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Research Article

Delay-Dependent Stability Criterion of Arbitrary Switched Linear Systems with Time-Varying Delay

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This paper deals with the problem of delay-dependent stability criterion of arbitrary switched linear systems with time-varying delay. Based on switched quadratic Lyapunov functional approach and free-weighting matrix approach, some linear matrix inequality criterions are found to guarantee delay-dependent asymptotically stability of these systems. Simultaneously, arbitrary switched linear system can be expressed as a problem of uncertain liner system, so some delay-dependent stability criterions are obtained with the result of uncertain liner system. Two examples illustrate the exactness of the proposed criterions.

1. Introduction

Recently, switched linear systems have got more and more attention in the research community, which consists of a family of liner subsystems described by liner differential or difference equations and a switching law that orchestrates switching between them; see, for example, [1–4]. Simultaneously, systems with delays abound in the world and time-delay systems frequently appear in vast engineering systems [5–7]. Therefore, many papers consider switched linear systems with time constant delay or time-varying delay [8–24]. Naturally, stability is a fundamental property which has been investigated from the very beginning for this class of systems [25]. For stability analysis under arbitrary switching, even when all subsystems of a switched system are asymptotically stable or exponentially stable, it is still possible to construct a divergent trajectory from any initial state for such a switched system [4]. Thus, this paper aims to study the stability of arbitrary switched linear system with time-varying delay.

On one hand, many methods have been developed in the study of arbitrary switched systems such as common quadratic Lyapunov functional approach (CQLF), converse

Lyapunov theorem, and switched quadratic Lyapunov functional approach (SQLF) [4, 26–28]. On the other hand, Wu M. and He Y. develop free-weighting matrix approach for stability of liner system and uncertain liner system [29–33]. In this paper, Based on SQLF and free-weighting matrix approach, we consider the linear switched system:

$$x(k+1) = A_{r(k)}x(k) + A_{dr(k)}x(k-d(k)) + B_{r(k)}u(k), \quad k \in \mathbf{Z}^{+}, \; r(k) \in \Omega, \tag{1.1}$$

where $x(k) \in \mathbf{R}^n$ is the state, $u(k) \in \mathbf{R}^n$ is the control input, and $r(k)$ is a switching rule defined by $r(k) : N \to \Omega$ with $\Omega = \{1, 2, \ldots, N\}$. Moreover, $r(k) = i$ means the subsystem (A_i, A_{di}, B_i) is active. $d(k)$ is nonnegative differential time-varying functions which denote the time delays and satisfy $0 \leq d_1 \leq d(k) \leq d_2$.

At the same time, the uncertain linear system

$$x(k+1) = (A - \Delta A(k))x(k) + (A_d - \Delta A_d(k))x(k-d(k)) + (B - \Delta B(k))u(k), \quad k \in \mathbf{Z}^{+}, \tag{1.2}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^n$ is the control input, A, A_d, and B are given constant matrices, $\Delta A(k)$, $\Delta A_d(k)$, and $\Delta B(k)$ are the parameter uncertainties matrices which are assumed to be of the form

$$\begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta B(k) \end{bmatrix} = DF(k)\begin{bmatrix} E_a & E_{ad} & E_b \end{bmatrix}, \tag{1.3}$$

where E_a, E_{ad}, and E_b are given constant matrices of appropriate dimensions and $F(k)$ is the uncertain matrix such that

$$F^T(k)F(k) \leq I. \tag{1.4}$$

From (1.1) and (1.2), we know that when one subsystem switches to another subsystem, there exist matrixes A, A_d, and B such that

$$\begin{bmatrix} A_{r(k)} & A_{dr(k)} & B_{r(k)} \end{bmatrix} = \begin{bmatrix} A - \Delta A(k) & A_d - \Delta A_d(k) & B - \Delta B(k) \end{bmatrix} \tag{1.5}$$

so system (1.1) be equivalent to system (1.2). The key ideas of this paper are that SQLF is connected with free-weighting matrix approach and arbitrary switched linear system can be expressed as a problem of uncertain liner system.

This paper is organized as follows. In Section 2, we give some basic definitions. We analyze the stability of the system (1.1) with the SQLF and free-weighting matrix approach in Section 3. Based on uncertain liner system, we study the stability of the system (1.1) in Section 4. Some examples are given in Section 5. The last section offers the conclusions of this paper.

2. Preliminaries

In this section, with the switched quadratic Lyapunov functional approach, we investigate the stability of the origin of an autonomous switched system given by

$$x(k+1) = A_{r(k)}x(k), \quad k \in \mathbf{Z}^{+}, \, r(k) \in \Omega. \tag{2.1}$$

Define the indicator function

$$\xi(k) = [\xi_1(k), \ldots, \xi_i(k), \ldots, \xi_N(k)]^T, \tag{2.2}$$

with

$$\xi_i(k) = \begin{cases} 1 & \text{if } r(k) = i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then, the switched system (2.1) can also be written as

$$x(k+1) = \sum_{i=1}^{N} \xi_i(k) A_i x(k). \tag{2.4}$$

This corresponds to the switched Lyapunov function defined as

$$V(k, x(k)) = x^T(k) P_{r(k)} x(k) = x^T(k) \left(\sum_{i=1}^{N} \xi_i(k) P_i \right) x(k) \tag{2.5}$$

with P_i is symmetric positive definite matrices. If such a positive-definite Lyapunov function exists and

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) \tag{2.6}$$

is negative definite along the solutions of (2.1), then the origin of the switched system (2.1) is asymptotically stable. In order to represent, we give the following notation.

Throughout this paper, the superscript T stands for the inverse and transpose of a matrix; $R^{n \times m}$ is the set of all $n \times m$ real matrices; $P > 0$ means that the matrix P is positive definite; and the symmetric terms in a symmetric matrix are denoted by $\star$, for example,

$$\begin{bmatrix} M & O \\ \star & N \end{bmatrix} = \begin{bmatrix} M & O \\ O^T & N \end{bmatrix}. \tag{2.7}$$

Lemma 2.1 (see [4]). *If there exist positive definite symmetric matrices $P_i \in R^{n\times n}(P_i = P_i^T)$, satisfying*

$$\begin{bmatrix} P_i & A_i^T P_j \\ \star & P_j \end{bmatrix} < 0 \tag{2.8}$$

for all $i, j \in \Omega$, then the switched linear system (2.1) *is asymptotically stable.*

Lemma 2.2 (see [4]). *If there exist positive definite symmetric matrices $P_i \in R^{n\times n}(P_i = P_i^T)$ and matrices $F_i, G_i \in R^{n\times n}(i \in \Omega)$, satisfying*

$$\begin{bmatrix} A_i F_i^T + F_i A_i^T - P_i & A_i G_i - F_i \\ \star & P_j - G_i - G_i^T \end{bmatrix} < 0 \tag{2.9}$$

for all $i, j \in \Omega$, then the switched linear system (2.1) *is asymptotically stable.*

Lemma 2.3 (see [33]). *Let d_1 and d_2 be positive integers such that $0 \le d_1 \le d_2$. When $u(k) = 0$, the systems* (1.2) *is asymptotically stability if there exist symmetric matrices $P = P^T > 0$, $Q = Q^T > 0$, $Z = Z^T > 0$, $X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \ge 0$ and any appropriate dimensional matrices N_1, N_2 and $\lambda > 0$ such that the following LMIs hold,*

$$\begin{gathered} \begin{bmatrix} \Psi_{11} + \lambda E_a^T E_a & \Psi_{12} + \lambda E_a^T E_{ad} & (A - I)^T H & PD \\ \star & \Psi_{22} + \lambda E_{ad}^T E_{ad} & A_d^T H & 0 \\ \star & \star & -H & HD \\ \star & \star & \star & -\lambda I \end{bmatrix} < 0, \\ \begin{bmatrix} X_{11} & X_{12} & N_1 \\ \star & X_{22} & N_2 \\ \star & \star & Z \end{bmatrix} \ge 0, \end{gathered} \tag{2.10}$$

where $\Psi_{11} = (d_2 - d_1 + 1)Q + P(A - I) + (A - I)^T P + N_1 + N_1^T + d_2 X_{11}$, $\Psi_{12} = PA_d + N_Z^T - N_1 + d_2 X_{12}$, $\Psi_{22} = -Q - N_2^T - N_2 + d_2 X_{22}$, and $H = P + d_2 Z$.

3. Stability Analysis of System (1.1) with SQLF

In this section, firstly, when we do not consider the control input, the linear switched system (1.1) is rewritten as

$$x(k+1) = A_{r(k)} x(k) + A_{dr(k)} x(k - d(k)), \quad k \in Z^+,\ r(k) \in \Omega. \tag{3.1}$$

With SQLF and free-weighting matrix approach, we have the following theorem.

Theorem 3.1. *Let d_1 and d_2 be positive integers such that $0 \leq d_1 \leq d_2$; the systems (3.1) is asymptotically stability, if there exist symmetric matrices $P_i = P_i^T > 0$, $P_j = P_j^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T \geq 0$, $X^{ij} = \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} \\ \star & X_{22}^{ij} \end{bmatrix} \geq 0$ and any appropriate dimensional matrices N_1^{ij} and N_2^{ij} such that the following LMIs hold,*

$$\Phi^{ij} = \begin{bmatrix} \Phi_{11}^{ij} & \Phi_{12}^{ij} & d_2(A_i - I)^T Z \\ \star & \Phi_{22}^{ij} & d_2 A_{di}^T Z \\ \star & \star & -d_2 Z \end{bmatrix} < 0, \tag{3.2}$$

$$\Theta^{ij} = \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} & N_1^{ij} \\ \star & X_{22}^{ij} & N_2^{ij} \\ \star & \star & Z \end{bmatrix} \geq 0, \quad i, j \in \Omega, \tag{3.3}$$

where $\Phi_{11}^{ij} = (d_2 - d_1 + 1)Q + A_i^T P_j A_i - P_i + N_1^{ij} + (N_1^{ij})^T + d_2 X_{11}^{ij}$, $\Phi_{12}^{ij} = A_i^T P_j A_{di} + (N_2^{ij})^T - N_1^{ij} + d_2 X_{12}^{ij}$ *and* $\Phi_{22}^{ij} = A_{di}^T P_j A_{di} - Q - (N_2^{ij})^T - N_2^{ij} + d_2 X_{22}^{ij}$.

Proof. Suppose that $y(l) = x(l+1) - x(l)$, then we have $x(k+1) = x(k) + y(k)$ and $x(k) = x(k - d(k)) + \sum_{i=k-d(k)}^{k-1} y(l)$.

Combined with (2.2), we consider the following SQLF:

$$\begin{aligned} V(k, x(k)) &= V_1(k, x(k)) + V_2(k, x(k)) + V_3(k, x(k)), \\ V_1(k, x(k)) &= x^T(k) P_{r(k)} x(k) = x^T(k) \left(\sum_{i=1}^{N} \xi_i(k) P_i \right) x(k), \\ V_2(k, x(k)) &= \sum_{\theta=-d_2+1}^{0} \sum_{l=k-1+\theta}^{k-1} y^T(l) Z_{r(k)} y(l) = \sum_{\theta=-d_2+1}^{0} \sum_{l=k-1+\theta}^{k-1} y^T(l) \left(\sum_{i=1}^{N} \xi_i(k) Z_i \right) y(l), \\ V_3(k, x(k)) &= \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=k-1+\theta}^{k-1} x^T(l) Q_{r(k)} x(l) = \sum_{\theta=-d_2+1}^{-d_1+1} \sum_{l=k-1+\theta}^{k-1} x^T(l) \left(\sum_{i=1}^{N} \xi_i(k) Q_i \right) x(l), \end{aligned} \tag{3.4}$$

where $P_i = P_i^T > 0, Z_i = Z_i^T \geq 0,$ and $Q_i = Q_i^T \geq 0$.

With (2.6), we obtain

$$\begin{aligned}\Delta V_1(k,x(k)) &= x^T(k+1)P_{r(k+1)}x(k+1) - x^T(k)P_{r(k)}x(k)\\ &= x^T(k)\left[A_{r(k)}^T P_{r(k+1)}A_{r(k)} - P_{r(k)}\right]x(k)\\ &\quad + x^T(k-d(k))\left[A_{dr(k)}^T P_{r(k+1)}A_{dr(k)}\right]x(k-d(k))\\ &\quad + x^T(k)A_{r(k)}^T P_{r(k+1)}A_{dr(k)}x(k-d(k)) + x^T(k-d(k))A_{dr(k)}^T P_{r(k+1)}A_{r(k)}x(k),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Delta V_2(k,x(k)) &= \sum_{\theta=-d_2+1}^{0}\sum_{l=k+\theta}^{k} y^T(l)Z_{r(k+1)}y(l) - \sum_{\theta=-d_2+1}^{0}\sum_{l=k-1+\theta}^{k-1} y^T(l)Z_{r(k)}y(l)\\ &= d_2 y^T(k)Z_{r(k+1)}y(k) - \sum_{l=k-d_2}^{k-1} y^T(l)Z_{r(k)}y(l)\\ &\quad + \sum_{\theta=-d_2+1}^{0}\sum_{l=k+\theta}^{k-1} y^T(l)\left[Z_{r(k+1)} - Z_{r(k)}\right]y(l)\end{aligned} \tag{3.6}$$

when $Z_{r(k+1)} = Z_{r(k)}$,

$$\Delta V_2(k,x(k)) \le d_2 y^T(k)Z_{r(k+1)}y(k) - \sum_{l=k-d_2}^{k-1} y^T(l)Z_{r(k)}y(l), \tag{3.7}$$

$$\begin{aligned}\Delta V_3(k,x(k)) &= \sum_{\theta=-d_2+1}^{-d_1+1}\sum_{l=k+\theta}^{k} x^T(l)Q_{r(k+1)}x(l) - \sum_{\theta=-d_2+1}^{-d_1+1}\sum_{l=k-1+\theta}^{k-1} x^T(l)Q_{r(k)}x(l)\\ &= (d_2-d_1+1)x^T(k)Q_{r(k+1)}x(k) - \sum_{l=k-d_2}^{k-d_1} x^T(l)Q_{r(k)}x(l)\\ &\quad + \sum_{\theta=-d_2+1}^{-d_1+1}\sum_{l=k+\theta}^{k-1} x^T(l)\left[Q_{r(k+1)} - Q_{r(k)}\right]x(l)\end{aligned} \tag{3.8}$$

when $Q_{r(k+1)} = Q_{r(k)}$,

$$\Delta V_3(k,x(k)) \le (d_2-d_1+1)x^T(k)Q_{r(k+1)}x(k) - x^T(k-d(k))Q_{r(k)}x(k-d(k)). \tag{3.9}$$

Suppose that $r(k) = i$ and $r(k+1) = j$ mean that the subsystem i switches to the subsystem j in the switching system. As this has to be satisfied under arbitrary switching laws, it follows that this has to hold for the special configuration $\xi_i(k) = 1$, $\xi_{l \neq i}(k) = 0$, $\xi_j(k + 1) = 1$, and $\xi_{l \neq j}(k + 1) = 0$. And supposing that $Z_j = Z_i = Z$ and $Q_j = Q_i = Q$, we obtain

$$\begin{aligned}\Delta V(k, x(k)) \leq{}& x^T(k)\left[A_i^T P_j A_i - P_i\right]x(k) + x^T(k - d(k))\left[A_{di}^T P_j A_{di}\right]x(k - d(k)) + d_2 y^T(l) Z y(l)\\&+ x^T(k) A_i^T P_j A_{di} x(k - d(k)) + x^T(k - d(k)) A_{di}^T P_j A_i x(k)\\&- \sum_{l=k-d_2}^{k-1} y^T(l) Z y(l) + (d_2 - d_1 + 1) x^T(k) Q x(k) - x^T(k - d(k)) Q x(k - d(k)).\end{aligned} \tag{3.10}$$

By using the Leibniz-Newton formula, for any appropriately dimensioned matrices N_1^{ij} and N_2^{ij}, the following equation is true:

$$2\left[x^T(k) N_1^{ij} + x^T(k - d(k)) N_2^{ij}\right] \times \left[x(k) - x(k - d(k)) - \sum_{l=k-d(k)}^{k-1} y(l)\right] = 0. \tag{3.11}$$

In addition, for any semipositive definite matrix $X^{ij} = \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} \\ \star & X_{22}^{ij} \end{bmatrix} \geq 0$, the following equation holds:

$$\sum_{l=k-d_2}^{k-1} \phi_1^T(k) X^{ij} \phi_1(k) - \sum_{l=k-d(k)}^{k-1} \phi_1^T(k) X^{ij} \phi_1(k) = d_2 \phi_1^T(k) X^{ij} \phi_1(k) - \sum_{l=k-d(k)}^{k-1} \phi_1^T(k) X^{ij} \phi_1(k) \geq 0, \tag{3.12}$$

where $\phi_1(k) = [x^T \quad (k) x^T \quad (k - d(k))]^T$.

With (3.1), (3.10), (3.11), and (3.12), we have

$$\Delta V(k, x(k)) \leq \phi_1^T(k) \Gamma^{ij} \phi_1(k) - \sum_{l=k-d(k)}^{k-1} \phi_2^T(k, l) \Theta^{ij} \phi_2(k, l) \leq 0, \tag{3.13}$$

where

$$\begin{gathered}\Gamma^{ij} = \begin{bmatrix} \Phi_{11}^{ij} + d_2 (A_i - I)^T Z (A_i - I) & \Phi_{12}^{ij} + d_2 I (A_i - I)^T Z A_{di} \\ \star & \Phi_{22}^{ij} + d_2 A_{di} Z A_{di} \end{bmatrix},\\ \phi_2(k, l) = \left[\phi_1^T(k) \quad y^T(l)\right]^T.\end{gathered} \tag{3.14}$$

And $\phi_1(k)$ is defined in (3.12); Φ_{11}^{ij}, Φ_{12}^{ij}, and Φ_{22}^{ij} are defined in (3.2). Therefore, when $\Gamma^{ij} < 0$ and $\Theta^{ij} \geq 0$, the system (3.1) is asymptotically stability. Applying Schur's complement, $\Gamma^{ij} < 0$ is equivalent to $\Phi^{ij} < 0, i, j \in \Omega$. This completes the proof of Theorem 3.1. □

If we have $y(k)$, for any appropriately dimensioned matrices T_1^{ij}, T_2^{ij}, N_1^{ij}, N_2^{ij}, N_2^{ij}, and

$$X^{ij} = \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ \star & X_{22}^{ij} & X_{23}^{ij} \\ \star & \star & X_{33}^{ij} \end{bmatrix} \geq 0, \tag{3.15}$$

the following equations are also true:

$$\begin{gathered} 2\left[x^T(k)T_1^{ij} + y^T(k)T_2^{ij}\right] \times \left[y(k) - (A_i - I)x(k) - A_{di}x(k - d(k))\right] = 0, \\ 2\left[x^T(k)N_1^{ij} + y^T(k)N_2^{ij} + x^T(k - d(k))N_3^{ij}\right] \times \left[x(k) - x(k - d(k)) - \sum_{l=k-d(k)}^{k-1} y(l)\right] = 0, \\ \sum_{l=k-d_2}^{k-1} \eta_1^T(k)X^{ij}\eta_1(k) - \sum_{l=k-d(k)}^{k-1} \eta_1^T(k)X^{ij}\eta_1(k) = d_2\eta_1^T(k)X^{ij}\eta_1(k) - \sum_{l=k-d(k)}^{k-1} \eta_1^T(k)X^{ij}\eta_1(k) \geq 0, \end{gathered} \tag{3.16}$$

where $\eta_1(k) = [x^T(k) \quad y^T(l) \quad x^T(k - d(k))]^T$.

Considering (3.16), similar to the proof of Theorem 3.1, we can obtain the following corollary.

Corollary 3.2. *Let d_1 and d_2 be positive integers such that $0 \leq d_1 \leq d_2$; the systems (3.1) is asymptotically stability if there exist symmetric matrices $P_i = P_i^T > 0$, $P_j = P_j^T > 0$, $Q = Q^T \geq 0$, $Z = Z^T \geq 0$, and any appropriate dimensional matrices T_1^{ij}, T_2^{ij}, N_1^{ij}, N_2^{ij}, N_2^{ij}, and*

$$X^{ij} = \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} \\ \star & X_{22}^{ij} & X_{23}^{ij} \\ \star & \star & X_{33}^{ij} \end{bmatrix} \geq 0 \tag{3.17}$$

such that the following LMIs hold,

$$\begin{gathered} \Upsilon^{ij} = \begin{bmatrix} \Upsilon_{11}^{ij} & \Upsilon_{12}^{ij} & \Upsilon_{13}^{ij} \\ \star & \Upsilon_{22}^{ij} & \Upsilon_{23}^{ij} \\ \star & \star & \Upsilon_{33}^{ij} \end{bmatrix} < 0, \\ \begin{bmatrix} X_{11}^{ij} & X_{12}^{ij} & X_{13}^{ij} & N_1^{ij} \\ \star & X_{22}^{ij} & X_{23}^{ij} & N_2^{ij} \\ \star & \star & X_{33}^{ij} & N_3^{ij} \\ \star & \star & \star & Z \end{bmatrix} \geq 0, \quad i, j \in \Omega, \end{gathered} \tag{3.18}$$

where $\Upsilon_{11}^{ij} = (d_2 - d_1 + 1)Q + A_i^T P_j A_i - P_i - T_1^{ij}(A_i - I) - (A_i - I)^T (T_1^{ij})^T + (N_1^{ij})^T + N_1^{ij} + d_2 X_{11}^{ij}$, $\Upsilon_{12}^{ij} = T_1^{ij} - (A_i - I)^T (T_2^{ij})^T + (N_2^{ij})^T + d_2 X_{12}^{ij}$, $\Upsilon_{13}^{ij} = A_i^T P_j A_{di} - T_1^{ij} A_{di} + (N_3^{ij})^T - N_1^{ij} + d_2 X_{13}^{ij}$, $\Upsilon_{22}^{ij} = d_2 Z + (T_2^{ij})^T + T_2^{ij} + d_2 X_{22}^{ij}$, $\Upsilon_{23}^{ij} = -T_2^{ij} A_{di} - N_2^{ij} + d_2 X_{23}^{ij}$, *and* $\Upsilon_{33}^{ij} = A_{di}^T P_j A_{di} - Q - N_3^{ij} + (N_3^{ij})^T + d_2 X_{33}^{ij}$.

Next, we consider the design of a switched state feedback:

$$u(k) = K_{r(k)} x(k). \tag{3.19}$$

Ensuring stability of the closed-loop switched system:

$$x(k+1) = \left(A_{r(k)} + B_{r(k)} K_{r(k)}\right) x(k) + A_{dr(k)} x(k - d(k)), \quad k \in \mathbf{Z}^+,\ r(k) \in \Omega. \tag{3.20}$$

Based on Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *Let d_1 and d_2 be positive integers such that $0 \le d_1 \le d_2$. Under arbitrary switch, the systems* (1.1) *is asymptotically stability if there exist symmetric matrices $P_i = P_i^T > 0$, $P_j = P_j^T > 0$, $Q = Q^T \ge 0$, $Z = Z^T \ge 0$, $Y^{ij} = \begin{bmatrix} Y_{11}^{ij} & Y_{12}^{ij} \\ \star & Y_{22}^{ij} \end{bmatrix} \ge 0$, and any appropriate dimensional matrices M_1^{ij} and M_2^{ij} such that the following LMIs hold,*

$$\begin{gathered}
\Xi^{ij} = \begin{bmatrix} \Xi_{11}^{ij} & \Xi_{12}^{ij} & d_2 (A_i L_i + B_i V_i - V_i)^T \\ \star & \Xi_{22}^{ij} & d_2 L_i A_{di}^T \\ \star & \star & d_2 R \end{bmatrix} \ge 0, \\
\begin{bmatrix} Y_{11}^{ij} & Y_{12}^{ij} & M_1^{ij} \\ \star & Y_{22}^{ij} & M_2^{ij} \\ \star & \star & L R^{-1} L \end{bmatrix} \ge 0, \quad i, j \in \Omega,
\end{gathered} \tag{3.21}$$

where $\Xi_{11}^{ij} = (d_2 - d_1 + 1) W_i + (A_i L_i)^T P_j (A_i L_i) + (A_i L_i) P_j (B_i V_i) + (B_i V_i)^T P_j (A_i L_i) + (B_i V_i)^T P_j (B_i V_i) - L_i + (M_1^{ij})^T + M_1^{ij} + d_2 Y_{11}^{ij}$, $\Xi_{12}^{ij} = (A_i L_i)^T P_j (A_{di} L_i) + (B_i V_i)^T P_j (A_{di} L_i) + (M_2^{ij})^T - M_1^{ij} + d_2 Y_{12}^{ij}$, *and* $\Xi_{22}^{ij} = (A_{di} L_i)^T P_j (A_{di} L_i) - W - M_2^{ij} - (M_2^{ij})^T + d_2 Y_{22}^{ij}$.

Proof. To the system (3.1), A_i is replaced by $A_i + B_i K_i$ in (3.2). Simultaneously, two parts of inequality (3.2) multiply the same matrix $\operatorname{diag}[P_i^{-1}, P_i^{-1}, Z^{-1}]$ and two parts of inequality (3.3) multiply the same matrix $\operatorname{diag}[P_i^{-1}, P_i^{-1}, P_i^{-1}]$. Suppose that $L_i = P_i^{-1}$, $W_i = P_i^{-1} Q P_i^{-1}$, $Y^{ij} = \operatorname{diag}[P_i^{-1}, P_i^{-1}] X^{ij} \operatorname{diag}[P_i^{-1}, P_i^{-1}]$, $R = Z^{-1}$, $M_1^{ij} = P_i^{-1} N_1^{ij} P_i^{-1}$, $M_2^{ij} = P_i^{-1} N_2^{ij} P_i^{-1}$, and $V_i = K_i P_i^{-1}$; then we obtain (3.21).This completes the proof of Theorem 3.3. □

4. Stability Analysis of System (1.1) with Uncertain Liner System

In this section, results of uncertain liner system are extended to arbitrary switched linear system for arbitrary switched linear system can be expressed as a problem of uncertain liner system. When $u(k) = 0$, (1.5) are rewritten as

$$\left[\Delta A(k) \quad \Delta A_d(k)\right] = \left[A - A_{r(k)} \quad A_d - A_{dr(k)}\right]. \tag{4.1}$$

Then the system (3.1) is rewritten as

$$x(k+1) = \left(A - \left(A - A_{r(k)}\right)\right)x(k) + \left(A_d - \left(A_d - A_{dr(k)}\right)\right)x(k - d(k)), \quad k \in Z^+,\ r(k) \in \Omega. \tag{4.2}$$

Combined with Lemma 2.3, we easily have the following theorem.

Theorem 4.1. *Let d_1 and d_2 be positive integers such that $0 \le d_1 \le d_2$. Under arbitrary switch, the system* (4.2) *is asymptotically stability if there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, $Z = Z^T > 0$, $X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \ge 0$ and any appropriate dimensional matrices A,A_d,D, E_a,E_{ad},F_i, N_1, N_2, and $\lambda > 0$ such that the LMIs* (2.10) *and the following LMIs hold,*

$$\begin{gathered} \left[A - A_i \quad A_d - A_{di}\right] = DF_i\left[E_a \quad E_{ad}\right] \\ F_i^T F_i \le I, \quad i \in \Omega. \end{gathered} \tag{4.3}$$

Next, we consider the design of a switched state feedback. With (4.1) and (4.2), the system (3.20) is rewritten as

$$\begin{aligned} x(k+1) = {} & \left(A - \left(A - \left(A_{r(k)} + B_{r(k)}K_{r(k)}\right)\right)\right)x(k) \\ & + \left(A_d - \left(A_d - A_{dr(k)}\right)\right)x(k - d(k)), \quad k \in \mathbf{Z}^+,\ r(k) \in \Omega. \end{aligned} \tag{4.4}$$

Combined with Theorem 4.1, we easily have the following theorem.

Theorem 4.2. *Let d_1 and d_2 be positive integers such that $0 \le d_1 \le d_2$. Under arbitrary switch, the systems* (4.4) *is asymptotically stability if there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, $Z = Z^T > 0$, $X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \ge 0$, and any appropriate dimensional matrices A,A_d,D, E_a,E_{ad},F_i, N_1, N_2 and $\lambda > 0$ such that the LMIs* (2.10) *and* (4.3), *and the following LMIs hold,*

$$\left[A - (A_i + B_iK_i) \quad A_d - A_{di}\right] = DF_i\left[E_a \quad E_{ad}\right], \quad i \in \Omega. \tag{4.5}$$

5. Examples

Example 5.1. Consider the following switched delay systems with two subsystems

$$x(k+1) = A_i x(k) + A_{di} x(k - d(k)), \quad k \in \mathbf{Z}^+,\ i \in \Omega, \tag{5.1}$$

where

$$A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.2 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.1 & 0 \\ 0.3 & -0.2 \end{bmatrix}, \tag{5.2}$$

and $\Omega = \{1, 2\}$.

When $d_1 = d_2$ that is, $d(k) = d_2$, d_2 is without limit. To time-varying delay $d(k)$, when d_1 is given, d_2 is a maximum value of the solvability of LMIs (3.2) and (3.3), and some results are in Table 1.

In this example, the switching system has two subsystems, so there are there switches that are between subsystem 1 and subsystem 2, between subsystem 1 and subsystem 1, and between subsystem 2 and subsystem 2. According to Theorem 3.1, when $d_1 = 1$ and $d_2 = 3$, solving the LMIs (3.2) and (3.3) leads to

$$Q = \begin{bmatrix} 29.9101 & -3.0324 \\ -3.0324 & 14.4926 \end{bmatrix} Z = \begin{bmatrix} 5.3474 & 0.0341 \\ 0.0341 & 6.8596 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 143.7206 & -9.2029 \\ -9.2029 & 102.1555 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 144.9590 & -8.5971 \\ -8.5971 & 93.0312 \end{bmatrix}, \qquad N_1^{12} = \begin{bmatrix} -1.2592 & 0.5437 \\ -0.4002 & -2.1476 \end{bmatrix},$$

$$N_2^{12} = \begin{bmatrix} 1.9723 & -0.1401 \\ 0.4588 & 3.0624 \end{bmatrix}, \qquad N_1^{21} = \begin{bmatrix} -1.3023 & -0.8619 \\ 0.8282 & -2.1259 \end{bmatrix},$$

$$N_2^{21} = \begin{bmatrix} 2.2016 & 0.3203 \\ -0.6405 & 3.2331 \end{bmatrix}, \qquad N_1^{11} = \begin{bmatrix} -1.2553 & 0.6225 \\ -0.4427 & -2.2909 \end{bmatrix},$$

$$N_2^{11} = \begin{bmatrix} 1.9636 & -0.1949 \\ 0.5126 & 3.0790 \end{bmatrix}, \qquad N_1^{22} = \begin{bmatrix} -1.3199 & -0.7026 \\ 0.7078 & -2.0469 \end{bmatrix},$$

$$N_2^{22} = \begin{bmatrix} 2.2049 & 0.2493 \\ -0.5319 & 3.1800 \end{bmatrix}, \qquad X^{12} = \begin{bmatrix} 10.2348 & 0.0282 & -0.8722 & 0.3092 \\ 0.0282 & 8.2857 & 0.3092 & -0.5114 \\ -0.8722 & 0.3092 & 6.3470 & -0.8716 \\ 0.3092 & -0.5114 & -0.8716 & 4.7227 \end{bmatrix},$$

$$X^{21} = \begin{bmatrix} 10.1529 & 0.2694 & -0.4215 & -0.6588 \\ 0.2694 & 8.2610 & -0.6588 & -0.6227 \\ -0.4215 & -0.6588 & 5.0357 & 0.9594 \\ -0.6588 & -0.6227 & 0.9594 & 3.9515 \end{bmatrix},$$

$$X^{11} = \begin{bmatrix} 10.2369 & 0.0251 & -0.8788 & 0.3659 \\ 0.0251 & 7.7927 & 0.3659 & -0.4471 \\ -0.8788 & 0.3659 & 6.3347 & -0.8928 \\ 0.3659 & -0.4471 & -0.8928 & 4.7126 \end{bmatrix},$$

$$X^{22} = \begin{bmatrix} 10.1467 & 0.2491 & -0.4220 & -0.5494 \\ 0.2491 & 8.4372 & -0.5494 & -0.7209 \\ -0.4220 & -0.5494 & 5.2477 & 0.8376 \\ -0.5494 & -0.7209 & 0.8376 & 4.0014 \end{bmatrix}. \tag{5.3}$$

It can be seen from Figure 1 that when $d_1 = 1$ and $d_2 = 3$, all the state solutions corresponding to the 10 random initial points are convergent asymptotically to the unique equilibrium $x^* = \{0, 0\}$.

Example 5.2. Consider the following switched delay systems with two subsystems:

$$x(k+1) = A_i x(k) + A_{di} x(k - d(k)), \quad k \in \mathbf{Z}^+, \, i \in \Omega, \tag{5.4}$$

where

$$\begin{gathered} A_1 = \begin{bmatrix} 0.3850 & 0.0090 \\ 0.0180 & 0.5880 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} -0.4150 & 0.0090 \\ -0.0820 & -0.3120 \end{bmatrix}, \\ A_2 = \begin{bmatrix} 0.3970 & 0.0120 \\ 0.0150 & 0.5820 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} -0.4030 & 0.0120 \\ -0.0850 & -0.3180 \end{bmatrix}, \quad \Omega = \{1, 2\}. \end{gathered} \tag{5.5}$$

When there exit matrixes

$$\begin{gathered} A = \begin{bmatrix} 0.4000 & 0.0000 \\ 0.0000 & 0.6000 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -0.4000 & 0.0000 \\ -0.1000 & -0.3000 \end{bmatrix}, \\ D = \begin{bmatrix} 0.0300 & 0.0000 \\ 0.0000 & 0.0300 \end{bmatrix}, \qquad E_a = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}, \\ E_{ad} = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} -0.5000 & 0.3000 \\ 0.6000 & -0.4000 \end{bmatrix}, \\ F_2 = \begin{bmatrix} -0.1000 & 0.4000 \\ 0.5000 & -0.6000 \end{bmatrix}, \end{gathered} \tag{5.6}$$

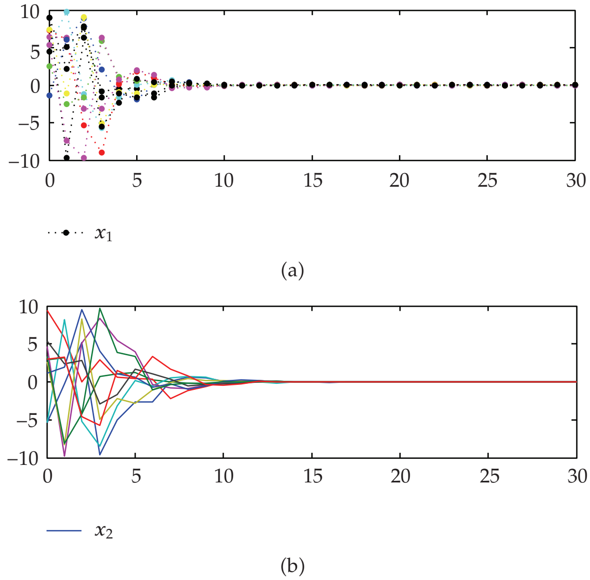
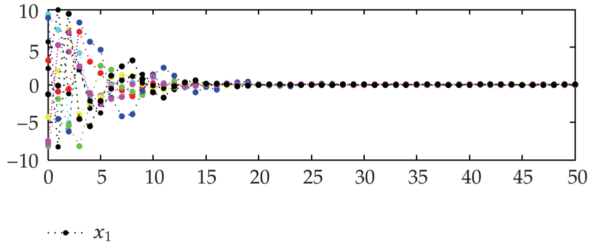


Figure 1: Global convergence of states x_1 and x_2 in Example 5.1, when $d_1 = 1$ and $d_2 = 3$.

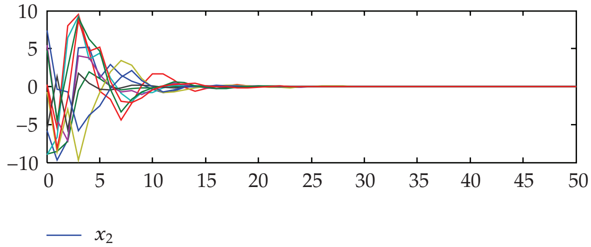
which satisfied (4.3), and $d_1 = 1$ and $d_2 = 3$, based on Theorem 4.1 solving the LMIs (2.10) leads to

$$P = (1.0e+004) * \begin{bmatrix} 1.0888 & -0.0129 \\ -0.0129 & 0.2758 \end{bmatrix}, \qquad Q = (1.0e+003) * \begin{bmatrix} 2.5193 & 0.0379 \\ 0.0379 & 0.4439 \end{bmatrix},$$
$$Z = (1.0e+003) * \begin{bmatrix} 1.4927 & 0.0809 \\ 0.0809 & 0.4826 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} -472.4815 & -6.3448 \\ -53.9979 & -181.8198 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 478.2798 & -0.0086 \\ 45.6698 & 185.5393 \end{bmatrix}, \tag{5.7}$$
$$X = \begin{bmatrix} 216.7792 & -41.8304 & -134.0982 & -18.7083 \\ -41.8304 & 113.2493 & -18.7083 & -62.9775 \\ -134.0982 & -18.7083 & 174.9568 & -10.6693 \\ -18.7083 & -62.9775 & -10.6693 & 84.3589 \end{bmatrix}, \quad \lambda = 148.1177.$$

It can be seen from Figure 2 that when $d_1 = 1$ and $d_2 = 3$, all the state solutions corresponding to the 10 random initial points are convergent asymptotically to the unique equilibrium $x^* = \{0, 0\}$.



(a)



(b)

Figure 2: Global convergence of states x_1 and x_2 in Example 5.2, when $d_1 = 1$ and $d_2 = 3$.

Table 1: Allowable upper bound of d_2 with given d_1.

d_1	0	1	2	3	4	5	6	7	8	9	50	100
d_2	11	12	13	14	15	16	17	18	19	20	61	111

6. Conclusions

This paper was dedicated to the delay-dependent stability of arbitrary switched linear systems with time-varying delay. We obtain two main results. Firstly, using switched quadratic Lyapunov functional approach and free-weighting matrix approach, less conservative LMI conditions have been proposed. Secondly, based on the result of uncertain liner system, some delay-dependent stability criterions are obtained.

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