_1, \ldots, 2^m x_n)}{2^m}, t\right) \\
&\quad \geq \min\left\{N'\left(\phi(2^m x_1), 2^m \frac{t}{n}\right), \ldots, N'\left(\phi(2^m x_n), 2^m \frac{t}{n}\right)\right\} \\
&\quad = \min\left\{N'\left(\phi(2^m x_1), \alpha^m \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \ldots, N'\left(\phi(2^m x_n), \alpha^m \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right)\right\} \\
&\quad = \min\left\{N'\left(\alpha^m \phi(x_1), \alpha^m \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \ldots, N'\left(\alpha^m \phi(x_n), \alpha^m \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right)\right\} \\
&\quad = \min\left\{N'\left(\phi(x_1), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \ldots, N'\left(\phi(x_n), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right)\right\},
\end{aligned} \tag{3.42}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. By letting $m \to \infty$ in (3.42), we find that $N(\Delta A(x_1, \ldots, x_n), t) = 1$ for all $t > 0$, which implies $\Delta A(x_1, \ldots, x_n) = 0$. Thus A satisfies (1.7). Hence the function $A : X \to Y$ is additive (see Lemma 2.1 of [53]).

The rest of the proof is similar to the proof of Theorem 3.1. □

The main result of the paper is the following.

Theorem 3.3. *Let $\phi : X \to (Z, N')$ be a function such that, $\phi(2x) = \alpha\phi(x)$ for some real number α with $|\alpha| < 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies* (3.1) *for all $x_1, \ldots, x_n \in X$ and all $t > 0$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\begin{aligned} &N\big(f(x) - Q(x) - A(x), t\big) \\ &\quad \geq \min\{M^e(x, (4-\alpha)t), M^e(-x, (4-\alpha)t), M^o(x, (2-\alpha)t), M^o(-x, (2-\alpha)t)\}, \end{aligned} \tag{3.43}$$

for all $x \in X$ and all $t > 0$, where $M^e(x, t)$ and $M^o(x, t)$ are defined as in Theorems 3.1 and 3.2.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$, then

$$\begin{aligned} N\big(\Delta f_e(x_1, \ldots, x_n), t_1 + \cdots + t_n\big) &= N\big(\Delta f(x_1, \ldots, x_n) + \Delta f(-x_1, \ldots, -x_n), 2t_1 + \cdots + 2t_n\big) \\ &\geq \min\{N'(\phi(x_1), t_1), N'(\phi(-x_1), t_1), \ldots, N'(\phi(x_n), t_n), N'(\phi(-x_n), t_n)\}, \end{aligned} \tag{3.44}$$

for all $x_1, \ldots, x_n \in X$ and $t > 0$. Hence, in view of Theorem 3.1, there exists a unique quadratic function $Q : X \to Y$ such that

$$N\big(f_e(x) - Q(x), t\big) \geq \min\{M^e(x, (4-\alpha)t), M^e(-x, (4-\alpha)t)\}, \tag{3.45}$$

for all $x \in X$ and $t > 0$. On the other hand, let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$, then, by using the above method from Theorem 3.2, there exists a unique additive function $A : X \to Y$ such that

$$N\big(f_o(x) - A(x), t\big) \geq \min\{M^o(x, (2-\alpha)t), M^o(-x, (2-\alpha)t)\}, \tag{3.46}$$

for all $x \in X$ and $t > 0$. Hence, (3.43) follows from (3.45) and (3.46). □

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