Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 364083, 13 pages
doi:10.1155/2010/364083

Research Article

On Global Attractivity of a Class of Nonautonomous Difference Equations

Wanping Liu,[1] Xiaofan Yang,[1,2] and Jianqiu Cao[2]

[1] *College of Computer Science, Chongqing University, Chongqing 400044, China*
[2] *School of Computer and Information, Chongqing Jiaotong University, Chongqing 400074, China*

Correspondence should be addressed to Wanping Liu, wanping_liu@yahoo.cn

Received 28 February 2010; Accepted 28 June 2010

Academic Editor: Guang Zhang

We mainly investigate the global behavior to the family of higher-order nonautonomous recursive equations given by $y_n = (p + ry_{n-s})/(q + \phi_n(y_{n-1}, y_{n-2}, \ldots, y_{n-m}) + y_{n-s})$, $n \in \mathbb{N}_0$, with $p \geq 0, r, q > 0, s, m \in \mathbb{N}$ and positive initial values, and present some sufficient conditions for the parameters and maps $\phi_n : (\mathbb{R}^+)^m \to \mathbb{R}^+, n \in \mathbb{N}_0$, under which every positive solution to the equation converges to zero or a unique positive equilibrium. Our main result in the paper extends some related results from the work of Gibbons et al. (2000), Iričanin (2007), and Stević (vol. 33, no. 12, pages 1767–1774, 2002; vol. 6, no. 3, pages 405–414, 2002; vol. 9, no. 4, pages 583–593, 2005). Besides, several examples and open problems are presented in the end.

1. Introduction

There has been a great interest in studying classes of nonlinear difference equations and systems, particularly those which model real situations in engineering and science, for example, [1–15]. On the other hand, non-autonomous difference equations also have a ubiquitous presence in applications from automatic controlling, ecology, economics, biology, population dynamics and so forth. Thus the main task when dealing them is to know the asymptotical behaviour of their solutions. For some recent advances in this area see [1, 16–24] and the references cited therein.

Gibbons et al. [25] discussed the behavior of nonnegative solutions to the rational recursive equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with $\alpha, \beta, \gamma \geq 0$, and also proposed an open problem, which had been solved by Stević in [4], concerning the particular case $\alpha = 0, \gamma = \beta$ in (1.1) (see also [26, 27] for the case of some related higher-order difference equations, as well as [28–30]).

In [3], Stević studied the behavior of nonnegative solutions of the following second-order difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{1 + g(x_n)}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $g: \mathbb{R}^+ \bigcup \{0\} \rightarrow \mathbb{R}$ is a nonnegative increasing mapping. Obviously (1.2) is a generalization of (1.1).

Later, Stević [6] extended (1.1) and (1.2) to the following more general equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{f(x_n, \dots, x_{n-k+1})}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $k \in \mathbb{N}$, $\alpha, \beta \geq 0$ and $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ is a continuous function nondecreasing in each variable such that $f(0, 0, \dots, 0) > 0$; and investigated the oscillatory behavior, the boundedness character and the global stability of nonnegative solutions to the equation.

Recently, Iričanin [2] studied the asymptotic behavior of the following class of autonomous difference equations:

$$x_n = \frac{\alpha x_{n-k}}{1 + x_{n-k} + f(x_{n-1}, \dots, x_{n-m})}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $\alpha > 0$, $k, m \in \mathbb{N}$ and f is a continuous mapping satisfying the condition

$$\beta \min\{u_1, \dots, u_m\} \leq f(u_1, u_2, \dots, u_m) \leq \beta \max\{u_1, \dots, u_m\}, \tag{1.5}$$

for certain $\beta \in (0, 1)$. In [2] he adopted the approach of frame sequences (a discrete analog of the method of frame curves used in the theory of differential equations), which has been used in the literature for many times, for example, [26–28, 30–38]; and showed that all positive solutions converge to zero if $0 < \alpha \leq 1$ and converge to the unique positive equilibrium if $\alpha > 1$.

Motivated by the above works, especially [2, 5], our aim in this paper is to study the global attractivity in the following family of non-autonomous difference equations:

$$y_n = \frac{p + r y_{n-s}}{q + \phi_n(y_{n-1}, \dots, y_{n-m}) + y_{n-s}}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $p \geq 0$, $r, q > 0$, $s, m \in \mathbb{N}$, and $\phi_n : (\mathbb{R}^+)^m \rightarrow \mathbb{R}^+$, $n \in \mathbb{N}_0$ are mappings satisfying the following condition

$$\beta \min\{x_1, \dots, x_m\} \leq \phi_n(x_1, x_2, \dots, x_m) \leq \beta \max\{x_1, \dots, x_m\}, \tag{1.7}$$

for some fixed $\beta \in (0, +\infty)$.

Through careful analysis, we find that the results in [2] also persist if the function f in (1.4) is replaced by variable functions such as ϕ_n satisfying condition (1.7). If $p = 0$, then (1.6) can be transformed into the following form

$$x_n = \frac{(r/q)x_{n-s}}{1 + G_n(x_{n-1}, \dots, x_{n-m}) + x_{n-s}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where $G_n(x_1, \dots, x_m) = \phi_n(qx_1, qx_2, \dots, qx_m)/q$, by setting $y_n = qx_n$. Then according to the results in [2], we have that if $r \le q$, then $\lim_{n\to\infty} y_n = 0$; and if $r > q$, then $\lim_{n\to\infty} y_n = q \lim_{n\to\infty} x_n = q((r/q - 1)/(1+\beta)) = (r-q)/(1+\beta)$, for some $\beta \in (0,1)$. Thus it suffices to consider the case when $p > 0$ in the following.

Note that if $p > 0$, then by relation (1.7), $\overline{y} = (\sqrt{(q-r)^2 + 4p(1+\beta)} + r - q)/2(1+\beta)$ is the unique positive equilibrium of (1.6). And in Section 3, we will prove the following main theorem.

Theorem 1.1. *Consider* (1.6), *where* $s, m \in \mathbb{N}$, $p, r, q > 0$ *with* $rq \ge p$, *and* $\phi_n : (\mathbb{R}^+)^m \to \mathbb{R}^+$ *are functions satisfying the condition*

$$\beta \min\{x_1, \dots, x_m\} \le \phi_n(x_1, x_2, \dots, x_m) \le \beta \max\{x_1, \dots, x_m\}, \tag{1.9}$$

for some fixed $\beta \in (0, +\infty)$. *If* $q \ge r$, $\beta \in (0, +\infty)$ *or* $q < r$, $\beta \in (0, \beta_0]$, *where* $\beta_0 = 4p/(q-r)^2 + 1$, *then the unique positive equilibrium* $\overline{y}$ *of* (1.6) *is a global attractor.*

2. Auxiliary Results

Before proving the main result of this paper, in this section we first confirm two preliminary lemmas.

Let $\Phi : \mathbb{R}^+ \bigcup \{0\} \times \mathbb{R}^+ \to \mathbb{R}$ be the mapping $\Phi(x, w) = (p + rw)/(q + \beta x + w)$, where $p, q, r, \beta > 0$ and $rq \ge p$, so as Φ is decreasing in the first variable and increasing in the second one. Then (1.6) can be simplified to the following form:

$$y_n = \Phi\left(\frac{\phi_n(y_{n-1}, \dots, y_{n-m})}{\beta}, y_{n-s}\right), \quad n \in \mathbb{N}_0. \tag{2.1}$$

Lemma 2.1. *Consider the following higher-order rational difference equation:*

$$w_n = \Phi(x, w_{n-s}), \quad n \in \mathbb{N}_0, \tag{2.2}$$

where $p, r, q, \beta \in \mathbb{R}^+$, $s \in \mathbb{N}$, *the parameter* $x \ge 0$ *and initial values* w_k, $k \in \{-1, \dots, -s\}$ *are arbitrary nonnegative numbers. Then every positive solution* $(w_n)_{n=-s}^{\infty}$ *to* (2.2) *converges to the unique positive equilibrium point*

$$S(x) = \frac{1}{2}\left(\sqrt{(q - r + \beta x)^2 + 4p} - (q - r + \beta x)\right). \tag{2.3}$$

Proof. First we show that (2.2) has a unique positive equilibrium. Assume that $\overline{w} > 0$ is an equilibrium point of (2.2), then $\overline{w} = (p + r\overline{w})/(q + \beta x + \overline{w})$ which implies only one positive root

$$\overline{w} = S(x) = \frac{1}{2}\left(\sqrt{(q-r+\beta x)^2 + 4p} - (q-r+\beta x)\right). \tag{2.4}$$

If $s \geq 2$, then (2.2) can be separated into s analogous first-order difference equations of the form

$$w_n^{(k)} = \frac{p + r w_{n-1}^{(k)}}{q + \beta x + w_{n-1}^{(k)}}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

with different initial values $w_{-1}^{(k)} = w_{-(k+1)}$, where $k \in \{0, 1, \ldots, s-1\}$. Note that the equation is Riccati, so it can be solved and the convergence of its solutions can be proved (see, e.g., [39] or a recent comment in [40]).

Let the symbol [ ] symbolize the greatest integer function and define a sequence $P(n) \equiv n(\mathrm{mod}\, s),\ n \in \mathbb{N}_0$. Obviously, for each positive solution $(w_n)_{n=-s}^{\infty}$ to (2.2) we have

$$w_n = w_{[n/s]}^{(s-P(n)-1)}, \quad n \geq -s. \tag{2.6}$$

From the above analysis, it suffices to prove the case when $s = 1$. Suppose that $s = 1$ for (2.2), then for all $n \in \mathbb{N}_0$, we have

$$w_{n+1} - w_n = \frac{(w_0 - w_{-1})\left(r(q+\beta x) - p\right)^{n+1}}{(q+\beta x + w_n)\left(\prod_{i=0}^{n-1}(q+\beta x + w_i)\right)^2 (q+\beta x + w_{-1})}, \tag{2.7}$$

$$w_{2k+2} - w_{2k} = \frac{\left(r(q+\beta x) - p\right)(w_{2k+1} - w_{2k-1})}{(q+\beta x + w_{2k+1})(q+\beta x + w_{2k-1})}, \quad k \in \mathbb{N}_0, \tag{2.8}$$

$$w_{2k+3} - w_{2k+1} = \frac{\left(r(q+\beta x) - p\right)(w_{2k+2} - w_{2k})}{(q+\beta x + w_{2k+2})(q+\beta x + w_{2k})}, \quad k \in \mathbb{N}_0. \tag{2.9}$$

Case 1. If $r(q+\beta x) \geq p$, then by (2.7) $(w_n)_{n=-1}^{\infty}$ is either nonincreasing or nondecreasing. On the other hand, we have that

$$\frac{\min\{p, r\}}{\max\{q+\beta x, 1\}} \leq w_n \leq \frac{\max\{p, r\}}{\min\{q+\beta x, 1\}} \tag{2.10}$$

for all $n \geq 0$. Therefore, the limit of $(w_n)_{n=-1}^{\infty}$ exists, and through simple calculations, we get $\lim_{n\to\infty} w_n = S(x)$.

Case 2. If $r(q + \beta x) < p$, then by (2.8) and (2.9) and inductively we have that (w_{2k}) is nonincreasing and (w_{2k-1}) nondecreasing, or (w_{2k}) is nondecreasing and (w_{2k-1})

nonincreasing. Again by (2.10), the limits of (w_{2k}) and (w_{2k-1}) exist, denoted by $\lim_{k\to\infty} w_{2k} = \alpha$ and $\lim_{k\to\infty} w_{2k-1} = \gamma$. From (2.2) we have

$$\alpha = \frac{p + r\gamma}{q + \beta x + \gamma}, \qquad \gamma = \frac{p + r\alpha}{q + \beta x + \alpha}, \tag{2.11}$$

which imply that $\alpha = \gamma = \overline{w}$. Hence $\lim_{n\to\infty} w_n = S(x)$.

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Suppose that the parameters, in* (2.3)*, satisfy* $p, r, q, \beta > 0$ *with* $t = q - r$*. Define two sequences* $(M_k)_{k=1}^{\infty}$ *and* $(m_k)_{k=1}^{\infty}$ *as follows:*

$$\begin{aligned} m_k &= S\left(M_k + \frac{\varepsilon}{k}\right), \quad k = 1, 2, \ldots, \\ M_k &= S\left(m_{k-1} - \frac{\varepsilon}{k-1}\right), \quad k = 2, 3, \ldots, \end{aligned} \tag{2.12}$$

where the initial value $M_1 = S(0)$*, and* $\varepsilon \in (0, \lambda)$*,*

$$\lambda = \frac{1}{2(1+\beta)}\left(\sqrt{(t + \beta M_1)^2 + 4p(1+\beta)} - (t + \beta M_1)\right). \tag{2.13}$$

If $q \geq r,\ \beta \in (0, +\infty)$ *or* $q < r,\ \beta \in (0, \beta_0]$*, where* $\beta_0 = 4p/(q-r)^2 + 1$*, then*

$$\lim_{k\to\infty} M_k = \lim_{k\to\infty} m_k. \tag{2.14}$$

Proof. By simple calculations, we have

$$\begin{aligned} M_2 - M_1 &= S(m_1 - \varepsilon) - S(0) \\ &= \frac{1}{2}\left(\sqrt{(t + \beta(m_1 - \varepsilon))^2 + 4p} - \sqrt{t^2 + 4p} - \beta(m_1 - \varepsilon)\right) \\ &= \frac{1}{2}\left(\frac{\beta^2(m_1 - \varepsilon)^2 + 2\beta t(m_1 - \varepsilon)}{\sqrt{(t + \beta(m_1 - \varepsilon))^2 + 4p} + \sqrt{t^2 + 4p}} - \beta(m_1 - \varepsilon)\right) \\ &= -\beta(m_1 - \varepsilon)\left(\frac{S(m_1 - \varepsilon) + S(0)}{\sqrt{(t + \beta(m_1 - \varepsilon))^2 + 4p} + \sqrt{t^2 + 4p}}\right). \end{aligned} \tag{2.15}$$

Obviously, $S(m_1 - \varepsilon) + S(0) > 0$.

Claim 1. $m_1 - \varepsilon > 0$.

Proof of Claim 1. Define a function $f(x) = 2(S(M_1 + x) - x)$. It suffices to prove that $f(x) > 0$ for all $x \in (0, \lambda)$. The derivative of $f(x)$ is

$$\frac{df}{dx} = \frac{\beta\left[t + \beta(M_1 + x)\right]}{\sqrt{\left[t + \beta(M_1 + x)\right]^2 + 4p}} - \beta - 2 < 0. \tag{2.16}$$

Since $f(\lambda) = 0$ and $f(0) > 0$, thus $f(x) > 0$ for $x \in (0, \lambda)$. □

Therefore, it follows from (2.15) and Claim 1 that

$$M_2 - M_1 < 0. \tag{2.17}$$

Denote

$$\begin{aligned} L_k &= \frac{2t + \beta(M_{k+1} + M_k + \varepsilon/k + \varepsilon/(k+1))}{\sqrt{\left[t + \beta(M_{k+1} + \varepsilon/(k+1))\right]^2 + 4p} + \sqrt{\left[t + \beta(M_k + \varepsilon/k)\right]^2 + 4p}}, \\ Q_k &= \frac{2t + \beta(m_k + m_{k-1} - \varepsilon/k - \varepsilon/(k-1))}{\sqrt{\left[t + \beta(m_k - \varepsilon/k)\right]^2 + 4p} + \sqrt{\left[t + \beta(m_{k-1} - \varepsilon/(k-1))\right]^2 + 4p}}. \end{aligned} \tag{2.18}$$

Simply, we obtain that $|L_k| < 1$ and $|Q_k| < 1$.

Observe that

$$\begin{aligned} 2(m_{k+1} - m_k) &= \beta(1 - L_k)\left[M_k - M_{k+1} + \frac{\varepsilon}{k(k+1)}\right], \quad k = 1, 2, \ldots, \\ 2(M_{k+1} - M_k) &= \beta(Q_k - 1)\left[m_k - m_{k-1} + \frac{\varepsilon}{k(k+1)}\right], \quad k = 2, 3, \ldots. \end{aligned} \tag{2.19}$$

With (2.17) and (2.19), it follows by induction that $(m_k)_{k=1}^{\infty}$, $(M_k)_{k=1}^{\infty}$ are strictly increasing and decreasing, respectively. In addition, $M_k > 0$, $k = 1, 2, \ldots$, hence $(M_k)_{k=1}^{\infty}$ possesses a finite limit denoted by $\varphi = \lim_{k \to \infty} M_k$. From (2.12), we know that the limit of $(m_k)_{k=1}^{\infty}$ (denoted by $\mu = \lim_{k \to \infty} m_k$) also exists. Therefore, taking limits on both sides of (2.12), we have

$$\begin{aligned} \mu &= S(\varphi), \\ \varphi &= S(\mu) \end{aligned} \tag{2.20}$$

which imply that

$$\begin{aligned} \mu^2 + t\mu + \beta\mu\varphi &= p, \\ \varphi^2 + t\varphi + \beta\mu\varphi &= p, \end{aligned} \tag{2.21}$$

$$(\mu - \varphi)(\mu + \varphi + t) = 0. \tag{2.22}$$

Claim 2. If $q \geq r,\ \beta \in (0, +\infty)$ or $q < r,\ \beta \in (0, \beta_0]$, then $\mu = \varphi$.

Proof of Claim 2. Suppose that $\mu \neq \varphi$, then it follows from (2.22) that $\mu = -\varphi - t$. By substituting $\mu = -\varphi - t$ into the second identity of (2.21), we get

$$(1 - \beta)\varphi^2 + t(1 - \beta)\varphi - p = 0. \tag{2.23}$$

(i) If $\beta = 1$, then $p = 0$ which is a contradiction to $p > 0$,

(ii) If $\beta \in (0, 1)$, then the unique positive root of (2.23) is

$$\varphi = \frac{\sqrt{t^2 + 4p/(1 - \beta)} - t}{2} > M_1. \tag{2.24}$$

However, $\varphi < M_1$ since (M_k) is strictly decreasing.

(iii) If $q = r,\ \beta \in (1, +\infty)$, then (2.23) reduces to $0 > (1 - \beta)\varphi^2 = p > 0$.

(iv) If $q \neq r,\ \beta \in (1, \beta_0)$, then for (2.23), $\Delta = t^2(1 - \beta)^2 + 4p(1 - \beta) < 0$ which implies that (2.23) has no real roots.

(v) For $q > r,\ \beta = \beta_0$, we have $\Delta = 0$. So, $\varphi = (r - q)/2 < 0$ which is contradictive to $\varphi \geq 0$.

(vi) For $q > r,\ \beta \in (\beta_0, +\infty)$, (2.23) has two negative roots.

(vii) For $q < r,\ \beta = \beta_0$. Solving (2.23), we get $\varphi = (r - q)/2$ implying $\mu = (r - q)/2$. Hence $\mu = \varphi$, which contradicts the assumption.

Obviously Claim 2 follows directly from (i)–(vii). □

Applying Claim 2 and (2.21), we conclude that

$$\lim_{k \to \infty} M_k = \lim_{k \to \infty} m_k = \frac{\sqrt{t^2 + 4p(1 + \beta)} - t}{2(1 + \beta)}. \tag{2.25}$$

Hence the lemma is complete. □

3. Main Results

Obviously, condition (1.7) in Section 1 guarantees the fact that (1.6) possesses a unique equilibrium point $\overline{y} = (\sqrt{t^2 + 4p(1+\beta)} - t))/(2(1+\beta))$, where $t = q - r$.

First, we present a proposition concerning the boundedness of all positive solutions to (1.6).

Proposition 3.1. *Consider* (1.6) *with condition* (1.7) *and* $p, q, r \in \mathbb{R}^+$, *then every positive solution to* (1.6) *has permanent bounds.*

Proof. Let (y_n) be a solution to (1.6) with positive initial values. Then, we have

$$\begin{aligned} y_n &= \frac{p + r y_{n-s}}{q + \phi_n(y_{n-1}, \ldots, y_{n-m}) + y_{n-s}} \le \frac{\max\{p, r\}}{\min\{q + \phi_n(y_{n-1}, \ldots, y_{n-m}), 1\}} \\ &\le \frac{\max\{p, r\}}{\min\{q, 1\}} = U, \quad \forall n \ge 0, \\ y_n &= \frac{p + r y_{n-s}}{q + \phi_n(y_{n-1}, \ldots, y_{n-m}) + y_{n-s}} \ge \frac{\min\{p, r\}}{\max\{q + \phi_n(y_{n-1}, \ldots, y_{n-m}), 1\}} \\ &\ge \frac{\min\{p, r\}}{\max\{q + \beta U, 1\}} = L, \quad \forall n \ge m. \end{aligned} \tag{3.1}$$

Thus we have $L \le y_n \le U$, for all $n \ge m$. □

In the following, we will give the proof of the main result (i.e., Theorem 1.1) presented in Section 1.

Proof of Theorem 1.1. Let $\varepsilon \in \mathbb{R}^+$ be an arbitrary fixed number satisfying $0 < \varepsilon < \lambda$ (λ defined by (2.13) in Lemma 2.2). Define two sequences $(M_k)_{k=1}^{\infty}, (m_k)_{k=1}^{\infty}$ as shown by (2.12) in Lemma 2.2. Let (y_n) be any positive solution to (1.6). In the following, we proceed by presenting two claims.

Claim 1. There exists $N_1 \in \mathbb{N}$, such that $m_1 - \varepsilon \le y_n \le M_1 + \varepsilon$ for all $n \ge \mathbb{N}_1$.

Proof of Claim 1. From (2.1), we have that

$$y_n = \Phi\left(\frac{\phi_n(y_{n-1}, \ldots, y_{n-m})}{\beta}, y_{n-s}\right) \le \Phi(0, y_{n-s}). \tag{3.2}$$

Suppose that (x_n) is a solution to the following difference equation

$$x_n = \Phi(0, x_{n-s}), \quad n \in \mathbb{N}_0, \tag{3.3}$$

with initial values $x_{-s} = y_{-s}, \ldots, x_{-1} = y_{-1}$. From this and in view of the monotonicity of the function $f(x) = (p + rx)/(q + x)$, $x \in \mathbb{R}^+$, by induction we can easily get that $y_n \le x_n$ for $n \ge -s$.

By Lemma 2.1, $\lim_{n\to\infty}x_n = M_1$. Hence, there exists $b_1 \in \mathbb{N}$ such that $x_n \le M_1 + \varepsilon$ for $n \ge b_1$, then

$$y_n \le M_1 + \varepsilon \tag{3.4}$$

for all $n \ge b_1$.

From (2.1), (1.7), and (3.4), it follows that

$$y_n \ge \Phi(\max\{y_{n-1}, \dots, y_{n-m}\}, y_{n-s}) \ge \Phi(M_1 + \varepsilon, y_{n-s}) \tag{3.5}$$

for all $n \ge b_1 + m$.

Suppose that (u_n) is a solution to the following difference equation:

$$u_n = \Phi(M_1 + \varepsilon, u_{n-s}), \quad n \in \mathbb{N}_0, \tag{3.6}$$

with initial values $u_{b_1+m-s} = y_{b_1+m-s}, \dots, u_{b_1+m-1} = y_{b_1+m-1}$.

Since the function $h(x) = \Phi(M_1 + \varepsilon, x)$ is increasing on the interval $(0, +\infty)$, we can easily get by induction that $y_n \ge u_n$ for $n \ge b_1 + m - s$, and by Lemma 2.1, $\lim_{n\to\infty}u_n = m_1$. Hence there exists a natural number $N_1 \ge b_1$ such that $u_n \ge m_1 - \varepsilon$ for $n \ge N_1$, then $m_1 - \varepsilon \le y_n \le M_1 + \varepsilon$ for $n \ge N_1$. □

Working inductively, we will eventually reach the following claim.

Claim 2. For each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that $m_k - \varepsilon/k \le y_n \le M_k + \varepsilon/k$ for all $n \ge N_k$.

Proof of Claim 2. By Claim 1, if $k = 1$, we have $N_1 \in \mathbb{N}$ such that $m_1 - \varepsilon \le y_n \le M_1 + \varepsilon$ for all $n \ge N_1$. Then by the method of induction, we can assume that for $k \in \mathbb{N}$ fixed, there exists $N_k \in \mathbb{N}$ such that $m_k - \varepsilon/k \le y_n \le M_k + \varepsilon/k$ for all $n \ge N_k$. Thus, it suffices to show that there exists $N_{k+1} \in \mathbb{N}$ such that $m_{k+1} - \varepsilon/(k+1) \le y_n \le M_{k+1} + \varepsilon/(k+1)$ for all $n \ge N_{k+1}$.

Let $z = \max\{s, m\}$. Define a sequence $(x_n^{(k+1)})$ as follows

$$x_n^{(k+1)} = \Phi\left(S^{-1}(M_{k+1}), x_{n-s}^{(k+1)}\right), \quad n \ge N_k + z, \tag{3.7}$$

with $x_n^{(k+1)} = y_n$, for $n = N_k, \dots, N_k + z - 1$.

By reasoning inductively on $n \ge N_k + z$, one has

$$\begin{aligned} y_n &\le \Phi(\min\{y_{n-1}, \dots, y_{n-m}\}, y_{n-s}) \le \Phi\left(S^{-1}(M_{k+1}), y_{n-s}\right) \\ &\le \Phi\left(S^{-1}(M_{k+1}), x_{n-s}^{(k+1)}\right) = x_n^{(k+1)}, \quad \forall n \ge N_k + z. \end{aligned} \tag{3.8}$$

By Lemma 2.1, $\lim_{n\to\infty}x_n^{(k+1)} = M_{k+1}$. Therefore, there is $b_{k+1} \ge N_k$ such that $y_n \le M_{k+1} + \varepsilon/(k+1)$ for all $n \ge b_{k+1}$.

Define the other sequence $(u_n^{(k+1)})$ as follows:

$$u_n^{(k+1)} = \Phi\left(S^{-1}(m_{k+1}), u_{n-s}^{(k+1)}\right), \quad \text{for } n \geq b_{k+1}, \tag{3.9}$$

where $u_n^{(k+1)} = y_n$, for $n = b_{k+1}, \ldots, b_{k+1} + z - 1$.

Once more, by induction on $n \geq b_{k+1} + z$,

$$\begin{aligned} y_n &\geq \Phi(\max\{y_{n-1}, \ldots, y_{n-m}\}, y_{n-s}) \geq \Phi\left(M_{k+1} + \frac{\varepsilon}{k+1}, y_{n-s}\right) \\ &\geq \Phi\left(S^{-1}(m_{k+1}), u_{n-s}^{(k+1)}\right) = u_n^{(k+1)}, \quad \forall n \geq b_{k+1} + z. \end{aligned} \tag{3.10}$$

By Lemma 2.1, $\lim_{n\to\infty} u_n^{(k+1)} = m_{k+1}$. Thus, let $N_{k+1} \geq b_{k+1}$ be greater enough so as $y_n \geq m_{k+1} - (\varepsilon/(k+1)) = S^{-1}(M_{k+2})$ for all $n \geq N_{k+1}$.

Therefore, we get that there exists $N_{k+1} \in \mathbb{N}$ such that

$$m_{k+1} - \frac{\varepsilon}{k+1} \leq y_n \leq M_{k+1} + \frac{\varepsilon}{k+1} \tag{3.11}$$

for all $n \geq N_{k+1}$. □

By Claim 2, we have

$$\lim_{k\to\infty} m_k = \lim_{k\to\infty}\left(m_k - \frac{\varepsilon}{k}\right) \leq \liminf_{n\to\infty} y_n \leq \limsup_{n\to\infty} y_n \leq \lim_{k\to\infty}\left(M_k + \frac{\varepsilon}{k}\right) = \lim_{k\to\infty} M_k. \tag{3.12}$$

This plus Lemma 2.2 leads to

$$\lim_{n\to\infty} y_n = \overline{y} = \frac{\sqrt{t^2 + 4p(1+\beta)} - t}{2(1+\beta)}. \tag{3.13}$$

The proof is complete. □

4. Applications and Future Work

Next, several examples are presented.

Example 4.1. Let $p_n \in (0, +\infty)$ for all $n \in \mathbb{N}_0$, and

$$\phi_n(x_1, x_2, \ldots, x_m) = \beta \sqrt[p_n]{\frac{\sum_{i=1}^m x_i^{p_n}}{m}}, \quad n \in \mathbb{N}_0 \tag{4.1}$$

for some $\beta > 0$. If $rq > p$ and $q \geq r$, $\beta \in (0, +\infty)$ or $q < r$, $\beta \in (0, \beta_0]$, where $\beta_0 = 4p/(q-r)^2 + 1$, then by Theorem 1.1 we conclude that every positive solution to the following non-autonomous difference equation:

$$y_n = \frac{p + r y_{n-s}}{q + \beta \sqrt[p_n]{\left(\sum_{i=1}^{m} y_{n-i}^{p_n}\right)/m} + y_{n-s}}, \quad n \in \mathbb{N}_0, \tag{4.2}$$

converges to the unique positive equilibrium point $\overline{y} = (\sqrt{(q-r)^2 + 4p(1+\beta)} + r - q)/2(1+\beta)$.

Example 4.1 extends Example 2.4 in [2].

Example 4.2. Let $\phi_n(x_1, x_2, \ldots, x_m) = \beta \max\{x_1, x_2, \ldots, x_m\}$ for all $n \in \mathbb{N}_0$, then under the conditions of Theorem 1.1, all positive solutions to the recursive equation

$$y_n = \frac{p + r y_{n-s}}{q + \beta \max\{y_{n-1}, y_{n-2}, \ldots, y_{n-m}\} + y_{n-s}}, \quad n \in \mathbb{N}_0, \tag{4.3}$$

converge to the unique positive equilibrium $\overline{y} = (\sqrt{(q-r)^2 + 4p(1+\beta)} + r - q)/2(1+\beta)$.

In this paper, the behavior of positive solutions to the case when $rq \geq p$, $q < r$, $\beta \in (\beta_0, +\infty)$, where $\beta_0 = 4p/(q-r)^2 + 1$, isn't investigated, since we have no further new ideas for the particular case. Through certain calculations, easily we know that the equation $S \circ S(x) = x$ has two different positive roots, if $q < r$, $\beta \in (\beta_0, +\infty)$, which implies $\lim_{k\to\infty} M_k > \lim_{k\to\infty} m_k$. From this we propose the following open problem.

Open Problem. Is there a positive solution (y_n) to (1.6) with condition (1.7) when $rq \geq p$, $q < r$, $\beta \in (\beta_0, +\infty)$, where $\beta_0 = 4p/(q-r)^2 + 1$, such that (y_n) eventually converges to a periodic solution?

Furthermore, the case $rq < p$ for (1.6) is also of extreme value to study.

Acknowledgments

The authors are grateful to the referees for their huge number of valuable suggestions, which considerably improved the presentation in the paper. Besides, the authors thank Professor. Iričanin for very valuable comments regarding this subject. This work was financially supported by National Natural Science Foundation of China (no. 10771227).

References

[1] J. S. Cánovas, A. Linero Bas, and G. Soler López, "On global periodicity of difference equations," *Taiwanese Journal of Mathematics*, vol. 13, no. 6B, pp. 1963–1983, 2009.

[2] B. D. Iričanin, "Dynamics of a class of higher order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 73849, 6 pages, 2007.

[3] S. Stević, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-1})/(1 + g(x_n))$," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 12, pp. 1767–1774, 2002.

[4] S. Stević, "On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$," *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 405–414, 2002.

[5] S. Stević, "The recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$," *Applied Mathematics Letters*, vol. 15, no. 3, pp. 305–308, 2002.

[6] S. Stević, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k})/f(x_n, \cdots, x_{n-k+1})$," *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 583–593, 2005.

[7] S. Stević, "Asymptotic behavior of a class of nonlinear difference equations," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 47156, 10 pages, 2006.

[8] S. Stević, "On monotone solutions of some classes of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 53890, 9 pages, 2006.

[9] S. Stević, "Global stability and asymptotics of some classes of rational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.

[10] S. Stević, "Asymptotics of some classes of higher-order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.

[11] S. Stević and K. S. Berenhaut, "The behavior of positive solutions of a nonlinear second-order difference equation," *Abstract and Applied Analysis*, vol. 2008, Article ID 653243, 8 pages, 2008.

[12] T. Sun and H. Xi, "Global behavior of the nonlinear difference equation $x_{n+1} = f(x_{n-s}, x_{n-t})$," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 760–765, 2005.

[13] T. Sun, H. Xi, and L. Hong, "On the system of rational difference equations $x_{n+1} = f(x_n, y_{n-k})$, $y_{n+1} = f(y_n, x_{n-k})$," *Advances in Difference Equations*, vol. 2006, Article ID 16949, 7 pages, 2006.

[14] X. Yang, L. Cui, Y. Y. Tang, and J. Cao, "Global asymptotic stability in a class of difference equations," *Advances in Difference Equations*, vol. 2007, Article ID 16249, 7 pages, 2007.

[15] X. Yang, Y. Y. Tang, and J. Cao, "Global asymptotic stability of a family of difference equations," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2643–2649, 2008.

[16] E. M. Elsayed and B. D. Iričanin, "On a max-type and a min-type difference equation," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 608–614, 2009.

[17] E. M. Elsayed, B. D. Iričanin, and S. Stević, "On the max-type equation $x_{n+1} = \max\{A_n/x_n, x_{n-1}\}$," *Ars Combinatoria*. In press.

[18] E. M. Elsayed and S. Stević, "On the max-type equation $x_{n+1} = \max\{A/x_n, x_{n-2}\}$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 910–922, 2009.

[19] J. Feuer, "On the eventual periodicity of $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ with a period-four parameter," *Journal of Difference Equations and Applications*, vol. 12, no. 5, pp. 467–486, 2006.

[20] B. Iričanin and S. Stević, "On a class of third-order nonlinear difference equations," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 479–483, 2009.

[21] C. M. Kent and M. A. Radin, "On the boundedness nature of positive solutions of the difference equation $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$ with periodic parameters," *Dynamics of Continuous, Discrete & Impulsive Systems. Series B*, pp. 11–15, 2003.

[22] S. Stević, "Global stability of a difference equation with maximum," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 525–529, 2009.

[23] S. Stević, "Global stability of a max-type difference equation," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 354–356, 2010.

[24] S. Stević, "On a generalized max-type difference equation from automatic control theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1841–1849, 2010.

[25] C. H. Gibbons, M. R. S. Kulenović, and G. Ladas, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-1})/(\gamma + x_n)$," *Mathematical Sciences Research Hot-Line*, vol. 4, no. 2, pp. 1–11, 2000.

[26] L. Berg, "On the asymptotics of the difference equation $x_n = x_n(1 + x_{n-1}x_{n-2})$," *Journal of Difference Equations and Applications*, vol. 14, no. 1, pp. 105–108, 2008.

[27] L. Berg and S. Stević, "On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$," *Journal of Difference Equations and Applications*. In press.

[28] L. Berg, "On the asymptotics of nonlinear difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.

[29] S. Stević, "More on the difference equation $x_{n+1} = x_{n-1}/(1 + x_{n-1}x_n)$," *Applied Mathematics E-Notes*, vol. 4, pp. 80–85, 2004.

[30] S. Stević and G. L. Karakostas, "On the recursive sequence $x_{n+1} = B + x_{n-k}/(\alpha_0 x_n + \cdots + \alpha_{k-1}x_{n-k+1} + \gamma)$," *Journal of Difference Equations and Applications*, vol. 10, no. 9, pp. 809–815, 2004.

[31] L. Berg, "Inclusion theorems for non-linear difference equations with applications," *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.

[32] L. Berg, "Corrections to: "Inclusion theorems for non-linear difference equations with applications"," *Journal of Difference Equations and Applications*, vol. 11, no. 2, pp. 181–182, 2005.

[33] L. Berg, *Asymptotische Darstellungen und Entwicklungen*, Deutscher Verlag der Wissenschaften, Berlin, Germany, 1968.

[34] K. S. Berenhaut, J. D. Foley, and S. Stević, "Quantitative bounds for the recursive sequence $y_{n+1} = A + y_n/y_{n-k}$," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 983–989, 2006.

[35] R. DeVault, G. Ladas, and S. W. Schultz, "On the recursive sequence $x_{n+1} = A/x_n + 1/x_{n-1}$," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3257–3261, 1998.

[36] P. Liu and X. Cui, "Hyperbolic logistic difference equation with infinitely many delays," *Mathematics and Computers in Simulation*, vol. 52, no. 3-4, pp. 231–250, 2000.

[37] S. Stević, "On positive solutions of a $(k+1)$th order difference equation," *Applied Mathematics Letters*, vol. 19, no. 5, pp. 427–431, 2006.

[38] S. Stević, "Existence of nontrivial solutions of a rational difference equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.

[39] H. Levy and F. Lessman, *Finite Difference Equations*, Dover, New York, NY, USA, 1961.

[40] B. Iričanin and S. Stević, "On some rational difference equations," *Ars Combinatoria*, vol. 92, pp. 67–72, 2009.