Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 263041, 29 pages
doi:10.1155/2010/263041

Research Article

Numerical Exploration of Kaldorian Interregional Macrodynamics: Enhanced Stability and Predominance of Period Doubling under Flexible Exchange Rates

Toichiro Asada,[1] Christos Douskos,[2] Vassilis Kalantonis,[2] and Panagiotis Markellos[2]

[1] *Faculty of Economics, Chuo University, 742-1 Higashinakano, Hachioji, Tokyo 192-0393, Japan*
[2] *Faculty of Engineering, University of Patras, Patras 26500, Greece*

Correspondence should be addressed to Panagiotis Markellos, p.markellos@des.upatras.gr

Received 27 October 2009; Accepted 27 April 2010

Academic Editor: Carl Chiarella

We present a discrete two-regional Kaldorian macrodynamic model with flexible exchange rates and explore numerically the stability of equilibrium and the possibility of generation of business cycles. We use a grid search method in two-dimensional parameter subspaces, and coefficient criteria for the flip and Hopf bifurcation curves, to determine the stability region and its boundary curves in several parameter ranges. The model is characterized by enhanced stability of equilibrium, while its predominant asymptotic behavior when equilibrium is unstable is period doubling. Cycles are scarce and short-lived in parameter space, occurring at large values of the degree of capital movement β. By contrast to the corresponding fixed exchange rates system, for cycles to occur sufficient amount of trade is required *together* with high levels of capital movement. Rapid changes in exchange rate expectations and decreased government expenditure are factors contributing to the creation of interregional cycles. Examples of bifurcation and Lyapunov exponent diagrams illustrating period doubling or cycles, and their development into chaotic attractors, are given. The paper illustrates the feasibility and effectiveness of the numerical approach for dynamical systems of moderately high dimensionality and several parameters.

1. Introduction

Aspects of international macroeconomics and regional economics are studied recently by methods of nonlinear economic dynamics (see, e.g., [1–3]). In particular, the Kaldorian business cycle theory (originated by [4]) has been developed by Lorenz [5], Gandolfo [6], and others. Recent developments in business cycle dynamics can be found in the work of Puu and Sushko [7], while a general discussion of analytical and numerical methods in the study of nonlinear dynamical systems in economics is in the work of Lorenz [8]. Interregional Kaldorian macrodynamic models of business cycles, based on trade interaction between the regions, have been studied by Lorenz [9] and Puu [2].

In this paper we study the economic interdependency between two regions. We consider a five-dimensional nonlinear discrete time model of the economic transactions between the regions with flexible exchange rates. The present work is a sequel to our previous study of the corresponding model of two-regional Macrodynamics with fixed exchange rates [10]. The economic structures of both regions, assumed similar, are characterized by the Kaldorian business cycle model, and the two regions interact economically through trade and capital movement. These two factors of economic interaction are expressed by separate terms of the model equations and quantified by means of the basic interaction parameters, δ for trade and β for capital movement. The present model is an extended two-regional version of the Kaldorian small open economy model with flexible exchange rates, which was expressed as a three-dimensional system of nonlinear difference equations and considered in Asada et al. [11]. The corresponding three-dimensional small open economy model with fixed exchange rates was considered in Asada et al. [12].

The Kaldorian model of nonlinear business cycles provides a surprisingly simple and fundamental mechanism of complex macroeconomic behavior due to the dynamic interaction of income and capital by using very standard textbook-like macroeconomic equations, and because of its simplicity has huge potential to extend in various ways by introducing more realistic factors. Our paper is one such attempt to extend the model, which is not yet investigated thoroughly by other authors. It is appropriate to quote here the following description by Gandolfo: "It is an impressive tribute to its author, and a demonstration of the importance of nonlinearity, that Kaldor's business cycle model still yields stimuli to research" [6, pages 441-442].

Here we explore our two-regional macrodynamic model with flexible exchange rates focusing on the stability of equilibrium under variations of the model parameters and on the asymptotic behavior of the system outside the stability region, and consider in particular the possibility of occurrence of business cycles. For a five-dimensional model with several parameters, this is a formidable task. However, by means of a numerical grid search method and analytical coefficient criteria, we determine the stability region in several two-dimensional sections of the parameter space and identify the flip bifurcation curve and the Hopf-Neimark bifurcation curve as parts of the boundary of this region.

Indeed, the main aspect of the work in terms of methodology is that by contrast to many contributions in the field of economic dynamics, we do not employ simplifying assumptions to reduce the dynamical system to fewer dimensions so as to enable the derivation of analytical results, but adopt an almost entirely numerical approach which can be a valuable alternative means of exploration for a nonlinear dynamical system of moderately high dimensionality and several parameters. It is our aim to illustrate the feasibility and effectiveness of the numerical approach for such systems. Our study of the present model completes a series of numerical studies of extended, 3- and 5-dimensional, Kaldorian Macrodynamic models.

Certain conclusions are drawn on the effects of the model parameters. These regard mainly the size of the region of stability of equilibrium in parameter space, the possibility of occurrence of business cycles at reasonably small values of the interaction parameters, and the type of predominant asymptotic dynamical behavior of the system outside the stability region. The model exhibits complex dynamics, and our results are presented mostly in the form of stability and bifurcation diagrams.

The paper is organized as follows. In Sections 2 and 3 we present the structure of the model and derive the fundamental dynamical system of equations. In Section 4 we present the functional forms and specifications adopted for our numerical study. In Sections 5 and 6 we determine the position of equilibrium, discuss its stability and the boundaries of the stability region in parameter space, identify the Hopf and flip bifurcation curves, in a variety of parameter ranges, and discuss the effects of variation of the basic model parameters. In Section 7 we consider the asymptotic dynamical behavior of the system outside the stability region. In Section 8 we focus on the effect of the speed of adaptation of the expected exchange rate, and in Section 9 we explore briefly the effect of a state parameter depending on government spending. Section 10 summarizes our findings and conclusions.

2. Structure of the Model

The following set of (2.1)–(2.9) is common to the two-regional Kaldorian macrodynamic model with flexible exchange rates in this paper and that with fixed exchange rates in Asada et al. [10].

The Kaldorian quantity adjustment process in the goods market is expressed as

$$Y_i(t+1) - Y_i(t) = \alpha_i[C_i(t) + I_i(t) + G_i + J_i(t) - Y_i(t)], \quad \alpha_i > 0, \tag{2.1}$$

where it is assumed that the real output of each region fluctuates according to whether the regional excess demand in the goods market is positive or negative. We can consider this process as a dynamic Keynesian multiplier process under fixed prices.

The capital accumulation equation becomes

$$K_i(t+1) - K_i(t) = I_i(t), \tag{2.2}$$

which implies that the net investment expenditure contributes to the changes of the capital stock.

Consumption function is given by

$$C_i(t) = c_i[Y_i(t) - T_i(t)] + C_{0i}, \quad 0 < c_i < 1, \; C_{0i} > 0, \tag{2.3}$$

which is a very standard textbook-like Keynesian consumption function that describes that current regional consumption is an increasing function of current regional income.

We formulate the investment function as

$$I_i(t) = I_i(Y_i(t), K_i(t), r_i(t)), \qquad \frac{\partial I_i}{\partial Y_i} > 0, \; \frac{\partial I_i}{\partial K_i} < 0, \; \frac{\partial I_i}{\partial r_i} < 0, \tag{2.4}$$

which means that the real regional investment expenditure is an increasing function of real regional income and a decreasing function of real regional capital stock and of the nominal regional rate of interest. This is a standard Keynesian/Kaldorian investment function.

We introduce the following quite conventional and simple tax function:

$$T_i(t) = \tau_i Y_i(t) - T_{0i}, \quad 0 < \tau_i < 1, \; T_{0i} > 0, \tag{2.5}$$

which means that the regional tax is an increasing function of regional income.

The equilibrium condition of the money market is given by

$$\frac{M_i(t)}{p_i} = L_i(Y_i(t), r_i(t)), \quad \frac{\partial L_i}{\partial Y_i} > 0, \ \frac{\partial L_i}{\partial r_i} < 0, \tag{2.6}$$

which is a standard textbook-like formulation of the "LM equation".

The current account function of region 1 is written as

$$J_1(t) = \delta H_1(Y_1(t), Y_2(t), E(t)), \quad \frac{\partial H_1}{\partial Y_1} < 0, \ \frac{\partial H_1}{\partial Y_2} > 0, \ \frac{\partial H_1}{\partial E} > 0, \ 0 \le \delta \le 1, \tag{2.7}$$

which means that the current account (net export) of a region depends on the incomes of the two regions and the exchange rate of the currencies of the two regions.

The capital account function of region 1 is given by

$$Q_1(t) = \beta\left[r_1(t) - r_2(t) - \frac{E^e(t) - E(t)}{E(t)}\right], \quad \beta > 0, \tag{2.8}$$

which describes the dynamic of the interregional capital movement. This means that the capital moves between regions depending on the interest rates differential between the regions and the expected rate of change of the exchange rate.

The following equation is simply the definition of the total balance of payments of region 1:

$$A_1(t) = J_1(t) + Q_1(t). \tag{2.9}$$

Here t denotes the time period and the subscript i $(= 1, 2)$ is the index number of a region. The meanings of the symbols are as follows: Y_i: real regional income, C_i: real private consumption expenditure, I_i: real net private investment expenditure on physical capital, G_i: real government expenditure (fixed), K_i: real physical capital stock, T_i: real income tax, r_i: nominal rate of interest, M_i: nominal money supply, p_i: price level (fixed), E: exchange rate (1 unit of currency of region 2 = E units of currency of region 1), E^e: expected exchange rate of near future, J_i: balance of current account (net export) in real terms ($Ep_2J_2 = -p_1J_1$), Q_i: balance of capital account in real terms ($Ep_2Q_2 = -p_1Q_1$), $A_i = J_i + Q_i$: total balance of payments in real terms ($Ep_2A_2 = -p_1A_1$), α_i: adjustment speed in the goods market, β: degree of capital mobility, and δ: degree of interregional trade.

The following set of (2.10)–(2.12) is peculiar to the model with flexible exchange rates in this paper:

$$A_1(t) = 0, \tag{2.10}$$

$$E^e(t+1) - E^e(t) = \gamma[E(t) - E^e(t)], \quad \gamma > 0, \tag{2.11}$$

$$M_i(t) = \overline{M}_i = \text{const.} \tag{2.12}$$

Furthermore, we fix price levels as follows:

$$p_1 = p_2 = 1. \tag{2.13}$$

Equation (2.10) means that the exchange rate $E(t)$ is determined endogenously to keep the equilibrium of the total balance of payments instantaneously.

Equation (2.11) formalizes the adaptive expectation hypothesis of the changes of the expected exchange rate $E^e(t)$.

It is worth noting that the nominal money supply of each region can be controlled by the monetary authority of each region independent of interregional trade and interregional capital movement in our model with flexible exchange rates.

Equation (2.12) means that nominal money supply of each region is fixed by the regional monetary authority.

In our model, price levels, except the exchange rate, are supposed to be fixed for simplicity. Equation (2.13) is the normalization procedure to simplify the notation.

3. Derivation of the Fundamental Dynamical System of Equations

Next, we derive the fundamental dynamical system of equations in this paper, which is a five-dimensional system of nonlinear difference equations. Substituting (2.12) and (2.13) into (2.6) and solving with respect to $r_i(t)$, we have the following "LM equation" (dependence of the rate of interest on income) in each region:

$$r_i(t) = r_i(Y_i(t)), \qquad \frac{\partial r_i}{\partial Y_i} = -\frac{\partial L_i/\partial Y_i}{\partial L_i/\partial r_i} > 0. \tag{3.1}$$

Substituting now (2.7), (2.8), and (2.9) into (2.10), we have

$$\delta H_1(Y_1(t), Y_2(t), E(t)) + \beta\left[r_1(Y_1(t)) - r_2(Y_2(t)) - \frac{E^e(t)}{E(t)} + 1\right] = 0. \tag{3.2}$$

Solving this equation with respect to $E(t)$, we obtain an expression of the exchange rate $E(t)$ as an endogenous variable:

$$E(t) = E\big(Y_1(t), Y_2(t), E^e(t); \beta, \delta\big), \tag{3.3}$$

and differentiating (3.2) with respect to Y_1, Y_2, and E^e we obtain

$$\begin{aligned} \frac{\partial E}{\partial Y_1} &= \frac{-(\partial H_1/\partial Y_1) - (\partial r_1/\partial Y_1)(\beta/\delta)}{(\partial H_1/\partial E) + (E^e/E^2)(\beta/\delta)}, \\ \frac{\partial E}{\partial Y_2} &= \frac{-(\partial H_1/\partial Y_2) + (\partial r_2/\partial Y_2)(\beta/\delta)}{(\partial H_1/\partial E) + (E^e/E^2)(\beta/\delta)}, \\ \frac{\partial E}{\partial E^e} &= \frac{(\beta/\delta)}{(\partial H_1/\partial E)E + (E^e/E)(\beta/\delta)} > 0. \end{aligned} \tag{3.4}$$

We note that due to (2.7), we have

$$\frac{\partial E}{\partial Y_1} > 0, \qquad \frac{\partial E}{\partial Y_2} < 0, \tag{3.5}$$

for sufficiently small values of β/δ, and

$$\frac{\partial E}{\partial Y_1} < 0, \qquad \frac{\partial E}{\partial Y_2} > 0, \tag{3.6}$$

for sufficiently large values of β/δ.

Substituting (2.2)–(2.5), (2.7), (3.1), and (3.3) into (2.1), (2.2), and (2.11), we obtain the following nonlinear five-dimensional system of difference equations, which is the fundamental system of dynamical equations in this paper:

$$\begin{aligned}
Y_1(t+1) &= Y_1(t) + \alpha_1 [c_1(1-\tau_1)Y_1(t) + c_1T_{01} + C_{01} + G_1 + I_1(Y_1(t), K_1(t), r_1(Y_1(t))) \\
&\qquad + \delta H_1(Y_1(t), Y_2(t), E(Y_1(t), Y_2(t), E^e(t); \beta, \delta)) - Y_1(t)] \\
&= F_1(Y_1(t), K_1(t), Y_2(t), E^e(t); \alpha_1, \beta, \delta), \\
K_1(t+1) &= K_1(t) + I_1(Y_1(t), K_1(t), r_1(Y_1(t))) = F_2(Y_1(t), K_1(t)), \\
Y_2(t+1) &= Y_2(t) + \alpha_2 \Bigg[c_2(1-\tau_2)Y_2(t) + c_2T_{02} + C_{02} + G_2 + I_2(Y_2(t), K_2(t), r_2(Y_2(t))) \\
&\qquad - \frac{\delta H_1(Y_1(t), Y_2(t), E(Y_1(t), Y_2(t), E^e(t); \beta, \delta))}{E(Y_1(t), Y_2(t), E^e(t); \beta, \delta)} - Y_2(t) \Bigg] \\
&= F_3(Y_1(t), Y_2(t), K_2(t), E^e(t); \alpha_2, \beta, \delta), \\
K_2(t+1) &= K_2(t) + I_2(Y_2(t), K_2(t), r_2(Y_2(t))) = F_4(Y_2(t), K_2(t)), \\
E^e(t+1) &= E^e(t) + \gamma [E(Y_1(t), Y_2(t), E^e(t); \beta, \delta) - E^e] = F_5(Y_1(t), Y_2(t), E^e(t); \beta, \delta).
\end{aligned} \tag{3.7}$$

4. Functional Forms and Specifications

For our numerical exploration, the fundamental system of dynamical equations is employed in the form of the following equations which are merely numerical specifications of the general model formalized in Sections 2 and 3. In particular, (3.7) becomes, respectively,

$$\begin{gathered}
Y_1(t+1) - Y_1(t) = \alpha_1\{\phi_1(t) + \delta H_1(t) + [c_1(1-\tau_1) - 1]Y_1(t) + Z_1\}, \\
K_1(t+1) - K_1(t) = \phi_1(t), \\
Y_2(t+1) - Y_2(t) = \alpha_2\left\{\phi_2(t) - \frac{\delta H_1(t)}{E(t)} + [c_2(1-\tau_2) - 1]Y_2(t) + Z_2\right\}, \\
K_2(t+1) - K_2(t) = \phi_2(t), \\
E^e(t+1) - E^e(t) = \gamma[E(t) - E^e(t)],
\end{gathered} \tag{4.1}$$

where

$$\begin{gathered}
\phi_i(t) = f(Y_i(t)) - \frac{3K_i(t)}{10} - r_i(t), \quad i = 1, 2, \\
r_i(t) = -M_i + 10Y_i(t)^{1/4}, \quad i = 1, 2, \\
H_1(t) = 100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}.
\end{gathered} \tag{4.2}$$

We also adopt the numerical specifications:

$$\begin{gathered}
M_1 = M_2 = 300, \qquad c_1 = c_2 = 0.8, \qquad \tau_1 = \tau_2 = 0.2, \\
Z_1 = c_1T_{01} + C_{01} + G_1 = Z_2 = c_2T_{02} + C_{02} + G_2 = 75,
\end{gathered} \tag{4.3}$$

corresponding to similarly structured regional economies. The function f is a particular case of the Kaldorian sigmoid direct dependence of the investment function on income (see, e.g., [13]), given by

$$f(x) = \frac{80}{\pi}\arctan\left[\frac{9\pi}{80}(x - 250)\right] + 35. \tag{4.4}$$

These functional forms and specifications are taken in the present case of flexible exchange rates to be essentially the same as in Asada et al. [10] for the case of fixed exchange rates, so as to facilitate comparison of that case with the present one. With the above forms and specifications, the right-hand sides of the system are

$$f_1 = \alpha_1\left\{300 + Z + f(Y_1(t)) - \frac{3K_1(t)}{10} - 10Y_1(t)^{1/4} - \frac{36}{100}Y_1(t) + \delta\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right]\right\}, \tag{4.5}$$

$$f_2 = 300 + f(Y_1(t)) - \frac{3K_1(t)}{10} - 10Y_1(t)^{1/4}, \tag{4.6}$$

$$f_3 = \alpha_2\left\{300 + Z + f(Y_2(t)) - \frac{3K_2(t)}{10} - 10Y_2(t)^{1/4} - \frac{36}{100}Y_2(t) - \frac{\delta}{E(t)}\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right]\right\}, \tag{4.7}$$

$$f_4 = 300 + f(Y_2(t)) - \frac{3K_2(t)}{10} - 10Y_2(t)^{1/4}, \tag{4.8}$$

$$f_5 = \gamma[E(t) - E^e(t)]. \tag{4.9}$$

The system is completed by (3.2) which takes the following form:

$$\delta\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right] + \beta\left[1 - \frac{E^e(t)}{E(t)} + 10Y_1(t)^{1/4} - 10Y_2(t)^{1/4}\right] = 0. \tag{4.10}$$

The expression (3.3) of the exchange rate is now

$$E(t) = \frac{10\left[100\delta + \beta E^e(t)\right]}{10\beta\left[1 + 10Y_1(t)^{1/4} - 10Y_2(t)^{1/4}\right] + \delta[1000 - 3Y_1(t) + 3Y_2(t)]}, \tag{4.11}$$

and the final recurrence system is obtained by substituting this into (4.5) and (4.7). For simplicity, in our numerical exploration we shall further assume equal speeds of adjustment of the goods markets in the two regions ($\alpha_1 = \alpha_2 = \alpha$), thus reducing the space of essential parameters of the model, from four-dimensional $(\alpha_1, \alpha_2, \beta, \delta)$ to three-dimensional (α, β, δ). However, in the following we shall also discuss variations of the quantities γ and Z, considered here as secondary parameters.

5. Position and Stability of Equilibrium

To find the equilibrium values of the system, denoted below by asterisks, we first observe that (4.9) implies $E^{e*} = E^*$, and substituting this into (4.11) we obtain

$$E^{e*} = E^* = -\frac{1000\delta}{100\beta\left[-(Y_1^*)^{1/4} + (Y_2^*)^{1/4}\right] + \delta\left(-1000 + 3Y_1^* - 3Y_2^*\right)}. \tag{5.1}$$

It is then found from (4.5)–(4.8) and (5.1) that the equilibrium values of our flexible exchange rates model under the above specifications are

$$Y_1^* = Y_2^* = \frac{25Z}{9},$$
$$K_1^* = K_2^* = \frac{10}{3}\left[300 + f\left(\frac{25Z}{9}\right) - 10\left(\frac{25Z}{9}\right)^{1/4}\right], \tag{5.2}$$
$$E^{e*} = E^* = 1.$$

In particular, for $Z = 75$ we obtain the following equilibrium values:

$$Y_1^* = Y_2^* = \frac{625}{3} \cong 208.333, \qquad K_1^* = K_2^* \cong 862.449. \tag{5.3}$$

Stability of the equilibrium is determined by the roots of the characteristic polynomial of the Jacobian of the mapping, that is, the following matrix:

$$J^* = I + \left(\frac{\partial f_i}{\partial x_j}\right), \quad i, j = 1, \ldots, 5, \qquad (x_1, x_2, x_3, x_4, x_5) = (Y_1, K_1, Y_2, K_2, E^e), \tag{5.4}$$

where I is the 5×5 unit matrix, and the superscript $(*)$ denotes evaluation at the equilibrium. The characteristic equation is a quintic:

$$P_5(\lambda) = \lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{5.5}$$

and for stability, all its roots, real or complex, must be inside the unit circle in the complex plane.

Our basic tool for the numerical determination of the region of stability is a two-dimensional grid-search technique. We compute the characteristic polynomial (5.5) and its roots at the node points of a dense grid covering a region of interest in a two-dimensional section of the space of parameters, and store for graphical representation the points at which the equilibrium is stable.

The technique is first employed to determine the stability diagrams of Figure 1 showing the stability region in the (β, δ) parameter plane for fixed $\gamma = 1.2$ and for different values of the common speed of adjustment of the goods markets α. The part of the stability region in which the roots of the characteristic equation are all real is shown dark-shaded, while the part in which some of the roots are complex conjugate is shown light-shaded. In these diagrams the flip bifurcation condition

$$g_1(\alpha, \beta, \gamma, \delta) = P_5(-1) = a_0 + a_2 + a_4 - (1 + a_1 + a_3) = 0 \tag{5.6}$$

is drawn in as a bold dashed curve. The Hopf bifurcation curve is also drawn in, as a continuous curve. We have found that in all stability diagrams of this paper the Hopf bifurcation curve can be determined by requiring that two of the roots of the characteristic polynomial have unit product (see [10]), and is given by the following relation of the coefficients of the characteristic polynomial:

$$g_2(\alpha,\beta,\gamma,\delta) = a_0^4 + a_0^2(a_1-2) - a_2^2 - a_0a_2\left(a_0^2+3a_1-2a_3-1\right) + (1+a_1-a_3)\left[(a_1-1)^2 + a_0^2a_3\right]$$
$$+ a_4\left\{\left(1+2a_0^2+a_1\right)a_2 - a_0\left[a_0^2+a_1^2+3a_3-a_1(2+a_3)-1\right]\right\}$$
$$- a_4^2\left(a_0^2+a_1+a_0a_2\right) + a_0a_4^3 = 0. \quad (5.7)$$

Each one of relations (5.6) and (5.7) is the implicit equation of a surface in the four-dimensional space of the parameters $(\alpha,\beta,\gamma,\delta)$, the contours of which for fixed γ and for various levels of α provide in the (β,δ) plane one or more curves. Segments of the curves arising from (5.6) form the part of the boundary of the stability region that is a flip bifurcation curve, and similarly segments of the curves arising from (5.7) form the part of the boundary of the stability region that is a Hopf bifurcation curve. Segments of the above curves that are not parts of the boundary of the stability region do not correspond to loss of stability and can be ignored. These two relations can therefore be employed, in combination with the grid search technique, as coefficient criteria for flip bifurcations and Hopf bifurcations in the present case of our five-dimensional discrete system (see [14], for a rigorous coefficient criterion in the case of a four-dimensional continuous system).

The above tools are similarly employed to determine the stability region in the (β,α) plane for $\gamma = 1.2$ and different values of the level of trade transactions δ, and some of the resulting stability region diagrams are shown in Figure 2.

6. Geometrical Aspects and Implications

Let us now consider the geometrical aspects of the boundary curves of the stability region. We begin by noting that in our model, (5.6) is equivalent to a quadratic with respect to α:

$$g_1(\alpha,\beta,\gamma,\delta) = (1-\alpha_{00}\alpha)\left\{10(1-\alpha_{00}\alpha)\left[\beta-50(\gamma-2)\delta\right] - \alpha\beta\delta\left[3-10\times 3^{3/4}(\gamma-2)\right]\right\} = 0, \quad (6.1)$$

with roots

$$\alpha = {\alpha_{00}}^{-1}, \quad (6.2)$$

$$\alpha = \left\{\alpha_{00} + \beta\delta\frac{3-10\times 3^{3/4}(\gamma-2)}{10\left[\beta-50\delta(\gamma-2)\right]}\right\}^{-1}, \quad (6.3)$$

where we have abbreviated

$$\alpha_{00} = \frac{9}{50} + \frac{3^{3/4}}{85} - \frac{23040}{4352+95625\,\pi^2} \cong 0.1825. \quad (6.4)$$

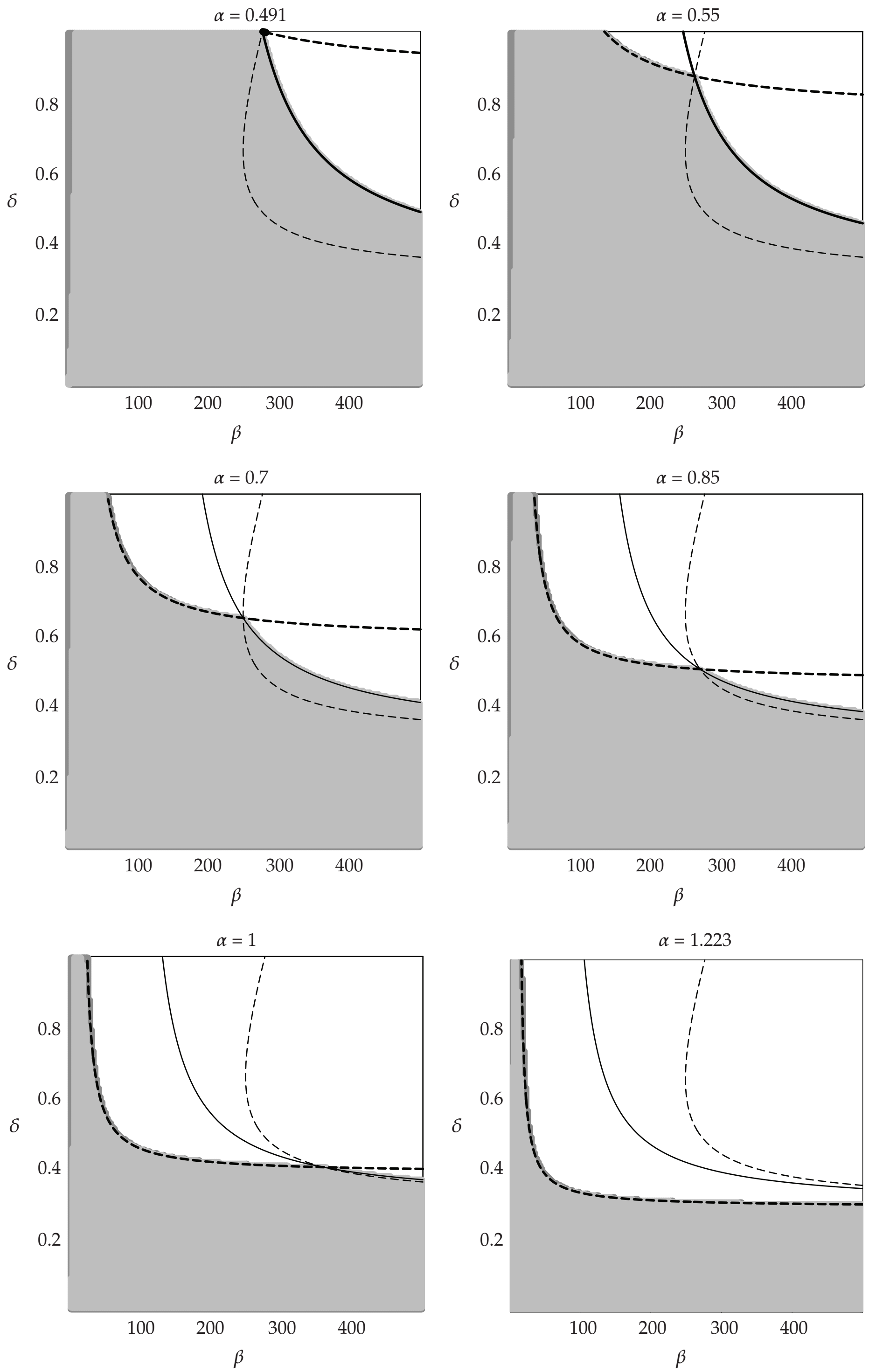

Figure 1: The region of stability of equilibrium in the (β, δ) plane for $\gamma = 1.2$ and sample values of α.

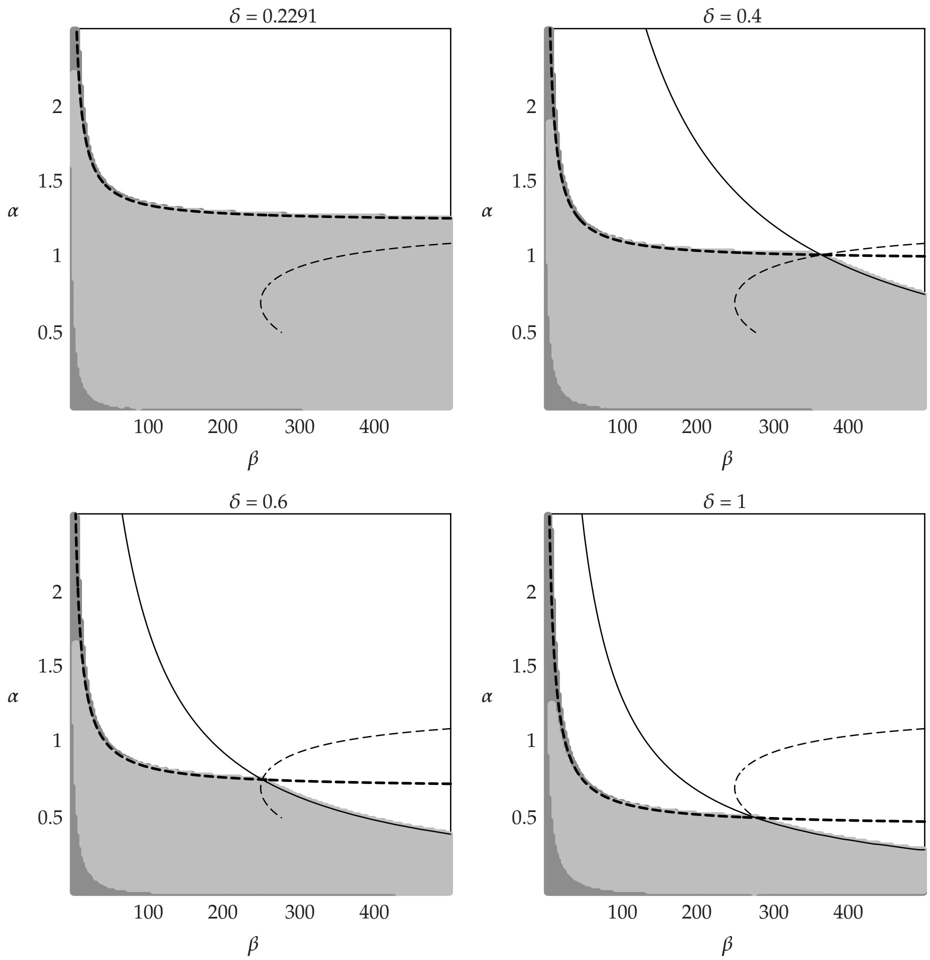

Figure 2: The region of stability of equilibrium in the (β, α) plane for $\gamma = 1.2$ and sample values of δ.

The constant root (6.2) corresponds to the leftmost point of the flip bifurcation curve in the (β, α) plane, while (6.3) represents the flip bifurcation condition (5.6) in the form $\alpha = \alpha(\beta, \gamma, \delta)$. Substituting this expression of α into (5.7), we obtain an equation

$$g_2(\alpha(\beta, \gamma, \delta), \beta, \gamma, \delta) = 0, \tag{6.5}$$

which, for a fixed value of γ, is satisfied by the values of β and δ representing the locus of the points of intersection of the flip bifurcation curve with the Hopf bifurcation curve in the (β, δ) plane. In Figure 1 this locus has been drawn in as a thin dashed curve.

Considering the existence of the locus for large values of β, we find that for $\beta \to \infty$ the expression in the left-hand side of (6.5) tends to a limit function $P_{\gamma,\delta}$, plots of which for three sample values of γ are shown in Figure 3. Its roots with respect to δ (denoted by δ_∞)

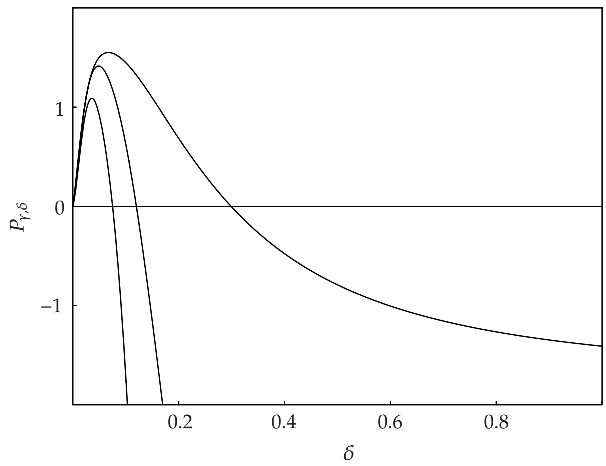

Figure 3: The limit function $P_{\gamma,\delta}$ for $\gamma = 1.2, 1.4, 1.6$ (from right to left).

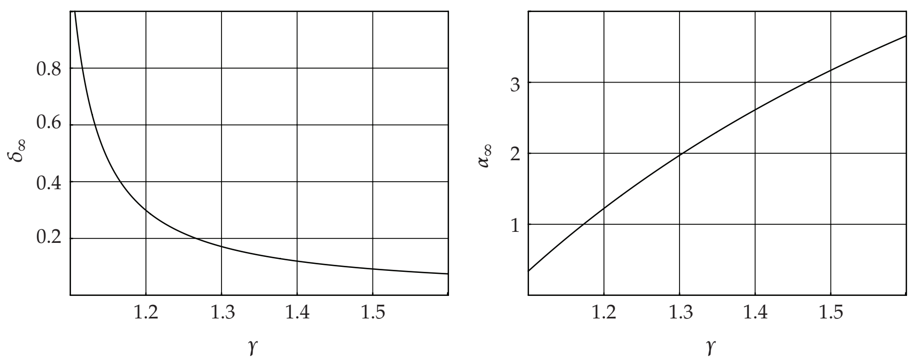

Figure 4: The limit functions $\delta_\infty(\gamma)$ and $\alpha_\infty(\gamma)$.

for these values of γ are given below together with the following corresponding values of α (denoted by α_∞ and obtained from (6.3) for $\delta = \delta_\infty$ and $\beta \to \infty$) :

$$\begin{aligned} \gamma = 1.2: &\quad \delta_\infty \cong 0.2991, \quad \alpha_\infty \cong 1.2231, \\ \gamma = 1.4: &\quad \delta_\infty \cong 0.1201, \quad \alpha_\infty \cong 2.6124, \\ \gamma = 1.6: &\quad \delta_\infty \cong 0.0751, \quad \alpha_\infty \cong 3.6556. \end{aligned} \tag{6.6}$$

The quantities δ_∞ and α_∞ can be expressed as functions of γ in the following simple forms, obtained by means of a numerical approximation technique using $P_{\gamma,\delta}$ and (6.3):

$$\delta_\infty(\gamma) \cong \frac{36}{987\gamma - 956}, \qquad \alpha_\infty(\gamma) \cong \frac{10(623\gamma - 664)}{3(189\gamma + 1)}, \tag{6.7}$$

with an accuracy of at least two decimal digits in the interval $1.2 < \gamma < 1.6$. These functions are shown as curves in Figure 4.

The flip bifurcation curve and the Hopf bifurcation curve approach each other asymptotically in the (β, δ) plane for $\beta \to \infty$ when $\delta = \delta_\infty$. It follows from Figure 1 that they do not intersect when $\delta < \delta_\infty$, but they do so at a finite value of β when $\delta > \delta_\infty$. Therefore, the locus of their points of intersection does not exist when $\delta < \delta_\infty$.

Similarly, in the (β, α) plane the flip bifurcation curve and the Hopf bifurcation curve approach each other asymptotically for $\beta \to \infty$ when $\alpha = \alpha_\infty$. It follows from Figure 2 that they do not intersect when $\alpha > \alpha_\infty$, but they do so at a finite value of β when $\alpha < \alpha_\infty$. Therefore, the locus of their points of intersection does not exist when $\alpha > \alpha_\infty$.

Of importance here is also the fact that for $\gamma = 1.2$ the locus of intersections of the flip bifurcation curve and the Hopf bifurcation curve attains its minimum with respect to the parameter β of capital movement at a considerably large value of β. Specifically, this minimum occurs at

$$\beta = \beta_{\min} \cong 249.13 \quad (\delta \cong 0.662, \ \alpha \cong 0.688). \tag{6.8}$$

To find the locus of intersections in the (β, α) plane, instead of in the (β, δ) plane, we can solve (6.3) for δ and substitute the resulting expression

$$\delta_1(\alpha, \beta, \gamma) = \frac{10(1 - \alpha_{00}\alpha)\beta}{500(1 - \alpha_{00}\alpha)(\gamma - 2) - \alpha\beta\left[10 \times 3^{3/4}(\gamma - 2) - 3\right]} \tag{6.9}$$

into (5.7). We then obtain the following equation which, for a fixed value of γ, is satisfied by the values of β and α representing the points of the locus curve in the (β, α) plane:

$$g_2(\alpha, \beta, \gamma, \delta_1(\alpha, \beta, \gamma)) = 0. \tag{6.10}$$

In Figure 2 the locus has been drawn in as in Figure 1 (thin dashed curve). However, since the maximum value of δ allowed in the model is 1, we can substitute $\delta = 1$ into (6.3) to find the following relation representing, for a fixed value of γ, the model restriction $\delta \leq 1$ on the flip bifurcation curve in the (β, α) plane:

$$\alpha \geq \left\{\alpha_{00} + \beta\frac{3 - 10 \times 3^{3/4}(\gamma - 2)}{10\left[\beta - 50(\gamma - 2)\right]}\right\}^{-1}. \tag{6.11}$$

For the value $\gamma = 1.2$, we thus obtain

$$\alpha \geq \left\{\alpha_{00} + \beta\frac{3 + 8 \times 3^{3/4}}{10(\beta + 40)}\right\}^{-1} \cong 0.433628 + \frac{15.9724}{\beta + 3.16579}. \tag{6.12}$$

This expression of α, with the equality sign, represents the final location of the flip bifurcation curve (corresponding to $\delta = 1$). It follows that the actual locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve is only the part of the curve (6.10) on which (6.12) is satisfied. In the diagrams of Figure 2, only that part of the curve is shown. We can now substitute α by its lowest possible value, as given by the right-hand side of (6.11), into (6.10) to obtain an equation:

$$g_2^*(\beta, \gamma) = 0, \tag{6.13}$$

giving (for a fixed value of γ) the value of β at the final point of the actual locus. For $\gamma = 1.2$ we obtain $\beta \cong 276.65$, and the value of α at this point is then found from (6.12) to be $\alpha \cong 0.491$. Note in Figure 1 that this is the value of α for the intersection of the flip bifurcation curve with the Hopf bifurcation curve to be at the top end point of the locus in the (β, δ) plane.

Let us now discuss the implications of the geometrical aspects of the boundary curves of the stability region, including consequences on the possibility of occurrence of interregional business cycles.

The minimum with respect to the parameter β of the locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve in Figures 1 and 2 means that no segment of the curve (5.7) can be part of the boundary of the stability region for $\beta < 249.13$, equivalently that no Hopf bifurcation curve exists, and no business cycles can occur, for lower values of β. For lower values of β, it is the flip bifurcation curve that forms the boundary of the stability region, and period doubling can be expected to be the only mode of asymptotic dynamical behavior of the system when exiting the stability region in parameter space.

Further, it follows from the above considerations, regarding the final point (corresponding to $\delta = 1$) of the said locus, that in a global sense (i.e., for all values of δ) period doubling does not occur when exiting the stability region at points of the (β, α) plane whose coordinates satisfy the inequalities $\beta > 276.65$ and $\alpha < 0.491$. In other words, only cycles may occur on exit at such points.

It follows from the above discussion that the inequalities:

$$\delta > \delta_\infty, \qquad \beta > \beta_{\min}, \qquad \alpha < \alpha_\infty \tag{6.14}$$

represent the threshold for the occurrence of cycles in our model. For $\delta > \delta_\infty$, $\alpha < \alpha_\infty$, and large but finite values of capital movement β, the locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve exists, and cycles occur when exiting the stability region in parameter space through the segment of the curve (5.7) which forms part of the boundary of the stability region for such large values of β.

The important difference between our present flexible exchange rates system and the corresponding fixed exchange rates system studied in Asada et al. [10] is that in our present system for cycles to occur sufficient amount of trade is required *together* with high levels of capital movement.

In Figure 5 we show the occurring cycles for $\delta = 0.6$, $\beta = 300$, and α as the bifurcation parameter varying between 0.627 and 0.629. The cycles are shown in their (Y_1, Y_2) and exchange rate (ER) versus Y_1 projections. The (Y_1, Y_2) projection in particular shows counter synchronization of regional incomes when interregional cycles appear. Note, also, that when the income of region 1 is sufficiently higher than the income of region 2, then the exchange rate is less than 1, that is, the currency of region 1 is "stronger" (1 unit of currency of region 2 is <1 unit of currency of region 1). However, the occurring cycles are small in amplitude and short-lived as α varies.

We conclude this section with the following basic conclusions concerning the enhanced stability of equilibrium characterizing our model of flexible exchange rates. From

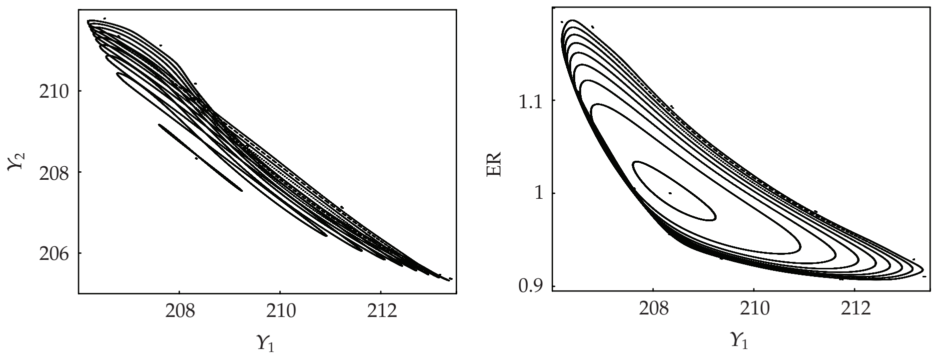

Figure 5: Interregional cycles occurring for $\delta = 0.6$ and $\beta = 300$ and α from 0.627 to 0.629, shown in their (Y_1, Y_2) and (Y_1, ER) projections.

the stability region diagrams, it can be seen that high levels of capital movement β do not induce instability of the system for low levels of the speed of adjustment of the goods markets α (i.e., for prudent reactions by firms); but high levels of α induce instability even at relatively low levels of β (Figure 2). For prudent reactions by firms ($\alpha < 1$) equilibrium remains stable at high levels of capital movement even for high levels of trade transactions δ (Figure 1).

7. Period Doubling and Second Generation Cycles

We now consider the asymptotic dynamical behavior of the system outside the stability region. For this we employ numerical simulations of the model mapping (4.5)–(4.9) to compute bifurcation and Lyapunov exponent diagrams with β as the bifurcation parameter. We choose parameter cases in which as β increases stability is lost by going through a flip bifurcation, such being the characteristic cases in our model (except for very large values of β). Our results for $\delta = 0.6$ (see the bottom left-hand diagram of Figure 2 for the relevant stability region), and sample values of α, are shown in Figure 6. As expected, period doubling occurs when stability is lost (at: $\beta \cong 7.96$ for $\alpha = 2$, and $\beta \cong 14.71$ for $\alpha = 1.5$).

However, in the present case the period doubling process does not develop directly into chaotic behavior. Instead, it first develops into an intermediate phase of "second generation", period-2, cycles as indicated by the characteristic flatness of the Lyapunov exponent diagram for $\alpha = 2$ and β approximately between 9.25 and 9.55. A similar situation, for a narrower interval of β, is seen to occur for $\alpha = 1.5$. Details of the bifurcation diagrams, showing clearly the occurrence of second generation cycles for $\alpha = 1.5$, are shown in Figure 7. The actual second generation cycles occurring for $\delta = 0.6$ and $\alpha = 2$ are shown in Figure 8, and their further development into chaotic attractors is shown in Figure 9 in some two-dimensional "projections".

8. Effect of the Speed of Adaptation of the Expected Exchange Rate

With regard to the effect of the speed of adaptation of the expected exchange rate γ, incorporated in our model in accordance with the adaptive expectation hypothesis of the changes of the expected exchange rate, it follows from the values (6.6) and Figures 3 and 4,

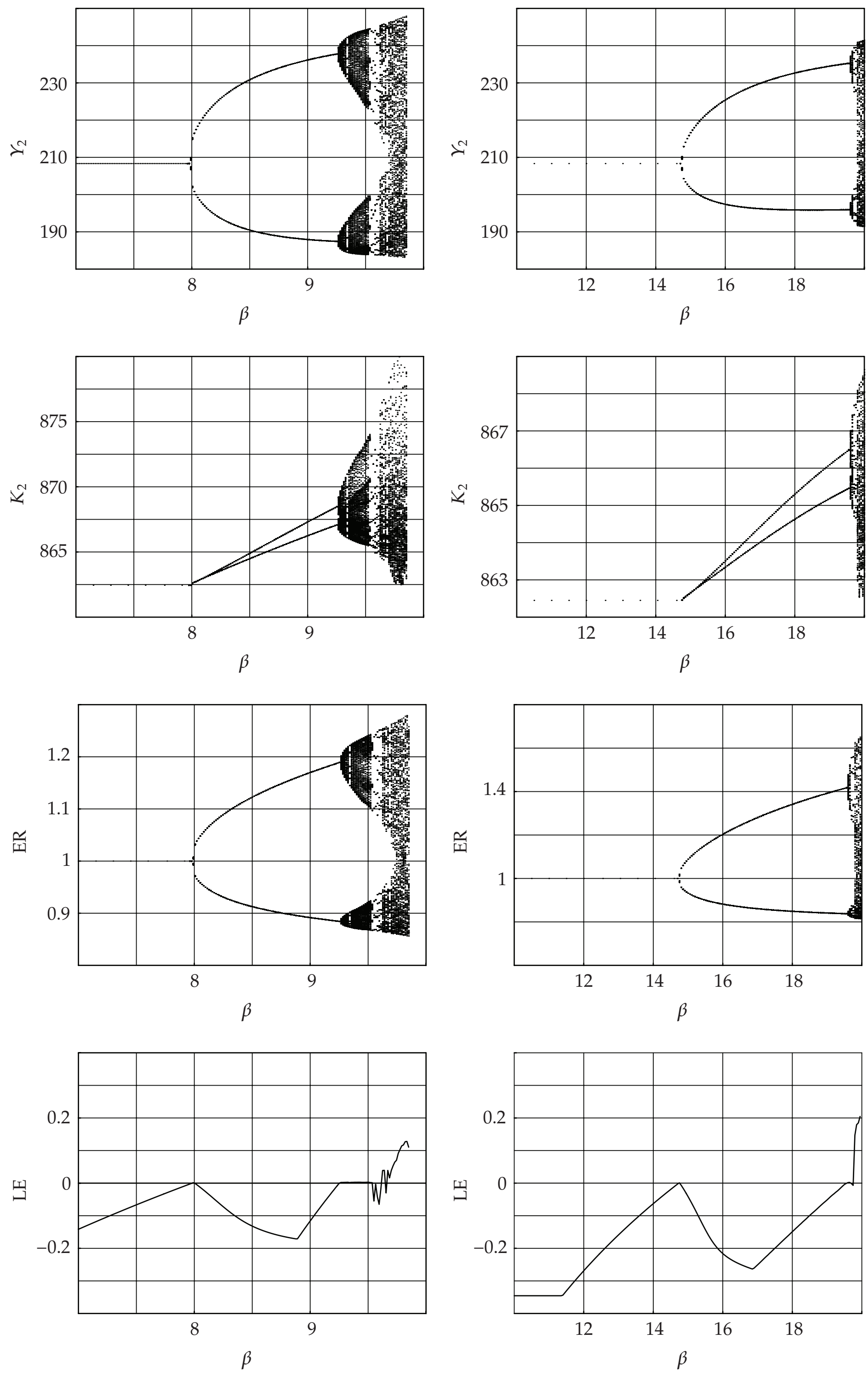

Figure 6: Bifurcation and Lyapunov exponent diagrams for $\delta = 0.6$ at: $\alpha = 2$ (left), $\alpha = 1.5$ (right).

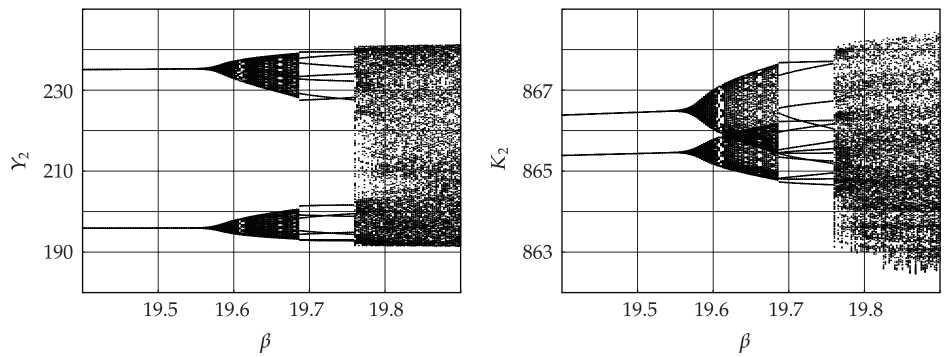

Figure 7: Details of bifurcation diagrams for $\alpha = 1.5$.

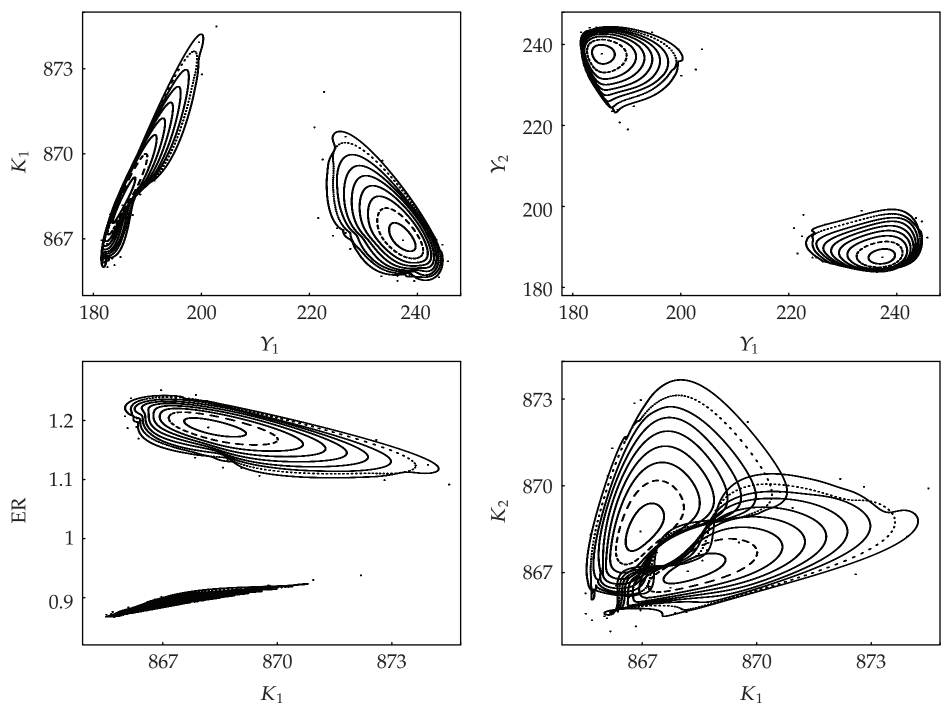

Figure 8: Period-2 cycles for $\delta = 0.6$, $\alpha = 2$.

that increase of γ relaxes the threshold restrictions (6.14) for the occurrence of cycles in so far as the parameters δ and α are concerned, since higher γ gives lower δ_∞ and higher α_∞.

To illustrate this conclusion, we present in Figure 10 the stability regions in the (β, α) plane for $\gamma = 1.4$ and $\gamma = 1.6$. Compared to the case $\gamma = 1.2$, we see that for $\gamma = 1.4$ the locus of intersections of the flip bifurcation curve and the Hopf bifurcation curve begins to exist (for $\beta \to \infty$) at lower values of δ and higher values of α, and thus cycles may occur for wider intervals of parameters. This is more pronounced for $\gamma = 1.6$.

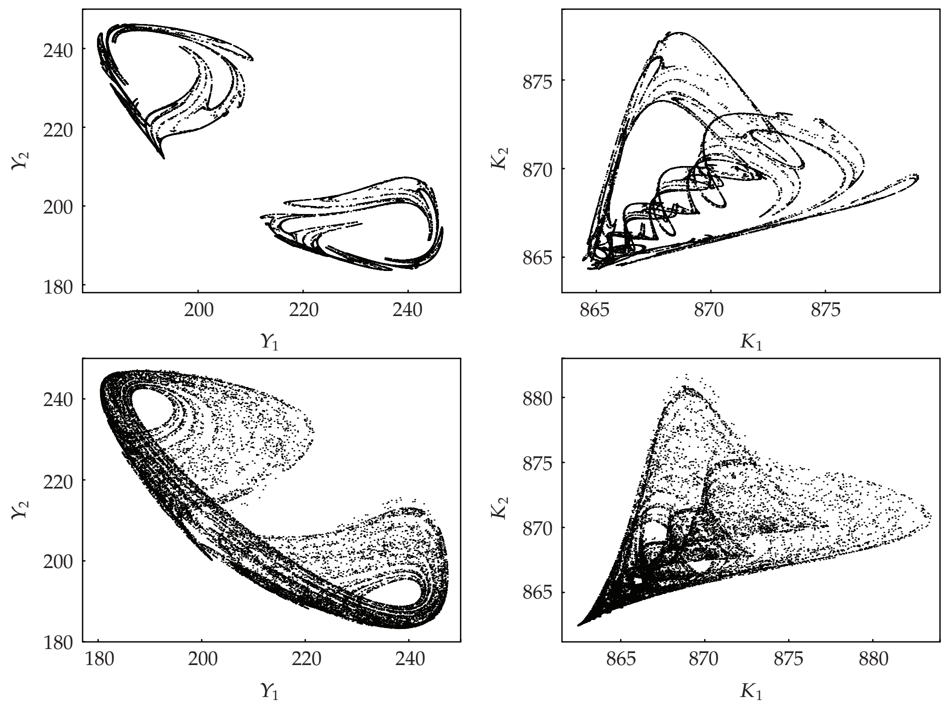

Figure 9: Development of the cycles of Figure 8 into chaotic attractors at: $\beta = 9.68$ (top) and $\beta = 9.76$ (bottom: "mask-like" and "heart-like" attractors).

Also, compared to the case $\gamma = 1.2$ the locus now attains its minimum w.r.t. β at considerably lower values of β:

$$\begin{aligned} \gamma = 1.4: &\quad \beta_{\min} \cong 36.804 \quad (\alpha \cong 1.715,\ \delta \cong 0.299), \\ \gamma = 1.6: &\quad \beta_{\min} \cong 11.962 \quad (\alpha \cong 2.743,\ \delta \cong 0.201). \end{aligned} \tag{8.1}$$

Thus, cycles now begin to occur at considerably lower levels of capital movement, that is, the threshold restriction for the occurrence of cycles is also relaxed with regard to the allowed values of β.

With regard to the final point of the locus, we can apply relations (6.11) and (6.13) as before. For the values of the speed of adaptation of the expected exchange rate employed here, (6.11) takes the following simple forms:

$$\gamma = 1.4: \quad \alpha \geq \left\{ \alpha_{00} + \beta \frac{3 + 6 \times 3^{3/4}}{10(\beta + 30)} \right\}^{-1} \cong 0.540476 + \frac{14.6148}{\beta + 2.95939}, \tag{8.2}$$

$$\gamma = 1.6: \quad \alpha \geq \left\{ \alpha_{00} + \beta \frac{3 + 4 \times 3^{3/4}}{10(\beta + 20)} \right\}^{-1} \cong 0.717195 + \frac{12.4663}{\beta + 2.61801}. \tag{8.3}$$

For $\gamma = 1.4$ the value of β at the final point of the actual locus is found from (6.13) to be $\beta \cong 63.117$, and the value of α at this point is found from (8.2) to be $\alpha \cong 0.762$. It follows that in a global sense (i.e., for all values of δ), period doubling does not occur when exiting the stability region at points of the (β, α) plane whose coordinates satisfy the following inequalities:

$$\gamma = 1.4: \quad \beta > 63.117, \quad \alpha < 0.762. \tag{8.4}$$

Only cycles may occur on exit at such points. The corresponding inequalities for $\gamma = 1.6$ are

$$\gamma = 1.6: \quad \beta > 25.918, \quad \alpha < 1.154. \tag{8.5}$$

In Figure 11 we illustrate the asymptotic behavior of the system outside the stability region by means of bifurcation and Lyapunov exponent diagrams for $\gamma = 1.6$ and $\delta = 0.4$. When exiting the stability region through the flip bifurcation curve, we get period doubling and second generation cycles, but when exiting through the Hopf bifurcation curve we get cycles for relatively small values of β. The actual cycles occurring in this case are similar to the cycles shown in Figure 5.

To explore further the effect of the speed γ of adaptation of the exchange rate, we employ the same techniques as in Section 5 to compute the region of stability of equilibrium and its boundary curves in the (β, γ) plane. This is done for sample values of the speed of adjustment α of the goods markets and sample values of the level δ of trade transactions between the economic regions. Some of the stability regions obtained are shown in Figure 12.

We see in these diagrams that for high values of α (>1) the stability region is restricted to relatively low values of β and to values of γ mainly below $\gamma = 2$. Also, as α increases the segment of the continuous curve which forms part of the boundary of the stability region, that is, the part which is truly a Hopf bifurcation curve, gets smaller in length; thus cycles are possible for fewer parameter constellations. The possibility of cycles becomes extinct eventually, shortly after a change of topology of the flip bifurcation curve (dashed) as described below.

The flip bifurcation curve is a hyperbola in the (β, γ) plane, which is represented by the expression found by solving (6.3) for γ, and has the following form:

$$\gamma = \frac{\psi_0(\alpha, \delta) + \psi_1(\alpha, \delta)\ \beta}{\zeta_0(\alpha, \delta) + \zeta_1(\alpha, \delta)\ \beta}, \tag{8.6}$$

with

$$\begin{gathered} \psi_0 = 1000(1 - \alpha_{00}\alpha)\delta, \qquad \psi_1 = 10(1 - \alpha_{00}\alpha) - \left(3 + 20 \times 3^{3/4}\right)\alpha\delta, \\ \zeta_0 = 500(1 - \alpha_{00}\alpha)\delta, \qquad \zeta_1 = -10 \times 3^{3/4}\alpha\delta. \end{gathered} \tag{8.7}$$

At the critical value, the determinant $\psi_0\zeta_1 - \psi_1\zeta_0$ of the coefficients is zero for the branches of the hyperbola to touch. On the other hand, Expression (8.6) is seen from (8.7) to take the value:

$$\gamma = \frac{\psi_0}{\zeta_0} = 2, \tag{8.8}$$

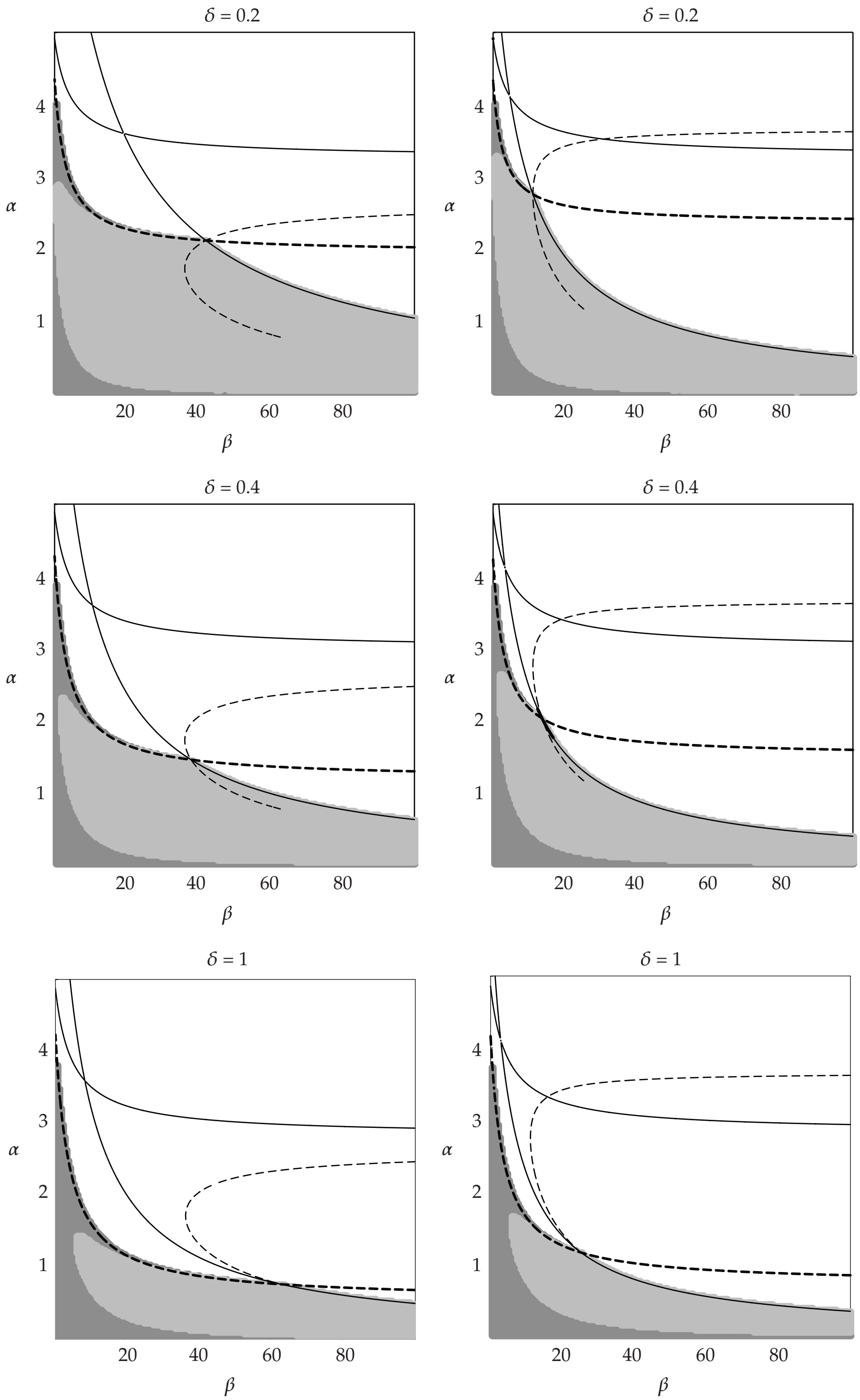

Figure 10: Regions of stability for $\gamma = 1.4$ (left) and $\gamma = 1.6$ (right).

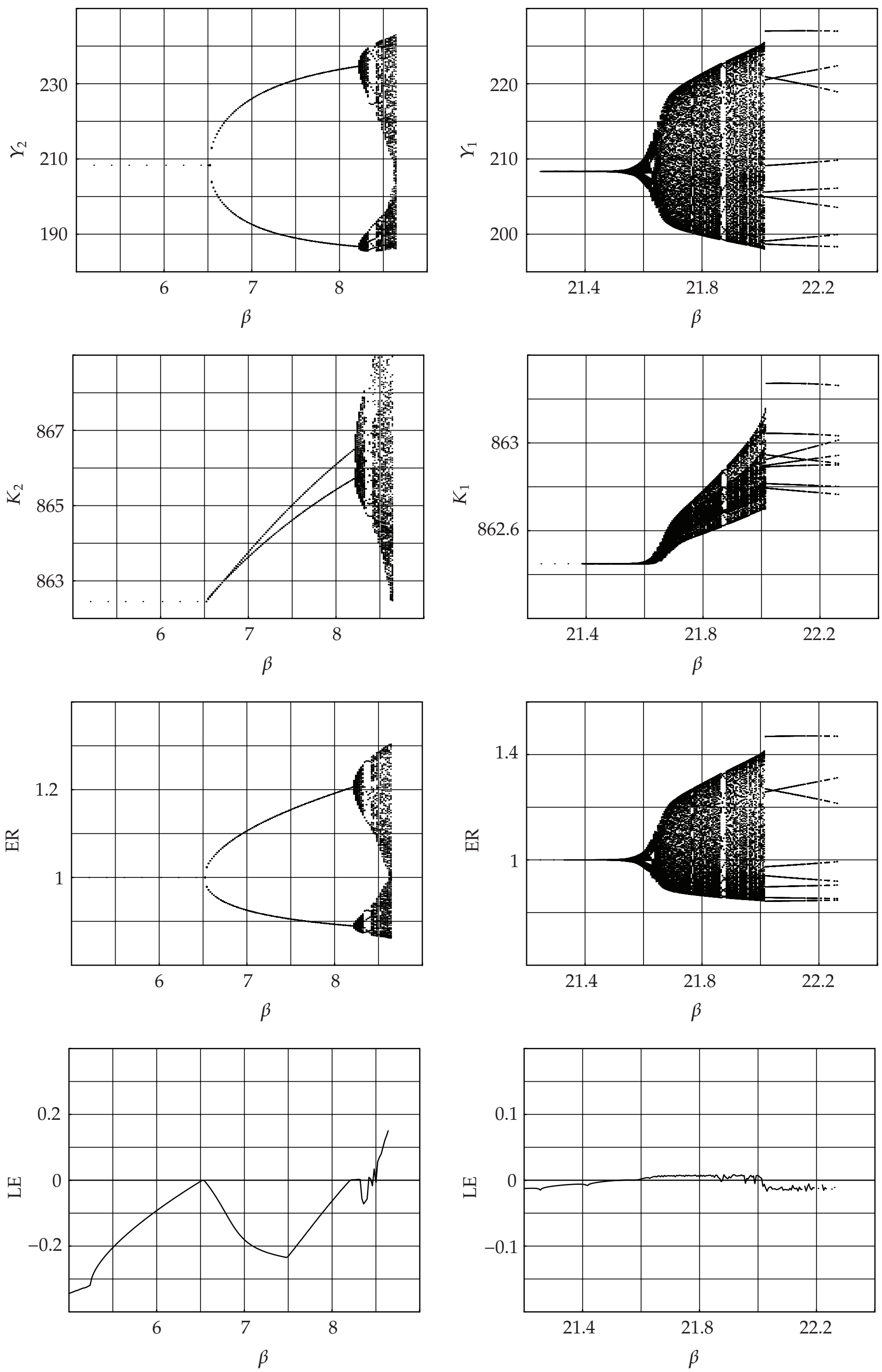

Figure 11: Bifurcation diagrams for $\gamma = 1.6$, $\delta = 0.4$, and $\alpha = 2.5$ (left), $\alpha = 1.5$ (right).

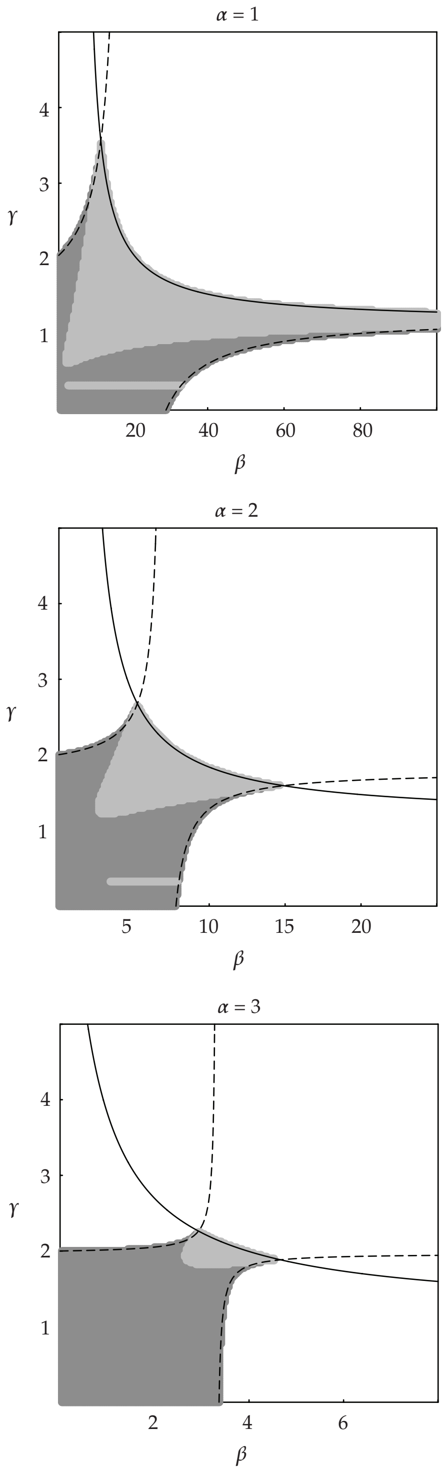

Figure 12: Regions of stability for $\delta = 0.4$ (left), and $\delta = 0.8$ (right). Note the different scales used on the β-axis.

for $\beta = 0$; thus the flip bifurcation hyperbola in Figure 12 always goes through the point $(0, 2)$. Therefore, when the hyperbola branches touch, we have

$$\frac{\psi_0 + \psi_1\beta}{\zeta_0 + \zeta_1\beta} = \frac{\psi_1}{\zeta_1} + \frac{\psi_0\zeta_1 - \psi_1\zeta_0}{\zeta_1(\zeta_0 + \zeta_1\beta)} = \frac{\psi_1}{\zeta_1} = 2, \tag{8.9}$$

and the last equation gives

$$\alpha = \alpha_{\text{crit.}}(\delta) = \left(\alpha_{00} + \frac{3\delta}{10}\right)^{-1}, \tag{8.10}$$

where α_{00} is given by (6.4). Note that (8.10) can also be directly obtained from (6.3) for $\gamma = 2$. Also, the root of the denominator of the expression (8.6) gives for $\alpha = \alpha_{\text{crit.}}$ the value of β when the hyperbola branches meet

$$\beta = \beta_{\text{crit.}}(\delta) = 5 \times 3^{1/4}\delta. \tag{8.11}$$

From (8.10) and (8.11), and for the values of δ used in Figure 12, we thus obtain

$$\begin{aligned} \delta = 0.4&: \quad \alpha_{\text{crit.}} \cong 3.306, \quad \beta_{\text{crit.}} \cong 2.632, \\ \delta = 0.8&: \quad \alpha_{\text{crit.}} \cong 2.367, \quad \beta_{\text{crit.}} \cong 5.264. \end{aligned} \tag{8.12}$$

For a small interval of $\alpha > \alpha_{\text{crit.}}$ the Hopf bifurcation curve continues to intersect the upper branch of the flip bifurcation curve. But for α being sufficiently larger than $\alpha_{\text{crit.}}$ (specifically $\alpha > 3.361$ for $\delta = 0.4$ and $\alpha > 2.406$ for $\delta = 0.8$), there is no longer any intersection of the two curves. For such high values of α, the entire boundary of the stability region is formed by the flip bifurcation curve, and Hopf bifurcations are impossible for any value of γ.

9. Effect of the State Parameter Z

We now explore briefly the effect of varying the value of the state parameter Z. To this end we employ the same techniques as in the previous sections. We note that this parameter involves the quantities G_i, $i = 1, 2$, of real public expenditure in the two economic regions; therefore, the effects of varying Z may also be of interest with regard to questions of economic policy. The stability regions in the (β, α) plane for $Z = 35$ are shown in Figure 13.

We find that in this case the locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve begins to exist (for $\beta \to \infty$) at $\delta = \delta_\infty \cong 0.1778$, $\alpha = \alpha_\infty \cong 1.1721$, and attains its minimum with respect to β at $\beta = \beta_{\min} \cong 114.53$, $\alpha \cong 0.6754$ for $\delta \cong 0.4059$. The final locus point is at $\beta \cong 166.61$, $\alpha \cong 0.3255$ (for $\delta = 1$).

Based on the same arguments as in the previous sections, we may reach the conclusion that when Z is reduced from $Z = 75$ to $Z = 35$, the threshold restriction for the occurrence of cycles is relaxed with respect to the relevant value of the parameter δ and is now described by the inequalities: $\delta > 0.1778$ and $\alpha < 1.1721$ (instead of the previous inequalities $\delta > 0.2991$, $\alpha < 1.2231$), while cycles now occur for considerably lower values of the parameter β of

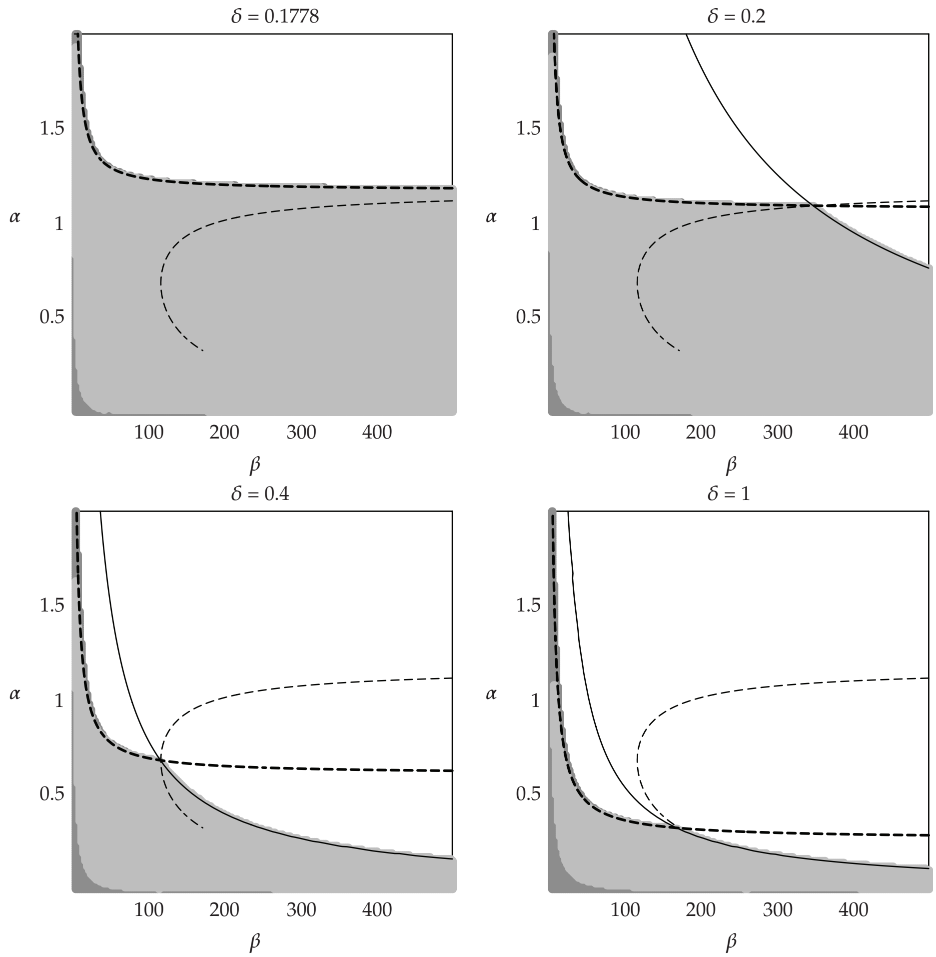

Figure 13: Regions of stability for $Z = 35$, $\gamma = 1.2$.

capital movement: $\beta > 114.53$ (instead of $\beta > 249.13$). Also, that in a global sense (for all values of δ) period doubling does not occur when exiting the stability region at points in parameter space satisfying the inequalities $\beta > 166.61$ and $\alpha < 0.3255$, only cycles may occur on exit at such points.

The emergence of period doubling when exiting the stability region through the flip bifurcation curve is illustrated in Figure 14 for $\delta = 0.4$, $\alpha = 1$, by means of bifurcation and Lyapunov exponent diagrams, and the emergence of cycles when exiting the stability region through the Hopf bifurcation curve is shown in the same Figure for $\delta = 0.7$, $\alpha = 0.2$. In the present case, the period doubling process leads directly to chaos without going through a phase of second generation cycles. Details of the period doubling process to chaos are shown in Figure 15, while the actual cycles occurring when exiting the stability region through the Hopf bifurcation curve are shown in some two-dimensional projections in Figure 16. Counter synchronization of regional incomes is observed again.

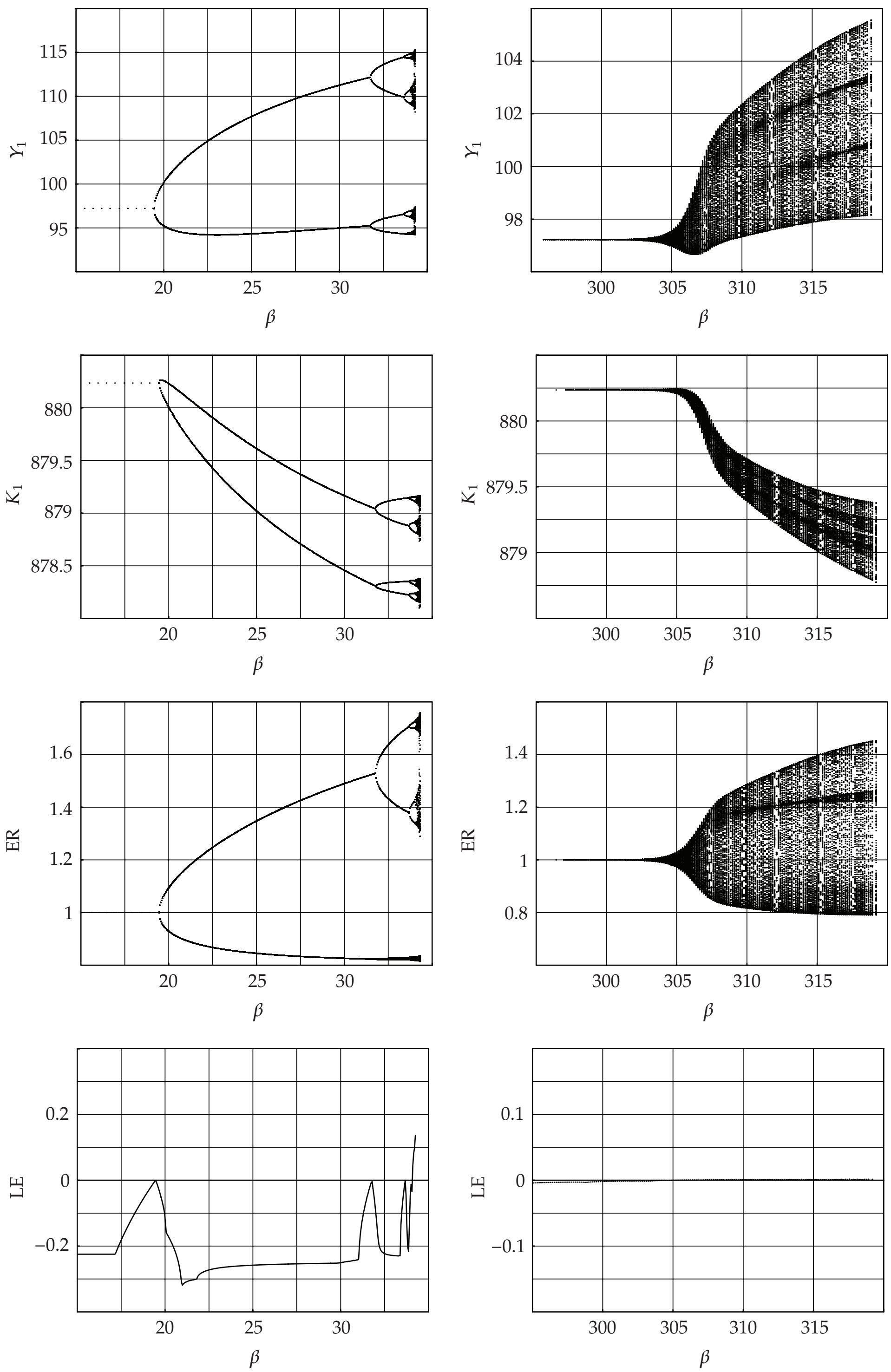

Figure 14: Left: Period doubling process for $Z = 35$, $\delta = 0.4$, $\alpha = 1$. Right: Cycles occurring for $Z = 35$, $\delta = 0.7$, $\alpha = 0.2$.

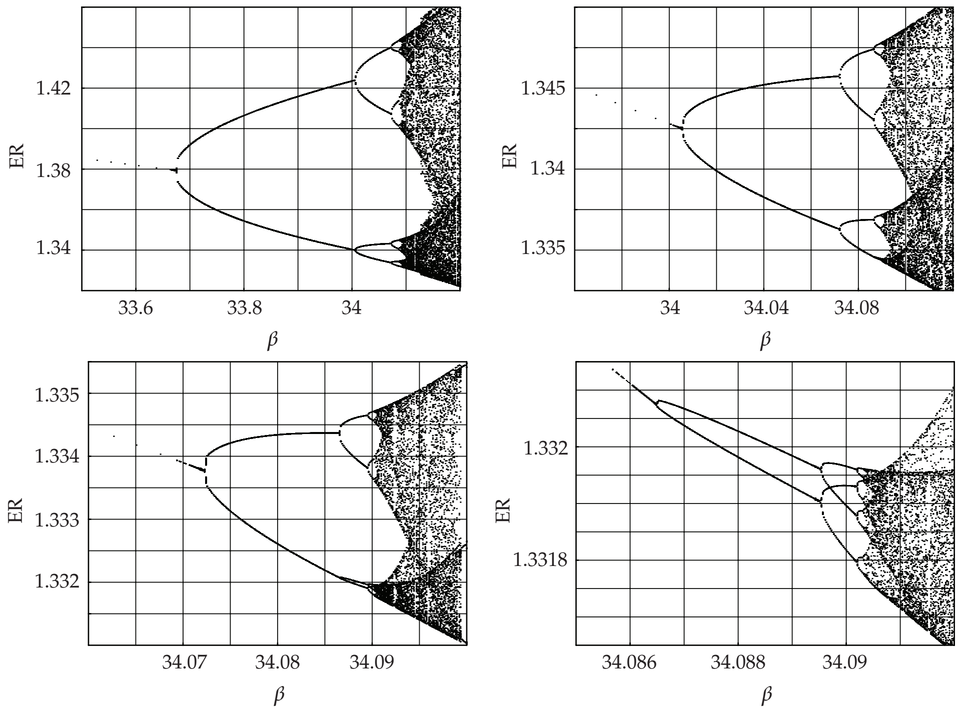

Figure 15: Details of Figure 14 (four levels of magnification). The period-doubling process to chaos for $Z = 35$, $\delta = 0.4$, $\alpha = 1$.

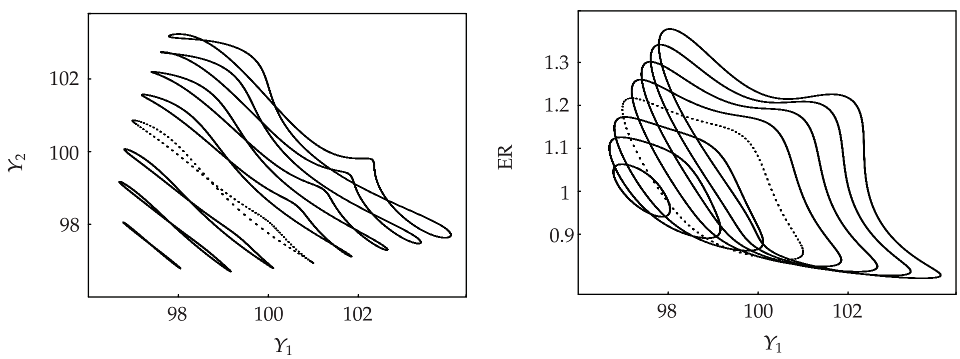

Figure 16: The bifurcating cycles for $Z = 35$, $\delta = 0.4$, $\alpha = 0.2$.

10. Summary: Conclusions

We presented a five-dimensional discrete two-regional Kaldorian macrodynamic model with flexible exchange rates, assuming similar economies of the two regions, and carried out a numerical exploration of its dynamical behavior, considering the effects of variation of the three basic parameters, namely, the common speed of adaptation of the goods markets α, the degree of capital mobility β, and the level of trade transactions between the regions δ. Further, we considered the effects of variation of two additional parameters, namely, the speed of adaptation γ of the expected exchange rate, incorporated in our model according to the adaptive expectation hypothesis, and a state parameter Z involving government expenditure.

We employed as our basic tools a grid search method and analytical coefficient criteria for the determination of the stability of equilibrium region and its boundary curves, the flip bifurcation curve and the Hopf bifurcation curve, in two-dimensional subsections of the parameter space. We considered in some detail the geometrical aspects of these boundary curves and of the curve-locus of their intersections, and the implications of these aspects on the stability region and the occurrence of period doubling or cycles in the parameter space. Our main findings are as follows.

Compared to the corresponding model of fixed exchange rates considered in a previous paper, our present model of flexible exchange rates is characterized by enhanced stability of equilibrium. High levels of capital movement β do not induce instability of the system for low levels of the speed of adjustment of the goods markets α (i.e., for prudent reactions by firms), although high levels of α induce instability even at relatively low levels of β. For prudent reactions by firms ($\alpha < 1$), equilibrium remains stable at high levels of capital movement even for high levels of trade transactions δ.

Business cycles are generally scarce and short-lived in parameter space, and occur at large values of the degree of capital movement β. We determined the threshold for the occurrence of cycles in the form of restrictions described by inequalities that must be satisfied by the parameters δ, α, and β. A characteristic difference between our present flexible exchange rates system and the corresponding fixed exchange rates system studied in Asada et al. [10] is that in our present system for cycles to occur sufficient amount of trade is required *together* with high levels of capital movement. The importance of trade as a generating factor for business cycles is significantly reduced by the flexibility of exchange rates.

Furthermore, it was demonstrated that the above threshold for the occurrence of cycles is relaxed, in the sense that cycles occur for larger regions in the basic parameters space (α, β, δ), when the speed of adaptation γ of the expected exchange rate is increased, and that it is similarly relaxed in the space of the economic interaction parameters (β, δ) when the state parameter Z is decreased. A plausible interpretation of these results is that rapid changes in exchange rates expectations and decreased government expenditure are factors contributing to the creation of Hopf bifurcations and interregional business cycles.

Concerning the plausibility of the above results and their economic interpretations, it may be commented that these do not contradict intuition and experience and are therefore suggestive of a degree of real world relevance of our formulation of the present model of Kaldorian interregional Macrodynamics under flexible exchange rates.

We also explored the asymptotic dynamical behavior of our system outside the stability of equilibrium region by means of numerical simulations resulting in bifurcation and Lyapunov exponent diagrams, and found that in several cases of the period doubling process occurring when exiting the stability region in parameter space through flip bifurcation, the process first develops into second generation (period 2) cycles. It was noted that when exiting the stability region through Hopf bifurcation the occurring cycles exhibit counter synchronization of regional incomes. Some examples of the occurring first and second generation cycles, and an example of the development of second generation cycles into chaotic attractors, were given in terms of two-dimensional projection diagrams.

We finally note that although our present model, like the corresponding model with fixed exchange rates studied in Asada et al. [10], achieves an extension of the Kaldorian model of business cycles by introducing some important and economically meaningful generalizations, and is complex enough compared to other variants of the Kaldorian model, it is still simple in the sense that all prices other than the exchange rate are assumed fixed. Thus, the financial aspect of this model would be inadequate to explain global economic phenomena such as, for example, the current financial crisis. To include the missing factors in the analysis will require a higher dimensional model and the numerical treatment will be much more difficult. The construction and exploration of such an extended model is a theme left for study in the future.

Acknowledgments

A preliminary version of this paper was presented in the 6th International Conference on Mathematical Modelling (MATHMOD 2009) held at Vienna, Austria, February 11–13, 2009. This work was supported by a University of Patras "K. Karatheodory" research grant.

References

[1] J. B. Rosser Jr., *From Catastrophe to Chaos: A General Theory of Economic Discontinuities*, Kluwer Academic Publishers, Boston, Mass, USA, 1991.
[2] T. Puu, *Attractors, Bifurcations, and Chaos*, Springer, Berlin, Germany, 2000.
[3] T. Asada, C. Chiarella, P. Flaschel, and R. Franke, *Open Economy Macrodynamics*, Springer, Berlin, Germany, 2003.
[4] N. Kaldor, "A model of the trade cycle," *Economic Journal*, vol. 50, pp. 69–86, 1940.
[5] H.-W. Lorenz, *Nonlinear Dynamical Economics and Chaotic Motion*, Springer, Berlin, Germany, 2nd edition, 1993.
[6] G. Gandolfo, *Economic Dynamics*, Springer, Berlin, Germany, 1996.
[7] T. Puu and I. Sushko, Eds., *Business Cycle Dynamics. Models and Tools*, Springer, Berlin, Germany, 2006.
[8] H.-W. Lorenz, "Analytical and numerical methods in the study of nonlinear dynamical systems in Keynesian economics," in *Business Cycles: Theory and Empirical Methods*, W. Semmler, Ed., pp. 73–112, Kluwer, Dordrecht, The Netherlands, 1994.
[9] H.-W. Lorenz, "International trade and the possible occurrence of chaos," *Economics Letters*, vol. 23, no. 2, pp. 135–138, 1987.
[10] T. Asada, C. Douskos, and P. Markellos, "Numerical exploration of kaldorian interregional macrodynamics: stability and the trade threshold for business cycles under fixed exchange rates," *Nonlinear Dynamics, Psychology and Life Sciences*. In press.
[11] T. Asada, C. Douskos, and P. Markellos, "Numerical exploration of Kaldorian macrodynamics: enhanced stability and predominance of period doubling and chaos with flexible exchange rates," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 529164, 23 pages, 2008.
[12] T. Asada, C. Douskos, and P. Markellos, "Numerical exploration of Kaldorian macrodynamics: Hopf-Neimark bifurcations and business cycles with fixed exchange rates," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 98059, 16 pages, 2007.
[13] A. Dohtani, T. Misawa, T. Inaba, M. Yokoo, and T. Owase, "Chaos, complex transients and noise: illustration with a Kaldor model," *Chaos, Solitons and Fractals*, vol. 7, no. 12, pp. 2157–2174, 1996.
[14] T. Asada and H. Yoshida, "Coefficient criterion for four-dimensional Hopf bifurcations: a complete mathematical characterization and applications to economic dynamics," *Chaos, Solitons and Fractals*, vol. 18, no. 3, pp. 525–536, 2003.

Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization