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Research Article

Adaptive Pinning Synchronization of Complex Networks with Stochastic Perturbations

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The adaptive pinning synchronization is investigated for complex networks with nondelayed and delayed couplings and vector-form stochastic perturbations. Two kinds of adaptive pinning controllers are designed. Based on an Lyapunov-Krasovskii functional and the stochastic stability analysis theory, several sufficient conditions are developed to guarantee the synchronization of the proposed complex networks even if partial states of the nodes are coupled. Furthermore, three examples with their numerical simulations are employed to show the effectiveness of the theoretical results.

1. Introduction

Recently, synchronization of all dynamical nodes in a network is one of the hot topics in the investigation of complex networks. It is well known that there are many useful synchronization phenomena in real life, such as the synchronous transfer of digital or analog signals in communication networks. Adaptive feedback control has witnessed its effectiveness in synchronizing a complex network [1–4]. By using the adaptive feedback control scheme, Chen and Zhou [1] studied synchronization of complex nondelayed networks, Cao et al. [2] investigated the complete synchronization in an array of linearly stochastically coupled identical networks with delays. In [3], Zhou et al. considered complex dynamical networks with uncertain couplings. In [4], Lu et al. studied the synchronization in arrays of delay-coupled neural networks. However, it is assumed that all the nodes need to be controlled in [1–4]. As we know, the real-world complex networks normally have a large number of nodes; it is usually impractical and impossible to control a complex network by adding the controllers to all nodes. To overcome this difficulty, pinning control, in which

controllers are only applied to a small fraction of nodes, has been introduced in recent years [5–9]. By using adaptive pinning control method, Zhou et al. [10] studied local and global synchronizations of complex networks without delays; authors of [11, 12] considered the global synchronizations of the complex networks with nondelayed and delayed couplings. Note that, in most of existing results of complex networks' synchronization, all of the states are coupled for connected nodes. However, adaptive pinning synchronization results in which only partial states of the nodes are coupled are few. Hence, in this paper, we consider two different adaptive pinning controllers, which synchronize complex networks with partial or complete couplings of the nodes' states.

In the process of studying synchronization of complex networks, two main factors should be considered: time delays and stochastic perturbations. Time delays commonly exist in the real world and even vary according to time. In the subsystems, time delay can give rise to chaos, such as delayed neural network and delayed Chua's circuit system in Section 4 of this paper. Moreover, time-delayed couplings between subsystems cannot be ignored, such as in long-distance communication and traffic congestions, and so forth. In [8], authors investigated synchronization of complex networks with delayed subsystems, while in [4, 12, 13], authors investigated synchronization of complex networks with coupling delays. On the other hand, in real world, due to random uncertainties, such as stochastic forces on the physical systems and noisy measurements caused by environmental uncertainties, a stochastic behavior should be produced instead of a deterministic one. In fact, signals transmitted between subsystems of complex networks are unavoidably subject to stochastic perturbations from environment, which may cause information contained in these signals to be lost [2]. Therefore, transmitted signals may not be fully detected and received by other subsystems. This can have a great influence on the behavior of complex networks. There are some works in the field of synchronization of complex networks [2, 14–16]. Noise perturbations in [2, 14, 15] are all one-dimensional, which means that the signal transmitted by subsystems is influenced by the same noise. In [16], Yang and Cao considered stochastic synchronization of coupled neural networks with intermittent control, in which noise perturbations are vector forms. Vector-form perturbation means that different subsystem is influenced by different noise, and hence is practical in the real world.

Based on the above analysis, in this paper, we study the synchronization of complex networks with time-varying delays and vector-form stochastic elements. Two different adaptive pinning controllers according to the different properties of inner couplings are considered. To obtain our main results, we first formulate a new complex network with nondelayed and delayed couplings and vector-form Wiener processes. Then, by virtue of an Lyapunov-Krasovskii functional and the stochastic stability analysis theory, we develop several theoretical results guaranteeing the synchronization of the new complex networks. Our adaptive pinning controllers are simple. The coupling matrices can be symmetric or asymmetric. Numerical simulations testify the effectiveness of our theoretical results.

Notations

In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. I_N denotes the $N \times N$ identity matrix. The Euclidean norm in $\mathbb{R}^n$ is denoted as $\|\cdot\|$; accordingly, for vector $x \in \mathbb{R}^n$, $\|x\| = x^T x$, where T denotes transposition. $G = (g_{ij})_{N\times N}$ denotes a matrix of N-dimension, $\|G\| = \sqrt{\lambda_{\max}(G^T G)}$, where $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of matrix A. $G^s = (1/2)(G+G^T)$. $G > 0$ or $G < 0$ denotes that the matrix G is symmetric and positive or negative definite matrix, respectively.

G_l denotes the matrix of the first l row-column pairs of G. G_{N-l} denotes the minor matrix of matrix G by removing all the first l row-column pairs of G. $E[\cdot]$ is the mathematical expectation.

The rest of this paper is organized as follows. In Section 2, new model of delayed complex networks with nondelayed and delayed couplings and vector-form stochastic Weiner processes is presented. Some necessary assumptions, definitions, and lemmas are also given in this section. Our main results and their rigorous proofs are described in Section 3. In Section 4, two directed networks and one Barabási-Albert (BA) network [17] with their numerical simulations are employed to show the effectiveness of our results. Finally, in Section 5, conclusions are given.

2. Preliminaries

Consider complex a network consisting of N identical nodes with nondelayed and delayed linear couplings and vector-form stochastic perturbations, which is described as

$$\begin{aligned} \mathrm{d}x_i(t) = &\left[f(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t) + d\sum_{j=1}^{N} u_{ij} D x_j(t-\tau(t)) \right] \mathrm{d}t \\ &+ h_i(x_1(t), \ldots, x_N(t))\mathrm{d}\omega_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{2.1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ represents the state vector of the ith node of the network at time t, $\tau(t) \geq 0$ is time-varying delay, and $f(t, x_i(t), x_i(t-\tau(t))) = [f_1(t, x_i(t), x_i(t-\tau(t))), f_2(t, x_i(t), x_i(t-\tau(t))), \ldots, f_n(t, x_i(t), x_i(t-\tau(t)))]^T$ is a continuous vector-form function. Constants $c > 0$, $d > 0$ are nondelayed coupling strength and delayed coupling strength, respectively. Matrices $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ are inner couplings of the networks at time t and $t-\tau(t)$, respectively, satisfying $\gamma_j \geq 0$, not all $\gamma_j = 0$, $d_j \geq 0$, $j = 1, 2, \ldots, n$. Matrices $G = (g_{ij})_{N\times N}$, and $U = (u_{ij})_{N\times N}$ are outer couplings of the networks at time t and $t-\tau(t)$, respectively, satisfying $g_{ij} \geq 0$ for $i \neq j$, $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$, and $u_{ij} \geq 0$ for $i \neq j$, $u_{ii} = -\sum_{j=1, j\neq i}^{N} u_{ij}$, $i = 1, 2, \ldots, N$. Matrix $h_i(x_1(t), x_2(t), \ldots, x_N(t)) \in \mathbb{R}^{n\times n}$ describes the unknown coupling of the complex networks satisfying $h_i(x(t), x(t), \ldots, x(t)) = 0$ (zero matrix of n dimension). $\omega_i(t) = (\omega_{i1}(t), \omega_{i2}(t), \ldots, \omega_{in}(t))^T$ is a bounded vector-form Weiner process. In this paper, we always assume that $\omega_i(t)$ and $\omega_j(t)$ are independent processes of one another for $i \neq j$, and matrix $G = (g_{ij})_{N\times N}$ is irreducible in the sense that there is no isolated node.

The initial condition of system (2.1) is given in the following form:

$$x_i(s) = \varphi_i(s), \quad -\tau \leq s \leq 0, \ i = 1, 2, \ldots, N, \tag{2.2}$$

where $\tau = \max\{\tau(t)\}$, $\varphi_i \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$, $L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable $\mathfrak{R}([-\tau, 0], \mathbb{R}^n)$-valued random variables satisfying $\sup_{-\tau\leq s\leq 0} E[\|\varphi_i(s)\|] < \infty$, and $\mathfrak{R}([-\tau, 0], \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued functions $\varphi_i(s)$ on $[-\tau, 0]$ with the norm $\|\varphi_i\| = \sup_{-\tau\leq s\leq 0} \varphi_i^T(s)\varphi_i(s)$. We always assume that system (2.1) has a unique solution with respect to initial condition.

Our objective of synchronization is to control network (2.1) to the trajectory $s(t) \in \mathbb{R}^n$ of the uncoupled system:

$$\mathrm{d}s(t) = f(t, s(t), s(t-\tau(t)))\mathrm{d}t, \tag{2.3}$$

where $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^T$ can be any desired state: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit. To achieve this goal, some adaptive pinning controllers will be added to part of its nodes. Without loss of generality, rearrange the order of the nodes in the network, and let the first l nodes be controlled. Thus, the pinning controlled network can be described by

$$\begin{aligned}
\mathrm{d}x_i(t) &= \left[f(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t) + d\sum_{j=1}^{N} u_{ij} D x_j(t-\tau(t)) \right]\mathrm{d}t \\
&\quad + h_i(t, x_1(t), \ldots, x_N(t))\mathrm{d}\omega_i(t) + R_i\mathrm{d}t, \quad i = 1, 2, \ldots, l, \\
\mathrm{d}x_i(t) &= \left[f(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t) + d\sum_{j=1}^{N} u_{ij} D x_j(t-\tau(t)) \right]\mathrm{d}t \\
&\quad + h_i(t, x_1(t), \ldots, x_N(t))\mathrm{d}\omega_i(t), \quad i = l+1, 2, \ldots, N,
\end{aligned} \tag{2.4}$$

where R_i, $i = 1, 2, \ldots, l$, are control inputs.

For convenience of writing, in the sequel, we denote $h_i(t, x_1(t), x_2(t), \ldots, x_N(t))$ as $h_i(t, x(t))$. Accordingly, denote $h_i(t, s(t), s(t), \ldots, s(t))$ as $h_i(t, s(t))$. Let $e_i(t) = x_i(t) - s(t)$, $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$, $i = 1, 2, \ldots, N$.

Subtracting (2.3) from (2.4) yields the following error dynamical system:

$$\begin{aligned}
\mathrm{d}e_i(t) &= \left[F(t, e_i(t), e_i(t-\tau(t))) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j(t) + d\sum_{j=1}^{N} u_{ij} D e_j(t-\tau(t)) \right]\mathrm{d}t \\
&\quad + \widetilde{h}_i(t, e(t))\mathrm{d}\omega_i(t) + R_i\mathrm{d}t, \quad i = 1, 2, \ldots, l, \\
\mathrm{d}e_i(t) &= \left[F(t, e_i(t), e_i(t-\tau(t))) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j(t) + d\sum_{j=1}^{N} u_{ij} D e_j(t-\tau(t)) \right]\mathrm{d}t \\
&\quad + \widetilde{h}_i(t, e(t))\mathrm{d}\omega_i(t), \quad i = l+1, 2, \ldots, N,
\end{aligned} \tag{2.5}$$

where $F(t, e_i(t), e_i(t-\tau(t))) = f(t, x_i(t), x_i(t-\tau(t))) - f(t, s(t), s(t-\tau(t)))$, and $\widetilde{h}_i(t, e(t)) = h_i(t, x(t)) - h_i(t, s(t))$.

Then the objective here is to find some appropriate adaptive pinning controllers R_i, $i = 1, 2, \ldots, l$, such that the trivial solution of error system (2.5) is globally asymptotically stable in the mean square, that is,

$$\lim_{t\to\infty} E\left[\|e_i(t)\|^2\right] = 0, \quad i = 1, 2, \ldots, N. \tag{2.6}$$

Definition 2.1. Function class QUAD1(P,Δ,η,θ): let $P = \operatorname{diag}(p_1,p_2,\ldots,p_n)$ be a positive-definite diagonal matrix, $\Delta = \operatorname{diag}(\delta_1,\delta_2,\ldots,\delta_n)$ a diagonal matrix, and constants $\eta > 0, \theta \geq 0$. QUAD1(P,Δ,η,θ) denotes a class of continuous functions $f(t,x,z) \in [0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$(x-y)^T P\left[f(t,x,z) - f(t,y,w) - \Delta(x-y)\right] \leq -\eta(x-y)^T(x-y) + \theta(z-w)^T(z-w) \tag{2.7}$$

for all $t \geq 0$, for all $x,y,z,w \in \mathbb{R}^n$.

Remark 2.2. The function class QUAD1(P,Δ,η,θ) includes almost all the well-known chaotic systems with delays or without delays such as Lorenz system, Rössler system, Chen system, delayed Chua's circuit as well as logistic delay differential system, delayed Hopfield neural networks and delayed CNNs, and so on. In fact, all those systems mentioned above satisfy the following condition [8, 18]: there exist positive $k_{ij} > 0$ such that for

$$\begin{aligned}&\left|f_i(t,x(t),x(t-\tau(t))) - f_i(t,y(t),y(t-\tau(t)))\right| \\ &\quad\leq \sum_{j=1}^{n} k_{ij}\left(\left|x_j(t)-y_j(t)\right| + \left|x_j(t-\tau(t)) - y_j(t-\tau(t))\right|\right), \quad i = 1,2,\ldots,n\end{aligned} \tag{2.8}$$

for any $x(t) = (x_1(t),x_2(t),\ldots,x_n(t))^T \in \mathbb{R}^n$, $y(t) = (y_1(t),y_2(t),\ldots,y_n(t))^T \in \mathbb{R}^n$.

From condition (2.8), we get

$$\begin{aligned}&(x-y)^T P\left(f(t,x(t),x(t-\tau(t))) - f(t,y(t),y(t-\tau(t)))\right) \\ &\quad\leq \sum_{i=1}^{n}\sum_{j=1}^{n} p_i k_{ij}\left|x_i(t)-y_i(t)\right|\left(\left|x_j(t)-y_j(t)\right| + \left|x_j(t-\tau(t)) - y_j(t-\tau(t))\right|\right) \\ &\quad= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\Big[k_{ij}^{1/2}\left|x_i(t)-y_i(t)\right|k_{ij}^{1/2}\left|x_j(t)-y_j(t)\right| \\ &\qquad\qquad + k_{ij}^{1/2}\left|x_i(t)-y_i(t)\right|k_{ij}^{1/2}\left|x_j(t-\tau(t)) - y_j(t-\tau(t))\right|\Big] \\ &\quad\leq \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\left[k_{ij}\left(x_i(t)-y_i(t)\right)^2 + \frac{1}{2}k_{ij}\left(x_j(t)-y_j(t)\right)^2 + \frac{1}{2}k_{ij}\left(x_j(t-\tau(t)) - y_j(t-\tau(t))\right)^2\right] \\ &\quad= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\left[k_{ij}\left(x_i(t)-y_i(t)\right)^2 + \frac{1}{2}k_{ji}\left(x_i(t)-y_i(t)\right)^2 + \frac{1}{2}k_{ji}\left(x_i(t-\tau(t)) - y_i(t-\tau(t))\right)^2\right] \\ &\quad= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\left[\left(k_{ij} + \frac{1}{2}k_{ji}\right)\left(x_i(t)-y_i(t)\right)^2 + \frac{1}{2}k_{ji}\left(x_i(t-\tau(t)) - y_i(t-\tau(t))\right)^2\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} p_i \left[\left(-\delta_i + \sum_{j=1}^{n} \left(k_{ij} + \frac{1}{2} k_{ji} \right) \right) \left(x_i(t) - y_i(t) \right)^2 + \frac{1}{2} \sum_{j=1}^{n} k_{ji} \left(x_i(t-\tau(t)) - y_i(t-\tau(t)) \right)^2 \right] \\
&\quad + \left(x(t) - y(t) \right)^T P \Delta \left(x(t) - y(t) \right) \\
&\le -\eta \left(x(t) - y(t) \right)^T \left(x(t) - y(t) \right) + \left(x(t) - y(t) \right)^T P \Delta \left(x(t) - y(t) \right) \\
&\quad + \theta \left(x(t-\tau(t)) - y(t-\tau(t)) \right)^T \left(x(t-\tau(t)) - y(t-\tau(t)) \right),
\end{aligned}
\tag{2.9}
$$

where $\eta = \min_{1\le i\le n}\{p_i(\delta_i - \sum_{j=1}^{n}(k_{ij} + (1/2)k_{ji}))\}$, $\theta = \max_{1\le i\le n}\{(1/2)p_i \sum_{j=1}^{n} k_{ji}\}$, and $2ab \le a^2 + b^2$ is used in the above inequality. Hence, condition (2.7) is satisfied. Note that function class QUAD1$(P, \Delta, \eta, \theta)$ is more general than the usual function class QUAD(P, Δ, η) (see [13, 19]) for chaotic systems and includes it as a special class when $\theta = 0$.

Before giving our main results, we present the following lemmas, which are needed in the next section.

Lemma 2.3 (see [12]). *If $A = (a_{ij})_{N\times N}$ is an irreducible matrix satisfying $a_{ij} = a_{ji} \ge 0$, for $i \ne j$, and $a_{ii} = -\sum_{i=1, i\ne j}^{N} a_{ij}$, for $1 = 1, 2, \ldots, N$, then all eigenvalues of the matrix $\overline{A} = A - \mathrm{diag}(k_1, \ldots, k_l, 0, \ldots, 0)$ are negative, where $k_1, \ldots, k_l$ are positive constants.*

Lemma 2.4 (see [19]). *Suppose that $G = (g_{ij})_{N\times N}$ is an irreducible matrix satisfying $g_{ij} \ge 0$, for $i \ne j$, and $g_{ii} = -\sum_{i=1, i\ne j}^{N} g_{ij}$, for $1 = 1, 2, \ldots, N$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ be the left eigenvector corresponding to the eigenvalue 0, that is, $\xi^T G = 0$. Then, $\xi_i > 0$, $i = 1, 2, \ldots, N$, and its multiplicity is 1.*

Lemma 2.5 (see [20]; (Schur Complement)). *The linear matrix inequality (LMI)*

$$
S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0 \tag{2.10}
$$

is equivalent to anyone of the following two conditions:

(L_1) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

(L_2) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$,

where $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$.

Now we list assumptions as follows.

(H_1) There exist nonnegative constants μ_{ij}, $i, j = 1, 2, \ldots, N$, such that

$$
\mathrm{trace}\left(\left(\widetilde{h}_i(t, e(t)) \right)^T \widetilde{h}_i(t, e(t)) \right) \le \sum_{j=1}^{N} \mu_{ij} \left\| e_j(t) \right\|^2, \tag{2.11}
$$

where $\widetilde{h}_i(t, e(t)) = h_i(t, x(t)) - h_i(t, s(t))$, $i = 1, 2, \ldots, N$.

(H_2) $\tau(t)$ is a continuous differentiable function on $\mathbb{R}^+$ with $\dot{\tau}(t) \le h < 1$.

3. Main Results

In this section, two different adaptive pinning feedback controllers corresponding to two kinds of properties of the inner coupling Γ are designed, and several sufficient conditions for synchronization of the complex networks (2.1) are derived.

For the the inner coupling matrix Γ, there exist the following two cases:

(a) $\gamma_j \geq 0$, $j = 1, 2, \ldots, n$, and there exists at least one $j_0 \in \{j = 1, 2, \ldots, n\}$ such that $\gamma_{j_0} > 0$.

(b) $\gamma_j > 0$, $j = 1, 2, \ldots, n$.

Obviously, case (b) is the special case of case (a). Case (b) means that all the states of connected nodes are coupled, while in case (a) this is not necessary. However, most of existing papers only consider case (b); for example, see references [7, 12, 13, 19]. In this paper, we will consider both of the two cases. We first study the general case (a).

Theorem 3.1. *Suppose that the assumptions* (H_1)-(H_2) *hold and* $f \in \mathrm{QUAD1}(P, \Delta, \eta, \theta)$. *For the case (a), if* $2c\overline{\gamma}\widehat{G}^s_{N-l} + W_{N-l} < 0$, *where* $\widehat{G} = (\widehat{g}_{ij})_{N\times N}$, $\widehat{g}_{ij} = g_{ij}, i \neq j$, $\widehat{g}_{ii} = (\underline{\gamma}/\overline{\gamma})g_{ii}$, $\overline{\gamma} = \max\{\gamma_j,\ j = 1, 2, \ldots, n\}$, $\underline{\gamma} = \min\{\gamma_j,\ j = 1, 2, \ldots, n\}$, $W = 2(-\eta + \overline{\delta})I_N + 2\Phi + ((5-h)/2(1-h))d\overline{d}\|V\|I_N$, $\overline{\delta} = \max\{p_j\delta_j,\ j = 1, 2, \ldots, n\}$, $\Phi = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_N)$, $\phi_j = (1/2)\sum_{i=1}^{N}\mu_{ij}$, $\overline{d} = \max\{d_j,\ j = 1, 2, \ldots, n\}$, $V = (|u_{ij}|)_{N\times N}$, *then the trivial solution of* (2.5) *is globally asymptotically stable in the mean square with the adaptive pinning controllers*

$$\begin{aligned} R_i &= -c\varepsilon_i e_i(t), \quad i = 1, 2, \ldots, l, \\ \dot{\varepsilon}_i &= v_i e_i^T(t) P e_i(t), \quad i = 1, 2, \ldots, l. \end{aligned} \tag{3.1}$$

Proof. According to (H_1) and (H_2), the origin is an equilibrium point of the error system (2.5). We define the Lyapunov-Krasovskii functional as

$$V(t) = V_1(t) + V_2(t), \tag{3.2}$$

where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) P e_i(t) + \frac{1}{2}\sum_{i=1}^{l}\frac{c(\varepsilon_i - k_i)^2}{v_i}, \qquad V_2(t) = \int_{t-\tau(t)}^{t} \zeta^T(s) Q \zeta(s)\mathrm{d}s, \tag{3.3}$$

$\zeta(t) = (\|e_1(t)\|, \|e_2(t)\|, \ldots, \|e_N(t)\|)^T$, Q is a positive definite matrix, k_i, $i = 1, 2, \ldots, l$, are positive constants, and Q and k_i, $i = 1, 2, \ldots, l$, are to be determined.

Differentiating $V_1(t)$ along the solution of (2.5) and using the properties of vector-form Weiner process [16], we get

$$
\begin{aligned}
\mathrm{d}V_1(t) &= \sum_{i=1}^{N}\left[e_i^T(t)P\mathrm{d}e_i(t) + \frac{1}{2}(\mathrm{d}e_i(t))^T P\mathrm{d}e_i(t)\right] + c\sum_{i=1}^{l}(\varepsilon_i - k_i)e_i^T(t)Pe_i(t)\mathrm{d}t \\
&= \sum_{i=1}^{N}e_i^T(t)P\left[F(t,e_i(t),e_i(t-\tau(t))) + c\sum_{j=1}^{N}g_{ij}\Gamma e_j(t) + d\sum_{j=1}^{N}u_{ij}De_j(t-\tau(t))\right]\mathrm{d}t \\
&\quad - c\sum_{i=1}^{l}k_i e_i^T(t)Pe_i(t)\mathrm{d}t + \sum_{i=1}^{N}e_i^T(t)\tilde{h}_i(t,e(t))\mathrm{d}\omega_i(t) \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\mathrm{trace}\left[\left(\tilde{h}_i(t,e(t))\right)^T\tilde{h}_i(t,e(t))\right]\mathrm{d}t \\
&= \left\{\sum_{i=1}^{N}e_i^T(t)P(F(t,e_i(t),e_i(t-\tau(t))) - \Delta e_i(t)) + \sum_{i=1}^{N}e_i^T(t)P\Delta e_i(t)\right. \\
&\quad - c\sum_{i=1}^{l}k_i e_i^T(t)Pe_i(t) + \sum_{i=1}^{N}e_i^T(t)P\left(c\sum_{j=1}^{N}g_{ij}\Gamma e_j(t) + d\sum_{j=1}^{N}u_{ij}De_j(t-\tau(t))\right) \\
&\quad \left. + \frac{1}{2}\sum_{i=1}^{N}\mathrm{trace}\left[(\tilde{h}_i(t,e(t))^T\tilde{h}_i(t,e(t))\right]\right\}\mathrm{d}t + \sum_{i=1}^{N}e_i^T(t)\tilde{h}_i(t,e(t))\mathrm{d}\omega_i(t) \\
&\le \left\{-\eta\sum_{i=1}^{N}\|e_i(t)\|^2 + \theta\sum_{i=1}^{N}\|e_i(t-\tau(t))\|^2 + \overline{\delta}\sum_{i=1}^{N}\|e_i(t)\|^2 - c\overline{p}\sum_{i=1}^{l}k_i\|e_i(t)\|^2\right. \\
&\quad + c\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}g_{ij}\overline{\gamma}\|e_i(t)\|\|e_j(t)\| + c\sum_{i=1}^{N}g_{ii}\underline{\gamma}\|e_i(t)\|^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mu_{ij}\|e_j(t)\|^2 \\
&\quad \left. + d\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}|u_{ij}|\overline{d}\|e_i(t)\|\|e_j(t-\tau(t))\|\right\}\mathrm{d}t + \sum_{i=1}^{N}e_i^T(t)\tilde{h}_i(t,e(t))\mathrm{d}\omega_i(t),
\end{aligned}
\tag{3.4}
$$

where $\overline{p} = \max\{p_j, j = 1,2,\ldots,n\}$.

Similarly, from $V_2(t)$ we have

$$
\begin{aligned}
\mathrm{d}V_2(t) &= \left[\zeta^T(t)Q\zeta(t) - (1-\dot{\tau}(t))\zeta^T(t-\tau(t))Q\zeta(t-\tau(t))\right] \\
&\le \left[\zeta^T(t)Q\zeta(t) - (1-h)\zeta^T(t-\tau(t))Q\zeta(t-\tau(t))\right]\mathrm{d}t.
\end{aligned}
\tag{3.5}
$$

Hence,

$$\begin{aligned}
\mathrm{d}V(t) \le & \Big[\left(\overline{\delta}-\eta\right)\zeta^T(t)\zeta(t)+\theta\zeta^T(t-\tau(t))\zeta(t-\tau(t))-c\overline{p}\zeta^T(t)K\zeta(t)+c\overline{\gamma}\zeta^T(t)\widehat{G}^s\zeta(t) \\
& +d\overline{d}\zeta^T(t)V\zeta(t-\tau(t))+\zeta^T(t)(R+Q)\zeta(t)-(1-h)\eta^T(t-\tau(t))Q\eta(t-\tau(t))\Big]\mathrm{d}t \\
& +\sum_{i=1}^{N}e_i^T(t)\widetilde{h}_i(t,e(t))\mathrm{d}\omega_i(t) \\
= & \frac{1}{2}\varpi^T(t)\mathbf{\Pi}\varpi(t)\mathrm{d}(t)+\sum_{i=1}^{N}e_i^T(t)\widetilde{h}_i(t,e(t))\mathrm{d}\omega_i(t),
\end{aligned} \tag{3.6}$$

where $K=\mathrm{diag}(k_1,k_2,\ldots,k_l,0,\ldots,0)$, $\varpi^T(t)=(\zeta^T(t),\zeta^T(t-\tau(t)))^T$, and

$$\mathbf{\Pi}=\begin{bmatrix} 2\left(\left(-\eta+\overline{\delta}\right)I_N-c\overline{p}K+c\overline{\gamma}\widehat{G}^s+\Phi+Q\right) & d\overline{d}V^T \\ d\overline{d}V & 2\theta I_N-2(1-h)Q \end{bmatrix}. \tag{3.7}$$

In view of Lemma 2.5, $\mathbf{\Pi}<0$ is equivalent to

$$\begin{gathered}
\Upsilon=2\left(\left(-\eta+\overline{\delta}\right)I_N-c\overline{p}K+c\overline{\gamma}\widehat{G}^s+\Phi+Q\right)+\frac{1}{2}d^2\overline{d}^2V^T((1-h)Q-\theta I_N)^{-1}V<0, \\
-(1-h)Q+\theta I_N<0.
\end{gathered} \tag{3.8}$$

Let $Q=(1/(1-h))(d\overline{d}\|V\|+\theta)I_N$, we obtain

$$\begin{aligned}
\Upsilon & \le 2\left(-\eta+\overline{\delta}\right)I_N-2c\overline{p}K+2c\overline{\gamma}\widehat{G}^s+2\Phi+\frac{5-h}{2(1-h)}d\overline{d}\|V\|I_N \\
& =\begin{bmatrix} -2c\overline{p}K_l+2c\overline{\gamma}\widehat{G}_l^s+W_l & 2c\overline{\gamma}\left(\widehat{G}_*^s\right)^T \\ 2c\overline{\gamma}\widehat{G}_*^s & 2c\overline{\gamma}\widehat{G}_{N-l}^s+W_{N-l} \end{bmatrix},
\end{aligned} \tag{3.9}$$

where $\widehat{G}_*^s$ is matrix with appropriate dimension.

Since $2c\overline{\gamma}\widehat{G}_{N-l}^s+W_{N-l}<0$, obviously, there exist positive constants $k_1,k_2,\ldots,k_l$ such that

$$-2c\overline{p}K_l+2c\overline{\gamma}\widehat{G}_l^s+W_l-2c^2\overline{\gamma}^2\left(\widehat{G}_*^s\right)^T\left(2c\overline{\gamma}\widehat{G}_{N-l}^s+W_{N-l}\right)^{-1}\widehat{G}_*^s<0. \tag{3.10}$$

Again, from Lemma 2.5 we obtain $\Upsilon < 0$. Hence, (3.8) holds, which means that $\mathbf{\Pi} < 0$. Taking the mathematical expectation of both sides of (3.6), we have

$$\begin{aligned}\frac{\mathrm{d}EV(t)}{\mathrm{d}t} &\le E\left[\frac{1}{2}\varpi^T(t)\mathbf{\Pi}\varpi(t)\right] \le -\frac{1}{2}\lambda_{\min}(-\mathbf{\Pi})E\left[\|\zeta(t)\|^2 + \|\zeta(t-\tau(t))\|^2\right] \\ &\le -\frac{1}{2}\lambda_{\min}(-\mathbf{\Pi})E\left[\|\zeta(t)\|^2\right] = -\frac{1}{2}\lambda_{\min}(-\mathbf{\Pi})E\left[\sum_{i=1}^{N}\|e_i(t)\|^2\right],\end{aligned} \tag{3.11}$$

where $\lambda_{\min}(-\mathbf{\Pi})$ is positive. In view of the LaSalle invariance principle of stochastic differential equation, which was developed in [21], we have $\lim_{t\to\infty}E[\sum_{i=1}^{N}\|e_i(t)\|^2] = 0$, which in turn illustrates that $\lim_{t\to\infty}E[\|e_i(t)\|^2] = 0$, and at the same time, $\lim_{t\to\infty}\varepsilon_i(t) = \varepsilon_i$ (constants). This completes the proof. □

Next, we consider the case (b). Obviously, the adaptive pinning controllers (3.1) can also synchronize (2.1) for this case. Here we use another kind of controllers, which is related to the inner matrix Γ, to synchronize the complex networks (2.1).

Assume that $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ is defined in Lemma 2.4; we have $\xi_i > 0$, $i = 1, 2, \ldots, N$. Define $\Xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N)$. It is easy to verify that $\Xi G + G^T\Xi$ is a symmetric and zero row sum matrix. Moreover, if G is irreducible, then ΞG is also irreducible; hence, $\Xi G + G^T\Xi$ is irreducible. Therefore, by Lemma 2.3, $(\Xi\overline{G})^s = (1/2)(\Xi\overline{G} + \overline{G}^T\Xi) = (1/2)(\Xi G + G^T\Xi) - \Xi K$ is negative definite, where $\overline{G} = G - K$, $K = \mathrm{diag}(k_1, \ldots, k_l, 0, \ldots, 0)$, and k_i, $i = 1, \ldots, l$, are positive constants.

Theorem 3.2. *Suppose that the assumptions* (H_1)-(H_2) *hold and* $f \in \mathrm{QUAD1}(P, \Delta, \eta, \theta)$. *If* $\gamma_j > 0$ *and* $2cp_j\gamma_j(\Xi G)^s_{N-l} + (W_j)_{N-l} < 0$, $j = 1, 2, \ldots, n$, *where* $W_j = -2\eta\Xi + 2p_j\delta_j\Xi + \widetilde{\Phi} + ((2-h)/(1-h))dp_jd_j\|\Xi U\|I_N + (2\theta/(1-h))\Xi$, $\widetilde{\Phi} = \mathrm{diag}(\widetilde{\phi}_1, \widetilde{\phi}_2, \ldots, \widetilde{\phi}_N)$, *and* $\widetilde{\phi}_j = \sum_{i=1}^{N}\xi_i\mu_{ij}$, *then the trivial solution of* (2.5) *is globally asymptotically stable in the mean square with the adaptive pinning controllers*

$$\begin{aligned} R_i &= -c\varepsilon_i\Gamma e_i(t), \quad i = 1, 2, \ldots, l, \\ \dot{\varepsilon}_i &= v_i e_i^T(t)P\Gamma e_i(t), \quad i = 1, 2, \ldots, l. \end{aligned} \tag{3.12}$$

Proof. Let $\widetilde{e}_j(t) = (e_{1j}, e_{2j}, \ldots, e_{Nj})^T$, $j = 1, 2, \ldots, n$. We define the Lyapunov-Krasovskii functional as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\xi_i e_i^T(t)Pe_i(t) + \frac{1}{2}\sum_{j=1}^{n}\int_{t-\tau(t)}^{t}\widetilde{e}_j^T(s)Q_j\widetilde{e}_j(s)\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{l}\frac{c\xi_i(\varepsilon_i - k_i)^2}{v_i}, \tag{3.13}$$

where Q_j, $j = 1, 2, \ldots, n$, are positive definite matrices, k_i, $i = 1, 2, \ldots, l$, are positive constants, and Q_j and k_i, $i = 1, 2, \ldots, l$, are to be determined.

Differentiating both sides of (3.13) along the solution of (2.5) yields

$$
\begin{aligned}
\mathrm{d}V(t) &= \sum_{i=1}^{N}\xi_i\left[e_i^T(t)P\mathrm{d}e_i(t)+\frac{1}{2}(\mathrm{d}e_i(t))^TP\mathrm{d}e_i(t)\right] \\
&\quad+\left\{\frac{1}{2}\sum_{j=1}^{n}\left[\tilde{e}_j^T(t)Q_j\tilde{e}_j(t)-(1-\dot{\tau}(t))\tilde{e}_j^T(t-\tau(t))Q_j\tilde{e}_j(t-\tau(t))\right]\right. \\
&\qquad\left.+c\sum_{i=1}^{l}\xi_i(\varepsilon_i-k_i)e_i^T(t)P\Gamma e_i(t)\mathrm{d}t\right\} \\
&= \sum_{i=1}^{N}\xi_ie_i^T(t)P\left[F(t,e_i(t),e_i(t-\tau(t)))+c\sum_{j=1}^{N}g_{ij}\Gamma e_j(t)+d\sum_{j=1}^{N}u_{ij}De_j(t-\tau(t))\right]\mathrm{d}t \\
&\quad-c\sum_{i=1}^{l}\xi_ik_ie_i^T(t)P\Gamma e_i(t)\mathrm{d}t+\sum_{i=1}^{N}\xi_ie_i^T(t)\tilde{h}_i(t,e(t))\mathrm{d}\omega_i(t) \\
&\quad+\frac{1}{2}\sum_{i=1}^{N}\xi_i\mathrm{trace}\left[\left(\tilde{h}_i(t,e(t))\right)^T\tilde{h}_i(t,e(t))\right]\mathrm{d}t \\
&\quad+\frac{1}{2}\sum_{j=1}^{n}\left[\tilde{e}_j^T(t)Q_j\tilde{e}_j(t)-(1-\dot{\tau}(t))\tilde{e}_j^T(t-\tau(t))Q_j\tilde{e}_j(t-\tau(t))\right]\mathrm{d}t \\
&= \left\{\sum_{i=1}^{N}\xi_ie_i^T(t)P(F(t,e_i(t),e_i(t-\tau(t)))-\Delta e_i(t))+\sum_{i=1}^{N}\xi_ie_i^T(t)P\Delta e_i(t)\right. \\
&\quad-c\sum_{i=1}^{l}\xi_ik_ie_i^T(t)P\Gamma e_i(t)+\sum_{i=1}^{N}\xi_ie_i^T(t)P\left(c\sum_{j=1}^{N}g_{ij}\Gamma e_j(t)+d\sum_{j=1}^{N}u_{ij}De_j(t-\tau(t))\right) \\
&\quad+\frac{1}{2}\sum_{i=1}^{N}\xi_i\mathrm{trace}\left[(\tilde{h}_i(t,e(t))^T\tilde{h}_i(t,e(t))\right] \\
&\quad\left.+\frac{1}{2}\sum_{j=1}^{n}\left[\tilde{e}_j^T(t)Q_j\tilde{e}_j(t)-(1-\dot{\tau}(t))\tilde{e}_j^T(t-\tau(t))Q_j\tilde{e}_j(t-\tau(t))\right]\right\}\mathrm{d}t \\
&\quad+\sum_{i=1}^{N}\xi_ie_i^T(t)\tilde{h}_i(t,e(t))\mathrm{d}\omega_i(t) \\
&\le \left\{-\eta\sum_{j=1}^{n}\tilde{e}_j^T(t)\Xi\tilde{e}_j(t)+\theta\sum_{j=1}^{n}\tilde{e}_j^T(t-\tau(t))\Xi\tilde{e}_j(t-\tau(t))+\sum_{j=1}^{n}p_j\delta_j\tilde{e}_j^T(t)\Xi\tilde{e}_j(t)\right. \\
&\quad+c\sum_{j=1}^{n}p_j\gamma_j\tilde{e}_j^T(t)\left(\Xi\overline{G}\right)^s\tilde{e}_j(t)+d\sum_{j=1}^{n}p_jd_j\tilde{e}_j^T(t)\Xi U\tilde{e}_j(t-\tau(t))+\frac{1}{2}\sum_{j=1}^{n}\tilde{e}_j^T(t)\tilde{\Phi}\tilde{e}_j(t) \\
&\quad\left.+\frac{1}{2}\sum_{j=1}^{n}\left[\tilde{e}_j^T(t)Q_j\tilde{e}_j(t)-(1-h)\tilde{e}_j^T(t-\tau(t))Q_j\tilde{e}_j(t-\tau(t))\right]\right\}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\xi_i e_i^T(t)\widetilde{h}_i(t,e(t))\mathrm{d}\omega_i(t)\\
&=\frac{1}{2}\sum_{j=1}^{n}\zeta_j^T\mathbf{\Pi}_j\zeta_j\mathrm{d}t+\sum_{i=1}^{N}\xi_i e_i^T(t)\widetilde{h}_i(t,e(t))\mathrm{d}\omega_i(t),
\end{aligned}
\tag{3.14}
$$

where $\zeta_j=(\widetilde{e}_j^T(t),\widetilde{e}_j^T(t-\tau(t)))^T$ and

$$
\mathbf{\Pi}_j=\begin{bmatrix}-2\eta\Xi+2p_j\delta_j\Xi+2cp_j\gamma_j\left(\Xi\overline{G}\right)^s+\Phi+Q_j & dp_jd_j\Xi U\\ dp_jd_j(\Xi U)^T & -(1-h)Q_j+2\theta\Xi\end{bmatrix},\quad j=1,2,\ldots,n. \tag{3.15}
$$

In view of Lemma 2.5, $\mathbf{\Pi}_j<0$ is equivalent to

$$
\begin{aligned}
\Upsilon_j=-2\eta\Xi+2p_j\delta_j\Xi+2cp_j\gamma_j\left(\Xi\overline{G}\right)^s+\Phi+Q_j+\left(dp_jd_j\right)^2(\Xi U)\left[(1-h)Q_j-2\theta\Xi\right]^{-1}(\Xi U)^T<0\\
-(1-h)Q_j+2\theta\Xi<0.
\end{aligned}
\tag{3.16}
$$

Let $Q_j=(1/(1-h))[dp_jd_j\|\Xi U\|I_N+2\theta\Xi]$; we have

$$
\begin{aligned}
\Upsilon_j&\le-2\eta\Xi+2p_j\delta_j\Xi+2cp_j\gamma_j\left(\Xi\overline{G}\right)^s+\Phi+\frac{2-h}{1-h}dp_jd_j\|\Xi U\|I_N+\frac{2\theta}{1-h}\Xi\\
&=\begin{bmatrix}2cp_j\gamma_j\left(\Xi\overline{G}\right)_l^s-2cp_j\gamma_j(\Xi K)_l+(W_j)_l & 2cp_j\gamma_j\left(\left(\Xi\overline{G}\right)_*^s\right)^T\\ 2cp_j\gamma_j\left(\Xi\overline{G}\right)_*^s & 2cp_j\gamma_j\left(\Xi\overline{G}\right)_{N-l}^s+(W_j)_{N-l}\end{bmatrix},
\end{aligned}
\tag{3.17}
$$

where $(\Xi\overline{G})_*^s$ is the matrix with appropriate dimension.

Since $2cp_j\gamma_j(\Xi G)_{N-l}^s+(W_j)_{N-l}<0$, $(\Xi\overline{G})_{N-l}^s=(\Xi G)_{N-l}^s$, there exist positive constants $k_1,k_2,\ldots,k_l$ such that

$$
2cp_j\gamma_j\left(\Xi\overline{G}\right)_l^s-2cp_j\gamma_j(\Xi K)_l+(W_j)_l-4c^2p_j^2\gamma_j^2\left(\left(\Xi\overline{G}\right)_*^s\right)^T\left[2c\gamma_j\left(\Xi\overline{G}\right)_{N-l}^s+(W_j)_{N-l}\right]^{-1}\left(\Xi\overline{G}\right)_*^s<0. \tag{3.18}
$$

Again, from Lemma 2.5 we obtain $\Upsilon_j<0$. Hence, (3.16) holds, which implies $\mathbf{\Pi}_j<0$. The rest steps are the same as those in the proof of Theorem 3.1. This completes the proof. □

Remark 3.3. If there is no noise perturbation in (2.1), then $\Phi=0$ in Theorem 3.1 and $\widetilde{\Phi}=0$ in Theorem 3.2, respectively. Hence, if (2.1) is synchronized with the first l nodes to be controlled by adding controllers (3.1) or (3.12), then (2.1) with no noise perturbation can sure be synchronized by using adapting pinning controllers (3.1) or (3.12) with no more than l nodes to be controlled.

Remark 3.4. Theorems 3.1 and 3.2 are applicable to directed networks and undirected networks. Although they do not point out which nodes should be controlled first; however, according to [22, 23], for the graph corresponding to G, nodes whose out-degrees are larger than their in-degrees [24] should be controlled first for directed networks, while the controlled nodes can be randomly selected and it is better to successively pin the most-highly connected nodes for undirected networks.

4. Numerical Examples

In this section, we provide three examples to illustrate the effectiveness of the results obtained above.

Example 4.1. Consider the following chaotic delayed neural networks:

$$\mathrm{d}s(t) = \left[-Cs(t) + Af(s(t)) + Bf(s(t-\tau))\right]\mathrm{d}t, \tag{4.1}$$

where $f(s) = \tanh(s)$, $\tau = 1$,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & -0.1 \\ -5 & 4.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix}. \tag{4.2}$$

In the case that the initial condition is chosen as $s_1(t) = 0.4$, $s_2(t) = 0.6$, for all $t \in [-1, 0]$, the chaotic attractor can be seen in Figure 1.

Taking $P = \mathrm{diag}(1, 1)$ and $\Delta = \mathrm{diag}(5, 11.5)$ and we have $\eta = 0.15$, $\theta = 3.25$. Hence the condition (2.7) is satisfied.

In order to verify our new results, consider the following coupled networks:

$$\begin{aligned} \mathrm{d}x_i(t) = {} & \left[-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + c\sum_{j=1}^{3} g_{ij}\Gamma x_j(t) + d\sum_{j=1}^{3} u_{ij} D x_j(t-\tau)\right]\mathrm{d}t \\ & + h_i(x_1(t), x_2(t), x_3(t))\mathrm{d}\omega_i(t), \quad i = 1, 2, 3, \end{aligned} \tag{4.3}$$

where Γ and D are 2-dimensional identity matrices, $c = 4$, $d = 2$, and G and U are asymmetric and zero-row sum matrices as the following:

$$G = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 4 & 6 & -10 \end{bmatrix}, \qquad U = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix}. \tag{4.4}$$

Through simple computation, we get that the left eigenvector of G corresponding to eigenvalue 0 is $(0.5494, 0.8242, 0.1374)^T$ and $\|\Xi U\| = 1.3901$.

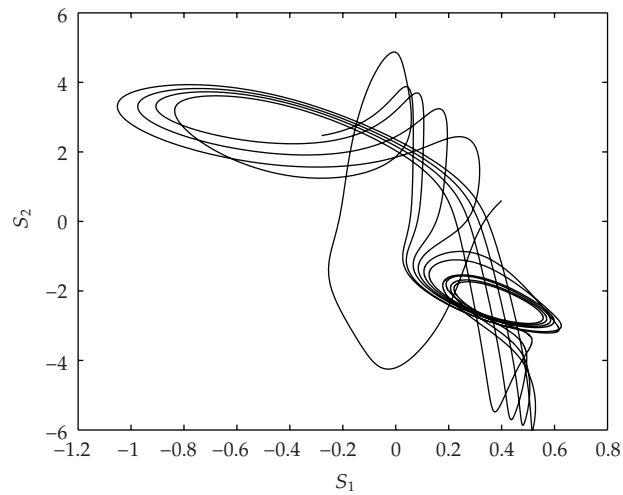

Figure 1: Chaotic trajectory of system (4.1).

Let $h_i(x_1(t), x_2(t), x_3(t)) = 0.2\,\mathrm{diag}(x_{i,1}(t) - x_{i+1,1}(t), x_{i,2}(t) - x_{i+1,2}(t))$, $i = 1, 2, 3$, where $x_4(t) = x_1(t)$. Obviously,

$$\begin{aligned}
\mathrm{trace}\left(\left(\widetilde{h}_i(e(t))\right)^T \widetilde{h}_i(e(t))\right) &= 0.04\left[(x_{i1}(t) - x_{i+1,1}(t))^2 + (x_{i2} - x_{i+1,2}(t))^2\right] \\
&= 0.04\left[(e_{i1}(t) - e_{i+1,1}(t))^2 + (e_{i2}(t) - e_{i+1,2}(t))^2\right] \\
&\le 0.08\left(\|e_i(t)\|^2 + \|e_{i+1}(t)\|^2\right), \quad i = 1, 2, 3,
\end{aligned} \tag{4.5}$$

where $e_4(t) = e_1(t)$. Hence, the assumption conditions (H_1) and (H_2) are satisfied. Selecting the first two nodes to be controlled, we have $2cp_1\gamma_1(\Xi G)^s_{3-2} + (W_1)_{3-2} < 2cp_2\gamma_2(\Xi G)^s_{3-2} + (W_2)_{3-2} = -1.2986 < 0$. According to Theorem 3.2, the complex networks (4.3) can be controlled to the state of (4.1) under the adaptive pinning controllers (3.12) with the first two nodes to be controlled.

The initial conditions of the numerical simulations are as follows: $x_i(s) = (-4 + 12i), -10 + 12i)^T$, $1 \le i \le 3$, for all $s \in [-1, 0]$, $\varepsilon_1(0) = \varepsilon_2(0) = 1$, $v_1 = v_2 = 0.04$. Figure 2 shows the time evolutions of synchronization errors $e_{ij}(t) = x_{ij}(t) - s_j(t)$, $i = 1, 2, 3$, $j = 1, 2$, with the first two nodes being controlled, which verify Theorem 3.2 perfectly. The trajectories of the gains are shown in Figure 3.

Example 4.2. Consider the following coupled neural networks:

$$\begin{aligned}
\mathrm{d}x_i(t) = &\left[-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + c\sum_{j=1}^{10} g_{ij}\Gamma x_j(t) + d\sum_{j=1}^{10} u_{ij} D x_j(t-\tau)\right]\mathrm{d}t \\
&+ h_i(x_1(t), x_2(t), \ldots, x_{10}(t))\mathrm{d}\omega_i(t), \quad i = 1, 2, 3, \ldots, 10,
\end{aligned} \tag{4.6}$$

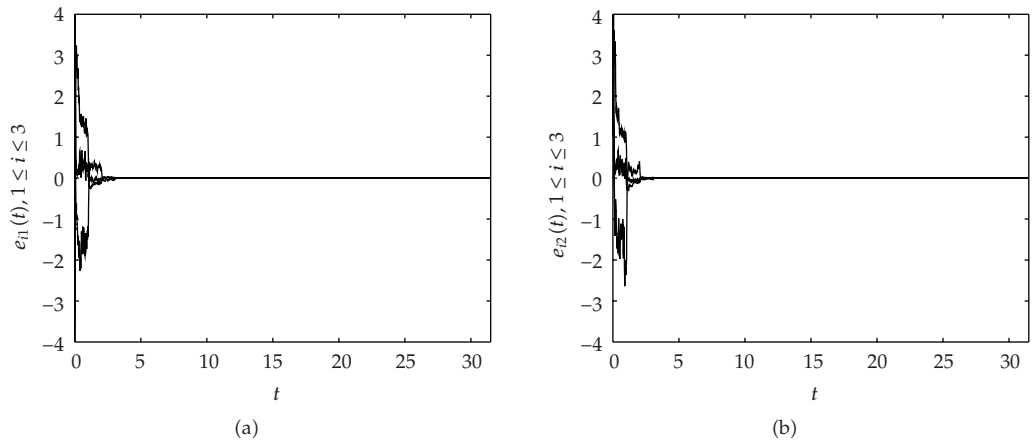

Figure 2: Time evolutions of synchronization errors with the first two nodes being controlled.

where $c = 1.7$, $d = 0.3$, $h_i(x_1(t), x_2(t), \ldots, x_{10}(t)) = 0.1\,\mathrm{diag}(x_{i,1}(t) - x_{i+1,1}(t), x_{i,2}(t) - x_{i+1,2}(t))$, $i = 1, 2, \ldots, 10$, $x_{11}(t) = x_1(t)$, and

$$G = U = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \qquad \Gamma = D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.7}$$

Take the first two nodes to be pinned. Through simple computation, we obtain that all the conditions of Theorem 3.1 are satisfied. Hence, system (4.6) is synchronized with adaptive pinning controllers (3.1).

The initial conditions of the numerical simulations are as follows: $x_i(s) = (-4 + 12i), -10 + 12i)^T$, $1 \le i \le 10$, for all $s \in [-1, 0]$, $\varepsilon_1(0) = \varepsilon_2(0) = 1$, $v_1 = v_2 = 1$. Figure 4 shows the time evolutions of synchronization errors with adaptive pinning control. The trajectories of the adaptive pinning control gains are shown in Figure 5.

Example 4.3. Consider the delayed Chua's circuit:

$$\mathrm{d}s(t) = \left[\overline{C}s(t) + g_1(s(t)) + g_2(s(t-\tau))\right]\mathrm{d}t, \tag{4.8}$$

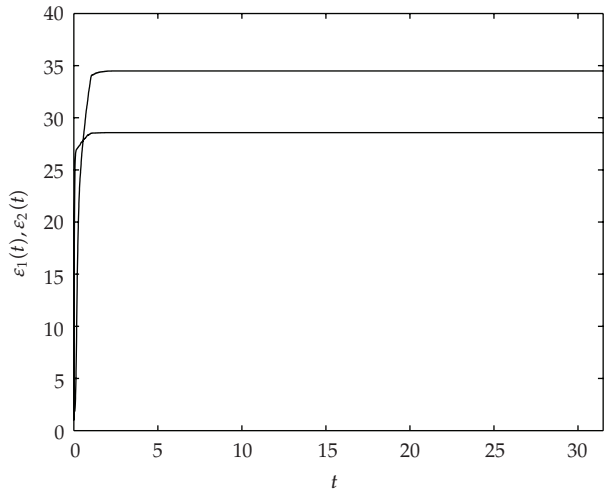

Figure 3: Trajectories of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ for (4.3) with adaptive pinning control.

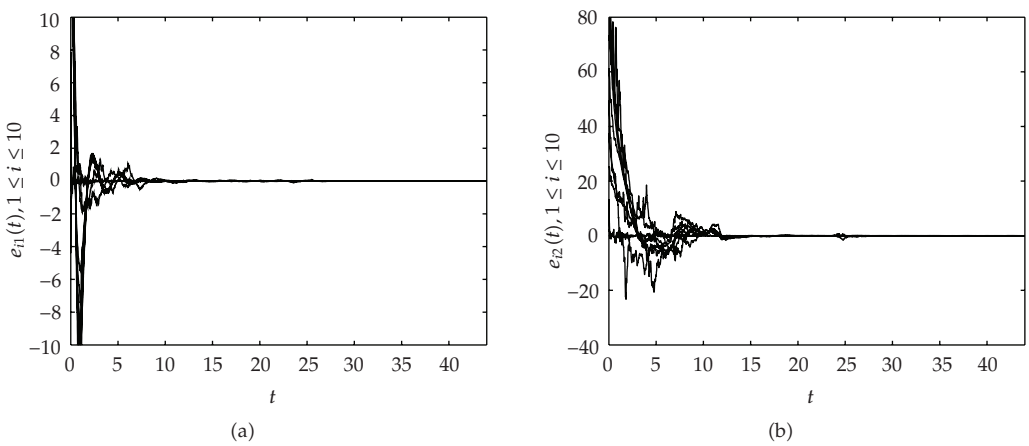

Figure 4: Trajectories of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ for (4.6) with adaptive pinning control.

where $s(t) = (s_1(t), s_2(t), s_3(t))^T$, $g_1(s(t)) = (-(1/2)\delta(a-b)(|s_1(t)+1| - |s_1(t)-1|), 0, 0)^T$, $g_2(s(t-\tau)) = (0, 0, -\xi\epsilon \sin(\nu s_1(t-\tau)))^T$,

$$\overline{C} = \begin{bmatrix} -\delta(1+b) & \delta & 0 \\ 1 & -1 & 1 \\ 0 & -\xi & -\omega \end{bmatrix}, \tag{4.9}$$

and $\delta = 10, \xi = 19.53, \omega = 0.1636, a = -1.4325, b = -0.8731, \nu = 0.5, \epsilon = 0.2, \tau = 0.02$.

In the case that the initial condition is chosen as $s_t(t) = 2$; $s_2(t) = 0.2$; $s_3(t) = 0.3$, for all $t \in [-0.02, 0]$, the chaotic attractor can be seen in Figure 6.

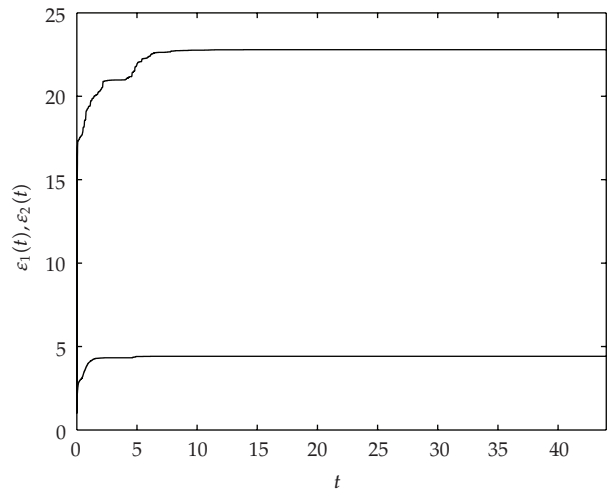

Figure 5: Trajectories of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ for (4.6) with adaptive pinning control.

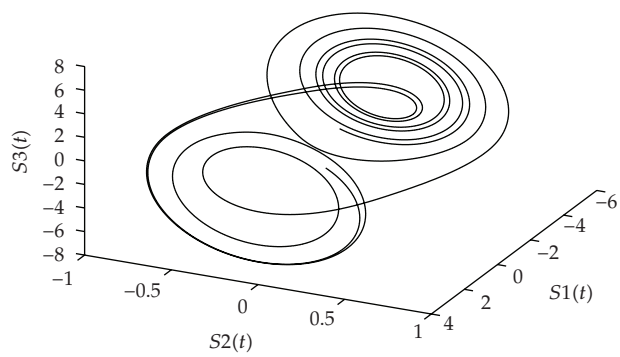

Figure 6: Chaotic trajectory of system (4.8).

Take $k_{11} = (1/2)\delta(b-1) + \delta(1+b) = 5.416$, $k_{12} = \delta = 10$, $k_{13} = 0$; $k_{21} = k_{22} = k_{23} = 1$, $k_{31} = \xi\epsilon\nu = 1.953$, $k_{32} = \xi = 19.53$; $k_{33} = \omega = 0.1636$, $P = \mathrm{diag}(1,1,1)$, $\Delta = \mathrm{diag}(19,18,23)$, we have $\eta = 0.035$, $\theta = 15.265$. Hence the condition (2.7) is satisfied.

Now we construct a complex network, which obeys the scale-free distribution of the Barabási-Albert model [17]. The parameters in the process of constructing are the following: initial graph is complete with $m_0 = 5$ nodes, $m = 5$ edges are added to the network when a new node is introduced, and the final number of nodes is $N = 100$. See Figure 7 for the BA scare-free network.

Consider the following complex networks:

$$\begin{aligned} \mathrm{d}x_i(t) = {} & \left[\overline{C}x_i(t) + g_1(x_i(t)) + g_2(x_i(t-\tau)) + 6\sum_{j=1}^{100} g_{ij}x_j(t) + 0.35\sum_{j=1}^{100} g_{ij}x_j(t-\tau)\right]\mathrm{d}t \\ & + 0.4h_i(x_1(t), x_2(t), \ldots, x_{100}(t))\mathrm{d}\omega_i(t), \quad i = 1, 2, \ldots, 100, \end{aligned} \tag{4.10}$$

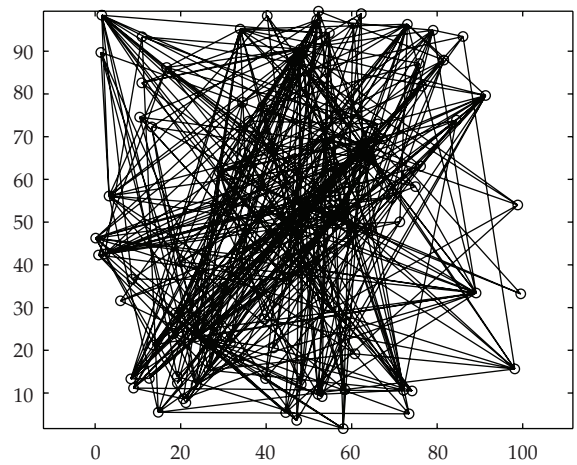

Figure 7: BA scale-free network initial graph is complete with $m_0 = 5$ nodes; $m = 5$ edges are added in the network when a new node is introduced and the final number of nodes is $N = 100$.

where $G = U = (g_{ij})_{100\times 100}$ is the graph laplacian of the BA scale-free network, $h_i(x_1(t), x_2(t), \ldots, x_{100}(t)) = \mathrm{diag}(x_{i,1}(t) - x_{i+1,1}(t), x_{i,2}(t) - x_{i+1,2}(t), x_{i,3}(t) - x_{i+1,3}(t))$ with $x_{101}(t) = x_1(t)$, $i = 1, 2, \ldots, 100$.

Without loss of generality, we take the first 10 nodes to be controlled. The initial conditions of the numerical simulations are as follows: $x_i(s) = (-8 + i, -5 + i, -10 + i)^T$, $s \in [-0.02, 0]$, $1 \le i \le 100$, $\varepsilon_j(0) = 1$, $v_j = 0.04$. For the synchronization errors $e_{ij}(t) = x_{ij}(t) - s_j(t)$, $1 \le i \le 100$, $j = 1, 2, 3$ and trajectories of the control gains; see Figures 8 and 9.

Remark 4.4. In Example 4.2, only the first state of all nodes is coupled; however, the complex network (4.6) is synchronized by adaptive pinning of the first two nodes, which verifies the effectiveness of our theoretical results. In [7, 12, 13, 19], however, all states of the nodes should be coupled. Therefore, our results extend some of existing results. On the other hand, Examples 4.1 and 4.2 are directed networks, while Example 4.3 is undirected network with 100 nodes. Examples with numerical simulations show that our theoretical results are applicable to directed and undirected networks, even networks of large size.

5. Conclusions

In this paper, we have studied the adaptive pinning synchronization of the complex networks with delays and vector-form stochastic perturbations, in which dynamical behaviors are more realistic and complicated. Since the new condition QUAD1, our proof is very simple. Via two simple adaptive pinning feedback control schemes, several sufficient conditions guaranteeing the synchronization of the proposed complex networks are obtained. Our results are valid even if only partial states of nodes are coupled. Numerical simulations verified the effectiveness of our results. Our results improve and extend some existing results. Models and results in this paper provide possible new applications for network designers.

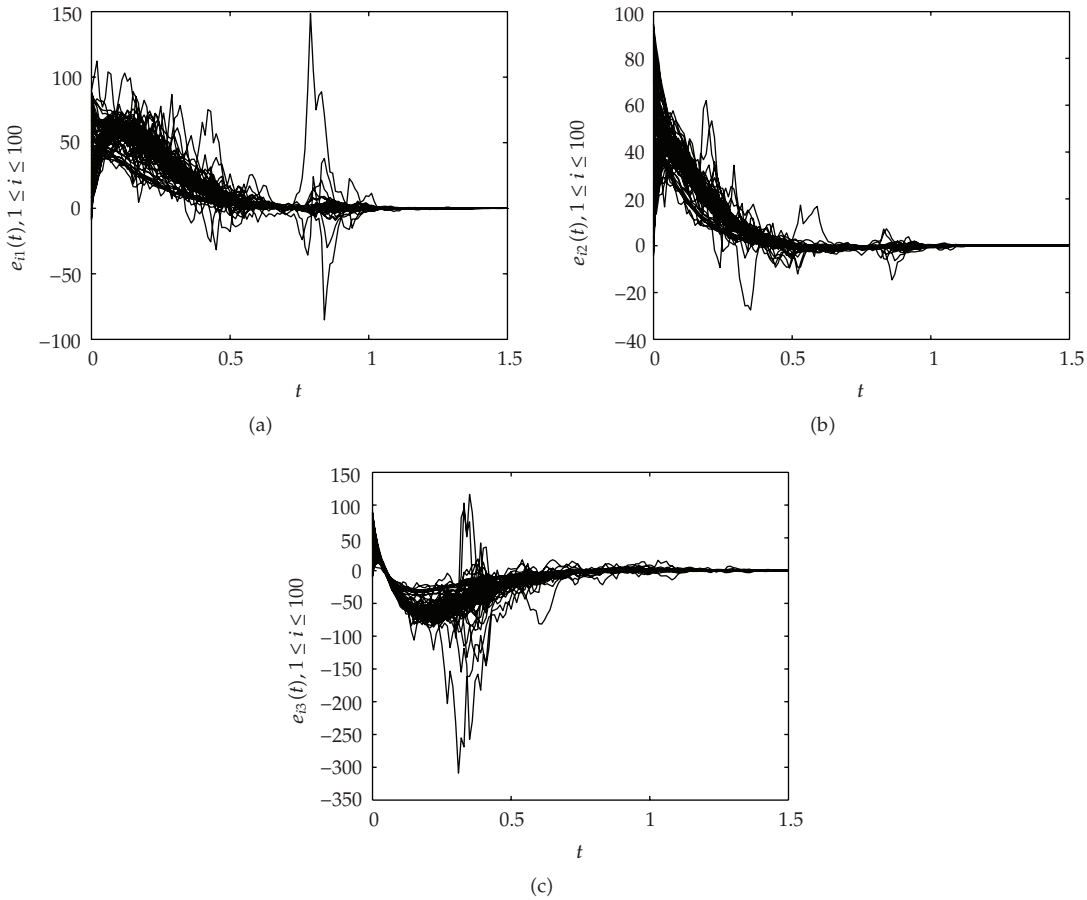

Figure 8: Synchronization errors $e_{i1}(t)$ (a), $e_{i2}(t)$ (b), and $e_{i3}(t)$ (c) $(1 \le i \le 100)$ of (4.10) with the first 10 nodes being controlled.

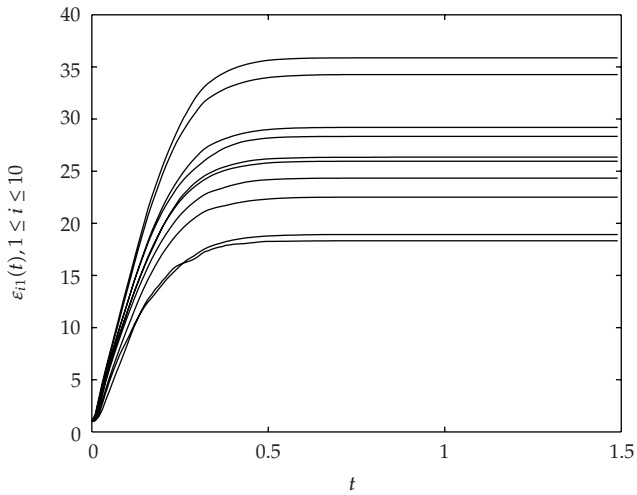

Figure 9: Trajectories of control gains $\varepsilon_i(t)$, $1 \le i \le 10$ for (4.10).

Acknowledgments

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