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Research Article

Quantitative Bounds for Positive Solutions of a Stević Difference Equation

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This paper studies the behavior of positive solutions to the following particular case of a difference equation by Stević $x_{n+1} = A + x_n^p / x_{n-k}^{p^{k+1}}$, $n \in \mathbb{N}_0$, where $A,p \in (0,+\infty)$, $k \in \mathbb{N}$, and presents theoretically computable explicit lower and upper bounds for the positive solutions to this equation. Besides, a concrete example is given to show the computing approaches which are effective for small parameters. Some analogous results are also established for the corresponding Stević max-type difference equation.

1. Introduction

The study regarding the behavior of positive solutions to the difference equation

$$x_n = A + \frac{x_{n-k}^p}{x_{n-m}^q}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A, p, q \in (0,+\infty)$ and $k, m \in \mathbb{N}, k \neq m$, was put forward by Stević at many conferences (see, e.g., [1–3]). For numerous papers in this area and some closely related results, see [1–39] and the references cited therein.

In [4, 24], the authors proved some conditions for the global asymptotic stability of the positive equilibrium to the difference equation given by

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with $A > 0, k \in \mathbb{N}$.

Motivated by these papers, the authors of [8] studied the quantitative bounds for the recursive equation (1.2) where $y_{-k}, \ldots, y_{-1}, y_0, A > 0$, and $k \in \mathbb{N}\backslash\{1\}$, and quantitative bounds of the form $R_i \leq y_i \leq S_i, i \geq k+1$ were provided. Exponential convergence was shown to persist for all solutions. The authors also took $A = k = 2$ as an example, and eventually obtained the concrete bounds as follows:

$$2 + \prod_{i=n-3}^{n-2}\left(1 - \left(\frac{12}{17}\right)^{[(i+2)/10]}\right) \leq y_n \leq 2 + \prod_{i=n-3}^{n-2}\left(1 + \left(\frac{2}{3}\right)^{2[(i-3)/10]+1}\right), \quad n > 6. \tag{1.3}$$

In [20], Stević investigated positive solutions of the following difference equation:

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $A, p, r \in (0, +\infty)$, and gave a complete picture concerning the boundedness character of the positive solutions to (1.4) as well as of positive solutions of the following counterpart in the class of max-type difference equations:

$$y_{n+1} = \max\left\{A, \frac{y_n^p}{y_{n-1}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where A, p, r are positive real numbers.

Motivated by the above work and works in [6, 9, 10, 12, 17, 21, 22], our aim in this paper is to discuss the quantitative bounds of the solutions to the following higher-order difference equation:

$$x_{n+1} = A + \frac{x_n^p}{x_{n-k}^{p^{k+1}}}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $A, p \in (0, +\infty), k \in \mathbb{N}$, and the initial values are positive. Following the methods and ideas from [8], we obtain theoretically computable explicit bounds of the form

$$A + \prod_{j=n-k-1}^{n-2} a\left(2\left[\frac{j+k-1}{4k+2}\right]\right)^{p^{n-j-1}} \leq x_n \leq A + \prod_{j=n-k-1}^{n-2} b\left(2\left[\frac{j-k-2}{4k+2}\right]+1\right)^{p^{n-j-1}} \tag{1.7}$$

which are independent of the positive initial values $x_{-k}, x_{-k+1}, \ldots, x_0$.

Our results extend those ones in [8], in which the case $p = 1$ was considered, and also in some way improve those in [20], in which the case $k = 1$ was considered.

On the other hand, inspired by the study in [19] we also investigate the quantitative bounds for the positive solutions to the following max-type recursive equation:

$$y_{n+1} = \max\left\{A, \frac{y_n^p}{y_{n-k}^{p^{k+1}}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where $A, p \in (0, +\infty), k \in \mathbb{N}$, and some similar results are established.

We want to point out that the boundedness characters of (1.1) and (1.8) for the case $k = 1$ and $m \in \mathbb{N}$, including our particular case, have been recently solved by Stević and presented at several conferences (see also [25]).

2. Auxiliary Results

In this section, we will present several preliminary lemmas needed to prove the main results in Section 3.

The following lemma can be easily proved.

Lemma 2.1. *Equation* (1.6) *has a unique positive equilibrium point* $\overline{x} > A$.

Now, let us define a first-order difference equation given by

$$u(n+1) = \frac{A}{(A+u(n)^r)^p} + \frac{1}{(A+u(n)^r)^{p^{k+1}}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $A, p > 0$ are identical to those of (1.6), $r = \sum_{i=1}^{k} p^i$, and the initial value $u(0) > 0$.

If $p = 1$, then (2.1) reduces to the sequence $\{x(i)\}$ defined in [8].

Lemma 2.2. *Equation* (2.1) *has a unique positive equilibrium if* $p > 1$ *and* $A \geq [rp(p^k - 1)]^{1/p}$ *or* $0 < p \leq 1$.

Proof. Suppose that $x > 0$ is an equilibrium point of (2.1), then we have

$$x = \frac{A}{(A+x^r)^p} + \frac{1}{(A+x^r)^{p^{k+1}}}. \tag{2.2}$$

Let $F(x) = x(A+x^r)^{p^{k+1}} - A(A+x^r)^{p(p^k-1)} - 1$, then it suffices to show that $F(x)$ has only one positive fixed point. The derivative of $F(x)$ is

$$\begin{aligned} F'(x) &= (A+x^r)^{p^{k+1}-p-1}\left[(A+x^r)^{p+1} + rx^r p^{k+1}(A+x^r)^p - rAp\left(p^k-1\right)x^{r-1}\right] \\ &= (A+x^r)^{p^{k+1}-p-1}\left\{(A+x^r)^{p+1} + rx^{r-1}\left[xp^{k+1}(A+x^r)^p - Ap\left(p^k-1\right)\right]\right\}. \end{aligned} \tag{2.3}$$

(i) If $p \leq 1$, then obviously $F'(x) > 0$ for $x \geq 0$.

(ii) If $p > 1$ and $x \geq 1$, then $F'(x) > 0$ follows from $(A+1)^p > A$.

(iii) If $p > 1$ and $0 < x < 1$, we have

$$\begin{aligned} &(A+x^r)^{p+1} + rx^{r-1}\left[xp^{k+1}(A+x^r)^p - Ap\left(p^k-1\right)\right] \\ &\quad > (A+x^r)^{p+1} - rpx^{r-1}A\left(p^k-1\right) > A\left(A^p - rp\left(p^k-1\right)\right) \geq 0. \end{aligned} \tag{2.4}$$

Hence $F'(x) > 0$.

Through above analysis, if $p > 1$ and $A > [rp(p^k - 1)]^{1/p}$ or $0 < p \leq 1$, then $F(x)$ is monotonically increasing on $(0, +\infty)$. Hence the uniqueness of positive equilibrium of (2.1) follows from $F(0) = -A^{p^{k+1}-p+1} - 1 < 0$, and $\lim_{x\to+\infty} F(x) = +\infty$. □

Lemma 2.3. *If $p > 1$ and $A \geq [rp(p^k - p)]^{1/p}$ or $0 < p \leq 1$, then the unique equilibrium point of* (1.6) *has the form $A + \lambda^r$, where $\lambda > 0$ is the unique positive equilibrium of* (2.1).

Proof. Defining $\rho(x) = x(A + x^r)^p - (A + x^r), x > 0$, simply we have that $\rho(x)$ has a unique positive zero denoted by λ, that is, $\lambda(A + \lambda^r)^p = (A + \lambda^r)$.

If $p = 1$, then $\lambda = 1$, and thus

$$\lambda = \frac{A}{(A+\lambda^r)^p} + \frac{1}{(A+\lambda^r)^{p^{k+1}}}, \qquad A + \lambda^r = A + \frac{(A+\lambda^r)^p}{(A+\lambda^r)^{p^{k+1}}}. \tag{2.5}$$

If $p > 0$ and $p \neq 1$, then

$$A + \lambda^r = \lambda^{1/(1-p)}, \qquad \lambda(A+\lambda^r)^p = A + \lambda^r = A + \frac{(A+\lambda^r)^p}{(A+\lambda^r)^{p^{k+1}}}. \tag{2.6}$$

Hence

$$\lambda = \frac{A}{(A+\lambda^r)^p} + \frac{1}{(A+\lambda^r)^{p^{k+1}}}. \tag{2.7}$$

From above analysis, we conclude that λ and $A + \lambda^r$ are the unique equilibriums of (2.1) and (1.6), respectively. □

3. Quantitative Bounds of Solutions to (1.6)

In this section, through analyzing the boundedness of (1.6) we mainly present two explicit bounds for the positive solutions to (1.6).

Let the positive sequence $\{x_i\}_{i=-k}^{\infty}$ be a solution to (1.6), then for $n \geq -k$ we define

$$\theta_n = \frac{x_{n+1}}{x_n^p}. \tag{3.1}$$

It follows from (3.1) and (1.6) that

$$\theta_n = \frac{A}{x_n^p} + \frac{1}{x_{n-k}^{p^{k+1}}}, \quad n \in \mathbb{N}_0. \tag{3.2}$$

Combining (3.1) and (1.6), we can simply obtain that

$$x_n = A + \frac{x_{n-1}^p}{x_{n-k-1}^{p^{k+1}}} = A + \frac{x_{n-1}^p}{x_{n-2}^{p^2}} \frac{x_{n-2}^{p^2}}{x_{n-3}^{p^3}} \cdots \frac{x_{n-k}^{p^k}}{x_{n-k-1}^{p^{k+1}}} = A + \theta_{n-2}^p \theta_{n-3}^{p^2} \cdots \theta_{n-k-1}^{p^k}, \quad n \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), the identity

$$\theta_n = \frac{A}{\left(A + \theta_{n-2}^p \theta_{n-3}^{p^2} \cdots \theta_{n-k-1}^{p^k}\right)^p} + \frac{1}{\left(A + \theta_{n-k-2}^p \theta_{n-k-3}^{p^2} \cdots \theta_{n-2k-1}^{p^k}\right)^{p^{k+1}}} \tag{3.4}$$

holds for all $n \geq k+1$.

Note that $x_i > A$ for $i \geq 1$, and hence it follows from (3.2) that

$$0 < \theta_i < A^{1-p} + \frac{1}{A^{p^{k+1}}}, \quad i \geq k+1. \tag{3.5}$$

Let us define two sequences $\{S_i\}_{i=k+1}^{\infty}$ and $\{B_i\}_{i=k+1}^{\infty}$ recursively in the following way:

$$\begin{aligned} B_i &= \frac{A}{\left(A + S_{i-2}^p S_{i-3}^{p^2} \cdots S_{i-k-1}^{p^k}\right)^p} + \frac{1}{\left(A + S_{i-k-2}^p S_{i-k-3}^{p^2} \cdots S_{i-2k-1}^{p^k}\right)^{p^{k+1}}}, \\ S_i &= \frac{A}{\left(A + B_{i-2}^p B_{i-3}^{p^2} \cdots B_{i-k-1}^{p^k}\right)^p} + \frac{1}{\left(A + B_{i-k-2}^p B_{i-k-3}^{p^2} \cdots B_{i-2k-1}^{p^k}\right)^{p^{k+1}}} \end{aligned} \tag{3.6}$$

for all $i \geq 3k+3$, and the initial values satisfy

$$S_i = 0, \quad B_i = A^{1-p} + \frac{1}{A^{p^{k+1}}}, \quad k+1 \leq i \leq 3k+2. \tag{3.7}$$

Apparently $S_i \leq \theta_i \leq B_i$ for $i \geq k+1$, and the problem of bounding (1.6) reduces to consideration of the recursive dependent sequences $\{S_i\}, \{B_i\}$.

Lemma 3.1. *The sequences $\{S_i\}$ and $\{B_i\}$ are nondecreasing and nonincreasing, respectively.*

Proof. It follows from (3.7) that for $k+1 \leq i \leq 3k+1$ we have $S_{i+1} = S_i = 0$ and

$$B_{i+1} = B_i = \frac{1}{A^{p-1}} + \frac{1}{A^{p^{k+1}}}. \tag{3.8}$$

Hence assume that $S_{i+1} \geq S_i$ and $B_{i+1} \leq B_i$ for $k+1 \leq i < M$ $(M \geq 3k+2)$. By induction, we have that

$$\begin{aligned} B_{M+1} &= \frac{A}{\left(A + S_{M-1}^{p} S_{M-2}^{p^2} \cdots S_{M-k}^{p^k}\right)^{p}} + \frac{1}{\left(A + S_{M-k-1}^{p} S_{M-k-2}^{p^2} \cdots S_{M-2k}^{p^k}\right)^{p^{k+1}}} \\ &\leq \frac{A}{\left(A + S_{M-2}^{p} S_{M-3}^{p^2} \cdots S_{M-k-1}^{p^k}\right)^{p}} + \frac{1}{\left(A + S_{M-k-2}^{p} S_{M-k-3}^{p^2} \cdots S_{M-2k-1}^{p^k}\right)^{p^{k+1}}} \\ &= B_M. \end{aligned} \tag{3.9}$$

Through similar calculations, we have $S_{M+1} \geq S_M$, and by induction the lemma is proved. □

Theorem 3.2. *For* (2.1) *with* $u(0) = 0$, *let* $S_n^* = u(2[(n+k-1)/(4k+2)])$ *and* $B_n^* = u(2[(n+k-2)/(4k+2)]+1)$ *for* $n \geq k+2$. *Then the inequality*

$$S_n^* \leq \theta_n \leq B_n^* \tag{3.10}$$

holds for all $n \geq k+2$.

Proof. From (3.7) and the definitions of S_n^*, B_n^*, we have that $S_i = S_i^*$ and $B_i = B_i^*$ for $k+2 \leq i \leq 3k+2$. Thus, assume that $S_i \leq S_i^*$ and $B_i \leq B_i^*$ for $k+2 \leq i < M$ $(M \geq 3k+3)$. Then

$$\begin{aligned} S_M &= \frac{A}{\left(A + B_{M-2}^{p} B_{M-3}^{p^2} \cdots B_{M-k-1}^{p^k}\right)^{p}} + \frac{1}{\left(A + B_{M-k-2}^{p} B_{M-k-3}^{p^2} \cdots B_{M-2k-1}^{p^k}\right)^{p^{k+1}}} \\ &\geq \frac{A}{\left(A + B_{M-k-1}^{p+\cdots+p^k}\right)^{p}} + \frac{1}{\left(A + B_{M-2k-1}^{p+\cdots+p^k}\right)^{p^{k+1}}} \geq \frac{A}{\left(A + B_{M-2k-1}^{r}\right)^{p}} + \frac{1}{\left(A + B_{M-2k-1}^{r}\right)^{p^{k+1}}} \\ &\geq \frac{A}{\left(A + \left(B_{M-2k-1}^{*}\right)^{r}\right)^{p}} + \frac{1}{\left(A + \left(B_{M-2k-1}^{*}\right)^{r}\right)^{p^{k+1}}} \\ &= \frac{A}{\left(A + \left(u(2[(M-3k-3)/(4k+2)]+1)\right)^{r}\right)^{p}} \\ &\quad + \frac{1}{\left(A + \left(u(2[(M-3k-3)/(4k+2)]+1)\right)^{r}\right)^{p^{k+1}}} \\ &= u\left(2\left[\frac{M+k-1}{4k+2}\right]\right) = S_M^*. \end{aligned} \tag{3.11}$$

Similar calculations lead to $B_M \leq B_M^*$, and inductively the theorem can be proved. □

Theorem 3.3. *If the solution* $u(i)$ *to* (2.1) *with* $u(0) = 0$ *converges to the unique equilibrium* λ *under the conditions in Lemma 2.2 and there exist two sequences* $\{a(i)\}$ *and* $\{b(i)\}$ *which are lower and*

upper bounds for (2.1) *such that* $a(i) \le u(i) \le b(i), i \ge k+2$, *and* $\lim_{i\to+\infty} a(i) = \lim_{i\to+\infty} b(i) = \lambda$, *then the solutions to* (1.6) *have explicit bounds of the following form:*

$$A + \prod_{j=n-k-1}^{n-2} a\left(2\left[\frac{j+k-1}{4k+2}\right]\right)^{p^{n-j-1}} \le x_n \le A + \prod_{j=n-k-1}^{n-2} b\left(2\left[\frac{j-k-2}{4k+2}\right]+1\right)^{p^{n-j-1}} \tag{3.12}$$

for all $n \ge 3k+3$.

Proof. The proof follows directly from Lemma 2.2 and Theorem 3.2, and thus is omitted. □

Note that Theorem 3.3 and Lemma 2.3 imply the following corollary.

Corollary 3.4. *If the solution* $u(i)$ *to* (2.1) *with* $u(0) = 0$ *converges to the unique equilibrium* λ *under the conditions in Lemma 2.2, then the unique equilibrium* $A + \lambda^r$ *of* (1.6) *is a global attractor.*

By Theorem 3.3, it suffices to determine the explicit bounds for (2.1). In the following, a simple case would be taken. For example,

if the parameters A, p, k are fixed, then by (2.1) we get

$$\begin{aligned} u_{n+2} = {} & \frac{A}{\left(A + \left(A/(A+u(n)^r)^p + 1/(A+u(n)^r)^{p^{k+1}}\right)^r\right)^p} \\ & + \frac{1}{\left(A + \left(A/(A+u(n)^r)^p + 1/(A+u(n)^r)^{p^{k+1}}\right)^r\right)^{p^{k+1}}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.13}$$

Denote $u(i) = \delta(i) + \lambda$ for $i \ge 0$ (λ the unique equilibrium of (2.1)), then we have that, being for $n \ge 0$,

$$\begin{aligned} \delta(n+2) = {} & \frac{A}{\left(A + \left(A/(A+(\delta(n)+\lambda)^r)^p + 1/(A+(\delta(n)+\lambda)^r)^{p^{k+1}}\right)^r\right)^p} \\ & + \frac{1}{\left(A + \left(A/(A+(\delta(n)+\lambda)^r)^p + 1/(A+(\delta(n)+\lambda)^r)^{p^{k+1}}\right)^r\right)^{p^{k+1}}} - \lambda = \gamma(\delta(n)), \end{aligned} \tag{3.14}$$

where the function γ is defined by

$$\begin{aligned} \gamma(x) = {} & \frac{A}{\left(A + \left(A/(A+(x+\lambda)^r)^p + 1/(A+(x+\lambda)^r)^{p^{k+1}}\right)^r\right)^p} \\ & + \frac{1}{\left(A + \left(A/(A+(x+\lambda)^r)^p + 1/(A+(x+\lambda)^r)^{p^{k+1}}\right)^r\right)^{p^{k+1}}} - \lambda \end{aligned} \tag{3.15}$$

for $x > -\lambda$.

Example 3.5 ($(A = 3, p = 1, k = 2)$). Then $\gamma(x)$ reduces to the following form:

$$\gamma(x) = \frac{4}{3 + \left(4/(3 + (x+1)^2)\right)^2} - 1, \quad x > -1. \tag{3.16}$$

Obviously, γ is monotonically increasing for $x > -1$ and $\gamma(0) = 0$. By simplifying γ, we have

$$\gamma(x) = \frac{x\left(x^3 + 4x^2 + 12x + 16\right)}{3x^4 + 12x^3 + 36x^2 + 48x + 64}. \tag{3.17}$$

Having the function φ defined via $\varphi(x) = \gamma(x)/x$, we get the derivative of φ as follows:

$$\varphi'(x) = -\frac{3x^6 + 24x^5 + 120x^4 + 384x^3 + 624x^2 + 640x}{\left(3x^4 + 12x^3 + 36x^2 + 48x + 64\right)^2} < 0 \tag{3.18}$$

for all $x > 0$. Thus $\varphi(x) < \varphi(0) = 1/4$ for $x > 0$ and

$$\frac{\gamma(t^n)}{t^{n+2}} = \frac{\gamma(t^n)}{t^n}\frac{1}{t^2} < \frac{1}{4t^2}. \tag{3.19}$$

Therefore $\gamma(t^n) < t^{n+2}$ whenever $t \geq 1/2$. In addition, for $-1 \leq x < 0, \gamma(x) < 0$ and both $\xi(x) = x^3 + 4x^2 + 12x + 16$ and $\eta(x) = 3x^4 + 12x^3 + 36x^2 + 48x + 64$ are monotonically increasing. Hence we have

$$\left|\frac{\gamma(-t^n)}{-t^{n+2}}\right| < \left|\frac{\xi(0)}{\eta(-1)}\right|\frac{1}{t^2} = \frac{16}{43}\frac{1}{t^2}, \tag{3.20}$$

and $\gamma(-t^n) > -t^{n+2}$ whenever $t^2 \geq 16/43$.

Now set $\pi^+(n) = (1/2)^n$, for $n \geq 0$, and $\pi^-(n) = (16/43)^{n/2}$. Note that $\delta(0) = -1 = -\pi^-(0)$ and $\delta(1) = 1/3 < \pi^+(1)$. Thus suppose that $-1 \leq -\pi^-(2i) \leq \delta(2i) \leq 0$ and $0 \leq \delta(2i+1) \leq \pi^+(2i+1)$, for $0 \leq i \leq N$ $(N \geq 0)$. Then by induction we have

$$\begin{aligned} \delta(2N+2) &= \gamma(\delta(2N)) \geq \gamma(-\pi^-(2N)) = \gamma\left(-\left(\frac{16}{43}\right)^N\right) \geq -\pi^-(2N+2), \\ \delta(2N+3) &= \gamma(\delta(2N+1)) \leq \gamma(\pi^+(2N+1)) = \gamma\left(\left(\frac{1}{2}\right)^{2N+1}\right) \geq \pi^+(2N+3). \end{aligned} \tag{3.21}$$

Therefore since the fact that $\delta(i) \leq 0$ for i even and $\delta(i) \geq 0$ for i odd, we obtain that, for $i \geq 0$,

$$-\left(\frac{16}{43}\right)^{i/2} \leq \delta(i) \leq \left(\frac{1}{2}\right)^i, \qquad 1 - \left(\frac{16}{43}\right)^{i/2} \leq u(i) \leq 1 + \left(\frac{1}{2}\right)^i. \tag{3.22}$$

Employing Theorem 3.3, we get the bounds

$$3+\prod_{i=n-3}^{n-2}\left(1-\left(\frac{16}{43}\right)^{[(i+1)/10]}\right)\le x_n\le 3+\prod_{i=n-3}^{n-2}\left(1+\left(\frac{1}{2}\right)^{2[(i-4)/10]+1}\right),\quad n\ge 9. \tag{3.23}$$

4. Quantitative Bounds for Solutions to (1.8)

In this section, the upper and lower bounds of solutions to (1.8) are given, and first we present a lemma concerning the equilibrium points of (1.8).

Lemma 4.1. *If $A>1$, then* (1.8) *has a unique positive equilibrium $\overline{y}=A$; and if $0<A\le 1$, then* (1.8) *has a unique positive equilibrium $\overline{y}=1$.*

The proof is simple and thus omitted.

Suppose that $\{y_i\}_{i=-k}^{p}$ is a positive solution to (1.8), and by the transformation

$$\beta_n=\frac{y_{n+1}}{y_n^p},\quad n\ge -k, \tag{4.1}$$

we have that

$$\beta_n=\max\left\{\frac{A}{y_n^p},\frac{1}{y_{n-k}^{p^{k+1}}}\right\},\quad n\in\mathbb{N}_0. \tag{4.2}$$

It follows from (4.1) and (1.8) that

$$y_n=\max\left\{A,\frac{y_{n-1}^p}{y_{n-k-1}^{p^{k+1}}}\right\}=\max\left\{A,\beta_{n-2}^p\beta_{n-3}^{p^2}\cdots\beta_{n-k-1}^{p^k}\right\},\quad n\in\mathbb{N}. \tag{4.3}$$

Employing (4.2) and (4.3), we obtain that, for $n\ge -k$,

$$\beta_n=\max\left\{\frac{A}{\max\left\{A,\beta_{n-2}^p\beta_{n-3}^{p^2}\cdots\beta_{n-k-1}^{p^k}\right\}^p},\frac{1}{\max\left\{A,\beta_{n-k-2}^p\beta_{n-k-3}^{p^2}\cdots\beta_{n-2k-1}^{p^k}\right\}^{p^{k+1}}}\right\}. \tag{4.4}$$

For two nonnegative sequences $\{L_i\}$ and $\{H_i\}$, let $L_i\le\beta_i\le H_i$ for $k+1\le i<T$ $(T\ge 3k+2)$.

Then according to (4.4) and the sequences

$$L_T = \max\left\{\frac{A}{\max\left\{A, H_{T-2}^{p} H_{T-3}^{p^2} \cdots H_{T-k-1}^{p^k}\right\}^{p}}, \frac{1}{\max\left\{A, H_{T-k-2}^{p} H_{T-k-3}^{p^2} \cdots H_{T-2k-1}^{p^k}\right\}^{p^{k+1}}}\right\},$$

$$H_T = \max\left\{\frac{A}{\max\left\{A, L_{T-2}^{p} H_{T-3}^{p^2} \cdots L_{T-k-1}^{p^k}\right\}^{p}}, \frac{1}{\max\left\{A, L_{T-k-2}^{p} L_{T-k-3}^{p^2} \cdots L_{T-2k-1}^{p^k}\right\}^{p^{k+1}}}\right\}, \tag{4.5}$$

we have that $L_T \le \beta_T \le H_T$.

Note that $y_i \ge A$ for $i \ge 1$, and thus from (4.2) we get

$$0 < \beta_i \le \max\left\{\frac{1}{A^{p-1}}, \frac{1}{A^{p^{k+1}}}\right\} = \begin{cases} \dfrac{1}{A^{p^{k+1}}}, & A < 1, \\ \dfrac{1}{A^{p-1}}, & A \ge 1 \end{cases} \tag{4.6}$$

for $k+1 \le i \le 3k+2$.

Lemma 4.2. *Let $L_i = 0$ and $H_i = \max\{1/A^{p-1}, 1/A^{p^{k+1}}\}$ for $k+1 \le i \le 3k+2$, then the sequences $\{L_i\}$ and $\{H_i\}$ are nondecreasing and nonincreasing, respectively.*

Proof. Assume that $L_i \le L_{i+1}$ and $H_{i+1} \le H_i$ for $k+1 \le i < T$ $(T \ge 3k+2)$. Then we have that

$$\begin{aligned} H_{T+1} &= \max\left\{\frac{A}{\max\left\{A, L_{T-1}^{p} L_{T-2}^{p^2} \cdots L_{T-k}^{p^k}\right\}^{p}}, \frac{1}{\max\left\{A, L_{T-k-1}^{p} L_{T-k-2}^{p^2} \cdots L_{T-2k}^{p^k}\right\}^{p^{k+1}}}\right\} \\ &\le \max\left\{\frac{A}{\max\left\{A, L_{T-2}^{p} L_{T-3}^{p^2} \cdots L_{T-k-1}^{p^k}\right\}^{p}}, \frac{1}{\max\left\{A, L_{T-k-2}^{p} L_{T-k-3}^{p^2} \cdots L_{T-2k-1}^{p^k}\right\}^{p^{k+1}}}\right\} \\ &= H_T. \end{aligned} \tag{4.7}$$

Analogous argument gives that $L_T \le L_{T+1}$, and the lemma can be proved inductively. □

Now we will take in the other first-order recursive equation

$$v(n+1) = \max\left\{ \frac{A}{\max\left\{A, (v(n))^r\right\}^p}, \frac{1}{\max\left\{A, (v(n))^r\right\}^{p^{k+1}}} \right\}, \quad n \in \mathbb{N}_0, \tag{4.8}$$

where A, p equal those of (1.8), $r = p + p^2 + \cdots + p^k$, and the initial value $v(0) = 0$.

Through Lemma 4.2, simpler bounds for $\{\beta_i\}$ are given below.

Theorem 4.3. *The inequality*

$$L_n^* \le \beta_n \le H_n^* \tag{4.9}$$

holds for $n \ge k+2$, *where* $L_n^* = v(2[(n+k-1)/(4k+2)])$ *and* $H_n^* = v(2[(n+k-2)/(4k+2)]+1)$.

Proof. Let $L_i = 0$ and $H_i = \max\{1/A^{p-1}, 1/A^{p^{k+1}}\}$ for $k+2 \le i \le 3k+2$, and by the definitions of $\{L_n^*\}$ and $\{H_n^*\}$ we have that $L_i^* = L_i$ and $H_i^* = H_i$ for $k+2 \le i \le 3k+2$.

Assume that $L_i^* \le L_i$ and $H_i^* \ge H_i$ for $k+2 \le \mathrm{i} < T$ $(T \ge 3k+3)$, then

$$\begin{aligned}
H_T &= \max\left\{ \frac{A}{\max\left\{A, L_{T-2}^p L_{T-3}^{p^2} \cdots L_{T-k-1}^{p^k}\right\}^p}, \frac{1}{\max\left\{A, L_{T-k-2}^p L_{T-k-3}^{p^2} \cdots L_{T-2k-1}^{p^k}\right\}^{p^{k+1}}} \right\} \\
&\le \max\left\{ \frac{A}{\max\left\{A, L_{T-k-1}^{p+\cdots+p^k}\right\}^p}, \frac{1}{\max\left\{A, L_{T-2k-1}^{p+\cdots+p^k}\right\}^{p^{k+1}}} \right\} \\
&\le \max\left\{ \frac{A}{\max\left\{A, L_{T-2k-1}^r\right\}^p}, \frac{1}{\max\left\{A, L_{T-2k-1}^r\right\}^{p^{k+1}}} \right\} \\
&\le \max\left\{ \frac{A}{\max\left\{A, (L_{T-2k-1}^*)^r\right\}^p}, \frac{1}{\max\left\{A, (L_{T-2k-1}^*)^r\right\}^{p^{k+1}}} \right\} \\
&= \max\left\{ \frac{A}{\max\left\{A, (v(2[(T-k-2)/(4k+2)]))^r\right\}^p}, \right. \\
&\qquad \left. \frac{1}{\max\left\{A, (v(2[(T-k-2)/(4k+2)]))^r\right\}^{p^{k+1}}} \right\} \\
&= v\left(2\left[\frac{T-k-2}{4k+2}\right]+1\right) = H_T^*.
\end{aligned} \tag{4.10}$$

Similar computations lead to the inequality $L_T^* \le L_T$, and the theorem follows by induction. □

Theorem 4.3 and (4.8) imply the following result.

Theorem 4.4. *Suppose that there exist two positive sequences* $\{c(j)\}$ *and* $\{d(j)\}$ *which are bounds for a positive solution to* (4.8) *with* $v(0) = 0$ *such that*

$$c(j) \le v(j) \le d(j), \quad j \ge k+2. \tag{4.11}$$

Then the solutions to (1.8) *have explicit upper and lower bounds of the following form:*

$$\max\left\{A, \prod_{i=n-k-1}^{n-2} c\left(2\left[\frac{i+k-1}{4k+2}\right]\right)^{p^{n-i-1}}\right\} \le y_n \le \max\left\{A, \prod_{i=n-k-1}^{n-2} d\left(2\left[\frac{i-k-2}{4k+2}\right]+1\right)^{p^{n-i-1}}\right\} \tag{4.12}$$

for all $n \ge 3k+3$.

5. Conclusion

In this paper, we investigate a particular case of a higher-order difference equation by Stević which is a natural extension of that one in [8], and mainly present improved results which give computable approaches for quantitative bounds of solutions to (1.6). However, the methods are only effective for small parameters, because complex polynomials will arise in the process of computing for large parameters A, p, k.

On the basis of Corollary 3.4 and Theorem 4.4, we suggest to study the behaviors, particularly the convergence and stability, of positive solutions to the following two recursive equations:

$$\begin{aligned} u(n+1) &= \frac{A}{\left(A+u(n)^r\right)^p} + \frac{1}{\left(A+u(n)^r\right)^{p^{k+1}}}, \quad n \in \mathbb{N}_0, \\ v(n+1) &= \max\left\{\frac{A}{\max\left\{A,(v(n))^r\right\}^p}, \frac{1}{\max\left\{A,(v(n))^r\right\}^{p^{k+1}}}\right\}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{5.1}$$

where $A, p \in (0,+\infty)$, $k \in \mathbb{N}$, and $r = \Sigma_{i=1}^{k} p^i$.

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