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Research Article

A Hybrid Method for a Countable Family of Multivalued Maps, Equilibrium Problems, and Variational Inequality Problems

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We introduce a new monotone hybrid iterative scheme for finding a common element of the set of common fixed points of a countable family of nonexpansive multivalued maps, the set of solutions of variational inequality problem, and the set of the solutions of the equilibrium problem in a Hilbert space. Strong convergence theorems of the purposed iteration are established.

1. Introduction

Let D be a nonempty convex subset of a Banach spaces E. Let F be a bifunction from $D \times D$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. The equilibrium problem for F is to find $x \in D$ such that $F(x,y) \geq 0$ for all $y \in D$. The set of such solutions is denoted by $EP(F)$. The set D is called *proximal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x-y\| = d(x,D)$, where $d(x,D) = \inf\{\|x-z\| : z \in D\}$. Let $CB(D)$, $K(D)$, and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \tag{1.1}$$

for $A, B \in CB(D)$. A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in D$. A multivalued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx,Ty) \leq$

$\|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ (resp., $T : D \to CB(D)$) if $p = Tp$ (resp., $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : D \to CB(D)$ is called *quasi-nonexpansive* [1] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multivalued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive; see [2].

The mapping $T : D \to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multivalued mapping $T : D \to CB(D)$ is *hemicompact*.

A mapping $T : D \to CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \tag{1.2}$$

for all $x \in D$.

In 1953, Mann [3] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

However, we note that Mann's iteration process (1.3) has only weak convergence, in general; for instance, see [4–6].

In 2003, Nakajo and Takahashi [7] introduced the method which is the so-called CQ method to modify the process (1.3) so that strong convergence is guaranteed. They also proved a strong convergence theorem for a nonexpansive mapping in a Hilbert space.

Recently, Tada and Takahashi [8] proposed a new iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H.

In 2005, Sastry and Babu [9] proved that the Mann and Ishikawa iteration schemes for multivalued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multivalued map with compact domain.

In 2007, Panyanak [10] extended the above result of Sastry and Babu [9] to uniformly convex Banach spaces but the domain of T remains compact.

Later, Song and Wang [11] noted that there was a gap in the proofs of Theorem 3.1 [10] and Theorem 5 [9]. They further solved/revised the gap and also gave the affirmative answer to Panyanak [10] question using the following Ishikawa iteration scheme. In the main results, domain of T is still compact, which is a strong condition (see [11, Theorem 1]) and T satisfies condition (I) (see [11, Theorem 1]).

In 2009, Shahzad and Zegeye [2] extended and improved the results of Panyanak [10], Sastry and Babu [9], and Song and Wang [11] to quasi-nonexpansive multivalued maps. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T, namely, $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak [10] question in a more general setting. In the main results,

T satisfies *Condition* (I) (see [2, Theorem 2.3]) and T is hemicompact and continuous (see [2, Theorem 2.5]).

A mapping $A : D \to H$ is called α-inverse-strongly monotone [12] if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in D. \tag{1.4}$$

Remark 1.1. It is easy to see that if $A : D \to H$ is α-inverse-strongly monotone, then it is a $(1/\alpha)$-Lipschitzian mapping.

Let $A : D \to H$ be a mapping. The classical variational inequality problem is to find a $u \in D$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in D. \tag{1.5}$$

The set of solutions of variational inequality (3.9) is denoted by $VI(D, A)$.

Question. How can we construct an iteration process for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of a variational inequality problem, and the set of common fixed points of nonexpansive multivalued maps ?

In the recent years, the problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points of single-valued nonexpansive mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; for instance, see [8, 13–20] and the references cited theorems.

In this paper, we introduce a monotone hybrid iterative scheme for finding a common element of the set of a common fixed points of a countable family of nonexpansive multivalued maps, the set of variational inequality, and the set of solutions of an equilibrium problem in a Hilbert space.

2. Preliminaries

The following lemmas give some characterizations and a useful property of the metric projection P_D in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let D be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D. \tag{2.1}$$

P_D is called the *metric projection* of H onto D. We know that P_D is a nonexpansive mapping of H onto D.

Lemma 2.1 (see [21]). *Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D. Given $x \in H$ and $z \in D$, then $z = P_D x$ if and only if the following holds:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D. \tag{2.2}$$

Lemma 2.2 (see [7]). *Let D be a nonempty, closed and convex subset of a real Hilbert space H and $P_D : H \to D$ the metric projection from H onto D. Then the following inequality holds:*

$$\|y - P_D x\|^2 + \|x - P_D x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \ \forall y \in D. \tag{2.3}$$

Lemma 2.3 (see [21]). *Let H be a real Hilbert space. Then the following equations hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$, *for all* $x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0,1]$ *and* $x, y \in H$.

Lemma 2.4 (see [22]). *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\left\{ v \in D : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a \right\} \tag{2.4}$$

is convex and closed.

For solving the equilibrium problem, we assume that the bifunction $F : D \times D \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in D$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in D$;

(A3) for each $x, y, z \in D$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in D$.

Lemma 2.5 (see [13]). *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in D$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in D. \tag{2.5}$$

Lemma 2.6 (see [18]). *For $r > 0$, $x \in H$, defined a mapping $T_r : H \to D$ as follows:*

$$T_r(x) = \left\{ z \in D : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in D \right\}. \tag{2.6}$$

Then the following holds:

(1) *T_r is a single value;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.7}$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

In the context of the variational inequality problem,

$$u \in VI(D, A) \Longleftrightarrow u = P_D(u - \lambda Au), \quad \forall\lambda > 0. \tag{2.8}$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply that $\langle f - g, x - y\rangle \geq 0$. A monotone mapping $T : H \to H$ is said to be maximal [23] if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$, $\forall(y, g) \in G(T)$ imply that $f \in Tx$. Let $A : D \to H$ be an inverse strongly monotone mapping and let $N_D v$ be the normal cone to D at $v \in D$, that is,

$$N_D v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in D\}, \tag{2.9}$$

and define

$$Tv = \begin{cases} Av + N_D v, & v \in D, \\ \emptyset, & v \notin D. \end{cases} \tag{2.10}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(D, A)$ (see, e.g., [24]).

In general, the fixed point set of a nonexpansive multivalued map T is not necessary to be closed and convex (see [25, Example 3.2]). In the next Lemma, we show that $F(T)$ is closed and convex under the assumption that $Tp = \{p\}$ for all $p \in F(T)$.

Lemma 2.7. *Let D be a closed and convex subset of a real Hilbert space H. Let $T : D \to CB(D)$ be a nonexpansive multivalued map with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Then $F(T)$ is a closed and convex subset of D.*

Proof. First, we will show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. We have

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(x_n, Tx) \\ &\leq d(x, x_n) + H(Tx_n, Tx) \\ &\leq 2d(x, x_n). \end{aligned} \tag{2.11}$$

It follows that $d(x, Tx) = 0$, so $x \in F(T)$. Next, we show that $F(T)$ is convex. Let $p = tp_1 + (1 - t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Let $z \in Tp$; by Lemma 2.3, we have

$$\begin{aligned}
\|p - z\|^2 &= \|t(z - p_1) + (1 - t)(z - p_2)\|^2 \\
&= t\|z - p_1\|^2 + (1 - t)\|z - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
&= td(z, Tp_1)^2 + (1 - t)d(z, Tp_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\
&\leq tH(Tp, Tp_1)^2 + (1 - t)H(Tp, Tp_2)^2 - t(1 - t)\|p_1 - p_2\|^2 \\
&\leq t\|p - p_1\|^2 + (1 - t)\|p - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
&= t(1 - t)^2\|p_1 - p_2\|^2 + (1 - t)t^2\|p_1 - p_2\|^2 - t(1 - t)\|p_1 - p_2\|^2 \\
&= 0.
\end{aligned} \tag{2.12}$$

Hence $p = z$. Therefore, $p \in F(T)$. □

3. Main Results

In the following theorem, we introduce a new monotone hybrid iterative scheme for finding a common element of the set of a common fixed points of a countable family of nonexpansive multivalued maps, the set of variational inequality, and the set of solutions of an equilibrium problem in a Hilbert space, and we prove strong convergence theorem without the condition (I).

Theorem 3.1. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying (A1)–(A4), let $A : D \to H$ be an α-inverse strongly monotone mapping, and let $T_i : D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(D, A) \neq \emptyset$ and $T_i p = \{p\}$, $\forall p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$ for all $i \in \mathbb{N}$, $\{r_n\} \subset [b, \infty)$ for some $b \in (0, \infty)$, and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. For an initial point $x_0 \in H$ with $C_1 = D$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}$, $\{y_n\}$, $\{s_{i,n}\}$, and $\{u_n\}$ be sequences generated by*

$$\begin{gathered}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D, \\
y_n = P_D(u_n - \lambda_n A u_n), \\
s_{i,n} = \alpha_{i,n} y_n + (1 - \alpha_{i,n}) z_{i,n}, \\
C_{i,n+1} = \{z \in C_{i,n} : \|s_{i,n} - z\| \leq \|y_n - z\| \leq \|x_n - z\|\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{i,n+1}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{3.1}$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

Proof. We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}}x_0$ is well defined for every $x_0 \in H$.

Since $0 < c \le \lambda_n \le d < 2\alpha$ for all $n \in \mathbb{N}$, we get that $P_C(I - \lambda_n A)$ is nonexpansive for all $n \in \mathbb{N}$. Hence, $\bigcap_{n=1}^{\infty} F(P_C(I - \lambda_n A)) = VI(D, A)$ is closed and convex. By Lemma 2.6(4), we know that $EP(F)$ is closed and convex. By Lemma 2.7, we also know that $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex. Hence, $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(D, A)$ is a nonempty, closed and convex set. By Lemma 2.4, we see that $C_{i,n+1}$ is closed and convex for all $i, n \in \mathbb{N}$. This implies that C_{n+1} is also closed and convex. Therefore, $P_{C_{n+1}}x_0$ is well defined. Let $p \in \Omega$ and $i \in \mathbb{N}$. From $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \le \|x_n - p\| \tag{3.2}$$

for every $n \ge 0$. From this, we have

$$\begin{aligned}
\|s_{i,n} - p\| &= \|\alpha_{i,n}y_n + (1 - \alpha_{i,n})z_{i,n} - p\| \\
&\le \alpha_{i,n}\|y_n - p\| + (1 - \alpha_{i,n})\|z_{i,n} - p\| \\
&\le \alpha_{i,n}\|y_n - p\| + (1 - \alpha_{i,n})d(z_{i,n}, T_i p) \\
&\le \alpha_{i,n}\|y_n - p\| + (1 - \alpha_{i,n})H(T_i y_n, T_i p) \\
&\le \|y_n - p\| \\
&= \|P_D(u_n - \lambda_n A u_n) - P_D(p - \lambda_n A p)\| \\
&\le \|u_n - p\| \\
&\le \|x_n - p\|.
\end{aligned} \tag{3.3}$$

So, we have $p \in C_{i,n+1}$, hence $\Omega \subset C_{i,n+1}$, $\forall i \in \mathbb{N}$. This shows that $\Omega \subset C_{n+1} \subset C_n$.

Step 2. Show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Since Ω is a nonempty closed convex subset of H, there exists a unique $v \in \Omega$ such that

$$z_0 = P_\Omega x_0. \tag{3.4}$$

From $x_n = P_{C_n}x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \ge 0$, we get

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \quad \forall n \ge 0. \tag{3.5}$$

On the other hand, as $\Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \le \|z_0 - x_0\|, \quad \forall n \ge 0. \tag{3.6}$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 3. Show that $x_n \to q \in D$ as $n \to \infty$.

For $m > n$, by the definition of C_n, we see that $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By Lemma 2.2, we get

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.7}$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence, there exists $q \in D$ such that $x_n \to q$ as $n \to \infty$.

Step 4. Show that $q \in F$.

From Step 3, we get

$$\|x_{n+1} - x_n\| \longrightarrow 0 \tag{3.8}$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|s_{i,n} - x_n\| \leq \|s_{i,n} - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \longrightarrow 0 \tag{3.9}$$

as $n \to \infty$ for all $i \in \mathbb{N}$,

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \longrightarrow 0 \tag{3.10}$$

as $n \to \infty$. Hence, $y_n \to q$ as $n \to \infty$. It follows from (3.9) and (3.10) that

$$\|z_{i,n} - y_n\| = \frac{1}{1-\alpha_{i,n}}\|s_{i,n} - y_n\| \longrightarrow 0 \tag{3.11}$$

as $n \to \infty$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we have

$$\begin{aligned} d(q, T_i q) &\leq \|q - y_n\| + \|y_n - z_{i,n}\| + d(z_{i,n}, T_i q) \\ &\leq \|q - y_n\| + \|y_n - z_{i,n}\| + H(T_i y_n, T_i q) \\ &\leq \|q - y_n\| + \|y_n - z_{i,n}\| + \|y_n - q\|. \end{aligned} \tag{3.12}$$

From (3.11), we obtain $d(q, T_i q) = 0$. Hence $q \in F$.

Step 5. Show that $q \in EP(F)$.

By the nonexpansiveness of P_D and the inverse strongly monotonicity of A, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &\le \|u_n - \lambda_n Au_n - (p - \lambda_n Ap)\|^2 \\
&\le \|u_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Ap\|^2 \\
&= \|T_{r_n}x_n - T_{r_n}p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + c(d - 2\alpha)\|Au_n - Ap\|^2.
\end{aligned} \tag{3.13}$$

This implies

$$\begin{aligned}
c(2\alpha - d)\|Au_n - Ap\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).
\end{aligned} \tag{3.14}$$

It follows from (3.10) that

$$\lim_{n\to\infty} \|Au_n - Ap\| = 0. \tag{3.15}$$

Since P_D is firmly nonexpansive, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_D(u_n - \lambda_n Au_n) - P_D(p - \lambda_n Ap)\|^2 \\
&\le \langle u_n - \lambda_n Au_n) - (p - \lambda_n Ap), y_n - p\rangle \\
&= \frac{1}{2}\Big(\|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\|^2 \\
&\qquad + \|y_n - p\|^2 - \|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap) - (y_n - p)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) - \lambda_n(Au_n - Ap)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\|\Big).
\end{aligned} \tag{3.16}$$

This implies that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Ap\|. \tag{3.17}$$

It follows that

$$\begin{aligned}\|u_n - y_n\|^2 &\le \|x_n - y_n\|\left(\|x_n - p\| + \|y_n - p\|\right) \\ &\quad + 2d\|u_n - y_n\|\|Au_n - Ap\|.\end{aligned} \tag{3.18}$$

From (3.10) and (3.15), we get

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.19}$$

It follows from (3.10) and (3.19) that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.20}$$

Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in D. \tag{3.21}$$

From the monotonicity of F, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad \forall y \in D, \tag{3.22}$$

hence

$$\left\langle y - u_n, \frac{u_n - x_n}{r_n}\right\rangle \ge F(y, u_n), \quad \forall y \in D. \tag{3.23}$$

From (3.20) and condition (A4), we have

$$0 \ge F(y, q), \quad \forall y \in \mathrm{D}. \tag{3.24}$$

For t with $0 < t \le 1$ and $y \in D$, let $y_t = ty + (1-t)q$. Since $y, q \in D$ and D is convex, then $y_t \in D$ and hence $F(y_t, q) \le 0$. So, we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, q) \le tF(y_t, y). \tag{3.25}$$

Dividing by t, we obtain

$$F(y_t, y) \ge 0, \quad \forall y \in D. \tag{3.26}$$

Letting $t \downarrow 0$ and from (A3), we get

$$F(q, y) \geq 0, \quad \forall y \in D. \tag{3.27}$$

Therefore, we obtain $q \in EP(F)$.

Step 6. Show that $q \in VI(D, A)$.

Since T is the maximal monotone mapping defined by (2.10),

$$Tx = \begin{cases} Ax + N_D x, & x \in D, \\ \emptyset, & x \notin D. \end{cases} \tag{3.28}$$

For any given $(x, u) \in G(T)$, hence $u - Ax \in N_D x$. It follows that

$$\langle x - y_n, u - Ax \rangle \geq 0. \tag{3.29}$$

On the other hand, since $y_n = P_D(u_n - \lambda_n A u_n)$, we have

$$\langle x - y_n, y_n - (u_n - \lambda_n A u_n) \rangle \geq 0, \tag{3.30}$$

and so

$$\left\langle x - y_n, \frac{y_n - u_n}{\lambda_n} + A u_n \right\rangle \geq 0. \tag{3.31}$$

From (3.29), (3.31), and the α-inverse monotonicity of A, we have

$$\begin{aligned} \langle x - y_n, u \rangle &\geq \langle x - y_n, Ax \rangle \\ &\geq \langle x - y_n, Ax \rangle - \left\langle x - y_n, \frac{y_n - u_n}{\lambda_n} + A u_n \right\rangle \\ &= \langle x - y_n, Ax - Ay_n \rangle + \langle x - y_n, Ay_n - Au_n \rangle - \left\langle x - y_n, \frac{y_n - u_n}{\lambda_n} \right\rangle \\ &\geq \langle x - y_n, Ay_n - Au_n \rangle - \left\langle x - y_n, \frac{y_n - u_n}{\lambda_n} \right\rangle. \end{aligned} \tag{3.32}$$

It follows that

$$\lim_{n \to \infty} \langle x - y_n, u \rangle = \langle x - q, u \rangle \geq 0. \tag{3.33}$$

Again since T is maximal monotone, hence $0 \in Tq$. This shows that $q \in VI(D, A)$.

Step 7. Show that $q = z_0 = P_\Omega x_0$.

Since $x_n = P_{C_n} x_0$ and $\Omega \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - p\rangle \geq 0 \quad \forall p \in \Omega. \tag{3.34}$$

By taking the limit in (3.34), we obtain

$$\langle x_0 - q, q - p\rangle \geq 0 \quad \forall p \in \Omega. \tag{3.35}$$

This shows that $q = P_\Omega x_0 = z_0$.

From Steps 3 to 5, we obtain that $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$. This completes the proof. □

Theorem 3.2. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(D, A) \neq \emptyset$ and $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$ and $\{\lambda_n\} \subset [c,d]$ for some $c, d \in (0, 2\alpha)$. For an initial point $x_0 \in H$ with $C_1 = D$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}$, $\{y_n\}$, and $\{s_{i,n}\}$ be sequences generated by*

$$\begin{aligned}
y_n &= P_D(x_n - \lambda_n A x_n),\\
s_{i,n} &= \alpha_{i,n} y_n + (1 - \alpha_{i,n}) z_{i,n},\\
C_{i,n+1} &= \left\{z \in C_{i,n} : \|s_{i,n} - z\| \leq \|y_n - z\| \leq \|x_n - z\|\right\},\\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{i,n+1},\\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{3.36}$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

Proof. Putting $F(x, y) = 0$ for all $x, y \in D$ in Theorem 3.1, we obtain the desired result directly from Theorem 3.1. □

Theorem 3.3. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : D \to CB(D)$ be nonexpansive multivalued maps for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\alpha_{i,n} \in [0,1)$ with $\limsup_{n\to\infty} \alpha_{i,n} < 1$. For an initial point $x_0 \in H$ with $C_1 = D$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}$ and $\{s_{i,n}\}$ be sequences generated by*

$$\begin{aligned}
s_{i,n} &= \alpha_{i,n} x_n + (1 - \alpha_{i,n}) z_{i,n},\\
C_{i,n+1} &= \{z \in C_{i,n} : \|s_n - z\| \leq \|x_n - z\|\},\\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{i,n+1},\\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{3.37}$$

where $z_{i,n} \in T_i y_n$. Then, $\{x_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

Proof. Putting $A = 0$ in Theorem 3.2, we obtain the desired result directly from Theorem 3.2. □

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References

[1] C. Shiau, K. K. Tan, and C. S. Wong, "Quasi-nonexpansive multi-valued maps and selections," *Fundamenta Mathematicae*, vol. 87, pp. 109–119, 1975.
[2] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 838–844, 2009.
[3] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
[4] H. H. Bauschke, E. Matoušková, and S. Reich, "Projection and proximal point methods: convergence results and counterexamples," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 56, no. 5, pp. 715–738, 2004.
[5] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.
[6] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.
[7] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.
[8] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.
[9] K. P. R. Sastry and G. V. R. Babu, "Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point," *Czechoslovak Mathematical Journal*, vol. 55, no. 4, pp. 817–826, 2005.
[10] B. Panyanak, "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 54, no. 6, pp. 872–877, 2007.
[11] Y. Song and H. Wang, "Erratum to Mann and Ishikawa iterative process for multivalued mappings in Banach spaces," *Comput. Math. Appl.*, vol. 55, no. 6, pp. 2999–3002, 2008.
[12] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[13] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[14] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.
[15] P. Cholamjiak, "A hybrid iterative scheme for equilibrium problems, variational inequality problems, and fixed point problems in Banach spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 719360, 18 pages, 2009.
[16] P. Cholamjiak and S. Suantai, "A new hybrid algorithm for variational inclusions, generalized equilibrium problems, and a finite family of quasi-nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2009, Article ID 350979, 20 pages, 2009.
[17] W. Cholamjiak and S. Suantai, "Monotone hybrid projection algorithms for an infinitely countable family of Lipschitz generalized asymptotically quasi-nonexpansive mappings," *Abstract and Applied Analysis*, vol. 2009, Article ID 297565, 16 pages, 2009.
[18] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
[19] A. Kangtunyakarn and S. Suantai, "Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 3, pp. 296–309, 2009.

[20] J.-W. Peng, Y.-C. Liou, and J.-C. Yao, "An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions," *Fixed Point Theory and Applications*, vol. 2009, Article ID 794178, 21 pages, 2009.

[21] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[22] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 5, pp. 1140–1152, 2006.

[23] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[24] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 3, pp. 341–350, 2005.

[25] A. R. Khan, "Properties of fixed point set of a multivalued map," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2005, no. 3, pp. 323–331, 2005.