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Research Article

Global Stability of a Virus Infection Model with Time Delay and Absorption

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In this paper, a virus infection model with time delay and absorption is studied. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria of the model is established. By using comparison arguments, it is shown that the infection free equilibrium is globally asymptotically stable when the basic reproduction ratio is less than unity. When the basic reproduction ratio is greater than unity, sufficient conditions are derived for the global stability of the virus-infected equilibrium. Numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

Mathematical modelling has been proven to be valuable in studying the dynamics of virus infection, such as HIV/AIDS, HBV, and HCV. In recent years, great attention has been paid by many researchers to the pathogen infectious agent, or germ, which can cause disease or illness to its host (see, e.g., [1–3]). Based on the clinical experiment of chronic HBV carriers treated with various doses of lamivudine, Nowak and Bangham [4] and Bonhoeffer et al. [5] proposed a classical mathematical model describing the interaction between the susceptible host cells (x), infected host cells (y), and free virus particles (v), which is formulated by the following differential equations:

$$\begin{aligned} \dot{x}(t) &= \lambda - dx(t) - \beta x(t)v(t), \\ \dot{y}(t) &= \beta x(t)v(t) - ay(t), \\ \dot{v}(t) &= ky(t) - uv(t), \end{aligned} \tag{1.1}$$

where hepatocytes are produced at a rate λ, die at rate dx, and become infected at rate βxv; infected hepatocytes are produced at rate βxv and die at rate ay; free viruses are produced from infected cells at rate ky and are removed at rate uv. It is assumed that parameters a, d, k, u, λ, and β are positive constants. The basic reproductive ratio $\mathcal{R}_0$ was found in [4]. Furthermore, in [6], by constructing suitable Lyapunov function, Korobeinikov showed that the infection-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$, and the virus-infected equilibrium is globally asymptotically stable if $\mathcal{R}_0 > 1$.

It is assumed in model (1.1) that the infection process follows the principle of mass action [7], namely, for each uninfected cell and free virus particle is assumed to be constant β between the rate of infection. However, studies of parasitic infections have shown that the relationship between the dose and rate of infection is clearly nonlinear. In [8], a more general saturated infection rate $\beta xv^p/(1+\alpha v^q)$ was suggested, where p, q, and α are positive constants.

The virus life cycle plays a crucial role in disease progression. The binding of a viral particle to a receptor on a target cell initiates a cascade of events that can ultimately lead to the target cell becoming productively infected, that is, producing new virus. In model (1.1), it is assumed that as soon as virus contacts a target cell, the cell begins producing virus. This is not biologically sensible. In reality, there is a time delay between initial viral entry into a cell and subsequent viral production [9]. There have been some works on virus infection model in the literature (see, e.g., [9–14]). In [10], Herz et al. examined the effect of including a constant delay in the source term for productively infected T-cells. In [11], Li and Ma considered the following more general HIV-1 infection model with saturation incidence rate and time delay:

$$\begin{aligned} \dot{x}(t) &= \lambda - dx(t) - \frac{\beta x(t)v(t)}{1+v(t)}, \\ \dot{y}(t) &= \frac{\beta x(t-\tau)v(t-\tau)}{1+v(t-\tau)} - ay(t), \\ \dot{v}(t) &= ky(t) - uv(t). \end{aligned} \tag{1.2}$$

By analyzing the transcendental characteristic equations, sufficient conditions for the local asymptotic stability of the equilibria were studied, and by using Lyapunov-LaSalle invariance principal, the global asymptotic stability of the viral-free equilibrium was given.

We note that in (1.1) and (1.2), the loss of pathogens due to the absorption into uninfected cells are ignored. In reality, when a pathogen enters an uninfected cell, the number of pathogens in the blood decreases by one. This is called the absorption effect [7]. In [15], considering the erythrocytic cycle in the absence of an immunological response by the host, Anderson et al. presented the following model for malaria infection:

$$\begin{aligned} \dot{x}(t) &= \lambda - dx(t) - \beta x(t)v(t), \\ \dot{y}(t) &= \beta x(t)v(t) - ay(t), \\ \dot{v}(t) &= ky(t) - \beta x(t)v(t) - uv(t), \end{aligned} \tag{1.3}$$

where the term $-\beta xv$ in the third equation of (1.3) represents the absorption effect.

Motivated by the works of Anderson et al. [15], Li and Ma [11], and Song and Neumann [8], in this paper, we study the following virus infection model with a time delay and absorption:

$$\begin{aligned}\dot{x}(t) &= \lambda - dx(t) - \frac{\beta x(t)v(t)}{1+\alpha v(t)},\\ \dot{y}(t) &= \frac{\beta x(t-\tau)v(t-\tau)}{1+\alpha v(t-\tau)} - ay(t),\\ \dot{v}(t) &= ky(t) - \frac{\beta x(t)v(t)}{1+\alpha v(t)} - uv(t).\end{aligned} \tag{1.4}$$

The initial conditions for system (1.4) take the form

$$\begin{gathered}x(\theta) = \phi_1(\theta), \qquad y(\theta) = \phi_2(\theta), \qquad v(\theta) = \phi_3(\theta),\\ \phi_1(\theta) \geq 0, \quad \phi_2(\theta) \geq 0, \quad \phi_3(\theta) \geq 0, \quad \theta \in [-\tau, 0],\\ \phi_1(0) > 0, \qquad \phi_2(0) > 0, \qquad \phi_3(0) > 0,\end{gathered} \tag{1.5}$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_{+0})$, here $\mathbb{R}^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

It is easy to show that all solutions of system (1.4) with initial condition (1.5) is defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, we introduce some notations and state several lemmas which will be essential to our proofs. In Section 3, by analyzing the corresponding characteristic equations, the local stability of each of nonnegative equilibria of system (1.4) is discussed. In Section 4, by using an iteration technique, we study the global stability of the uninfected equilibrium of system (1.4). By comparison arguments, we discuss the global stability of the virus-infected equilibrium of system (1.4). Numerical simulations are carried out in Section 5 to illustrate the main theoretical results. The paper ends with conclusions in Section 6.

2. Preliminaries

In this section, based on the work developed by Xu and Ma [16], we introduce some notations and state several results which will be useful in next section.

Let $\mathbb{R}^n_+$ be the cone of nonnegative vectors in $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, we write $x \leq y (x < y)$ if $x_i \leq y_i (x_i < y_i)$ for $1 \leq i \leq n$. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$. Suppose, $r \geq 0$ and let $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with supremum norm. If $\phi, \psi \in C$, we write $\phi \leq \psi (\phi < \psi)$ when the indicated inequality holds at each point of $[-r, 0]$. Let $C^+ = \{\phi \in C : \phi \geq 0\}$, and let $\wedge$ denote the inclusion $\mathbb{R}^n \to C([-r, 0], \mathbb{R}^n)$ by $x \to \hat{x}$, $\hat{x}(\theta) = x$, $\theta \in [-r, 0]$. Denote the space of functions of bounded variation on $[-r, 0]$ by $\mathrm{BV}[-r, 0]$. If $t_0 \in \mathbb{R}$, $A \geq 0$ and $x \in C([-t_0 - r, t_0 + A], \mathbb{R}^n)$, then for any $t \in [t_0, t_0 + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

We now consider

$$\dot{x}(t) = f(t, x_t). \tag{2.1}$$

We assume throughout this section that $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, $f(t,\phi)$ is continuously differentiable in $\phi, f(t+T,\phi) = f(t,\phi)$ for all $(t,\phi) \in \mathbb{R} \times C^+$, and some $T > 0$. Then, by [17], there exists a unique solution of (2.1) through (t_0,ϕ) for $t_0 \in \mathbb{R}$, $\phi \in C^+$. This solution will be denoted by $x(t,t_0,\phi)$ if we consider the solution in $\mathbb{R}^n$, or by $x_t(t_0,\phi)$ if we work in the space C. Again, by [17], $x(t,t_0,\phi)(x_t(t_0,\phi))$ is continuously differentiable in ϕ. In the following, the notation $x_{t_0} = \phi$ will be used as the condition of the initial data of (2.1), by which we mean that we consider the solution $x(t)$ of (2.1) which satisfies $x(t_0+\theta) = \phi(\theta)$, $\theta \in [-r, 0]$.

To proceed further, we need the following results. Let $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$, $|r| = \max_i\{r_i\}$, and define

$$C_r = \prod_{i=1}^{n} C([-r_i, 0], \mathbb{R}). \tag{2.2}$$

We write $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ for a generic point of C_r. Let $C_r^+ = \{\phi \in C_r : \phi \geq 0\}$. Due to the ecological applications, we choose C_r^+ as the state space of (2.1) in the following discussions.

Fix $\phi_0 \in C_r^+$ arbitrarily. Then, we set $\mathrm{L}(t,\cdot) = D_{\phi_0} f(t,\phi_0)$, $D_{\phi_0} f(t,\phi_0)$ denotes the Frechet derivation of f with respect to ϕ_0. It is convenient to have the standard representation of $L = (L_1, L_2, \ldots, L_n)$ as

$$L_i(t,\phi) = \sum_{j=1}^{n} \int_{-r_j}^{0} \phi_j(\theta)\, d_\theta \eta_{ij}(\theta,t), \quad 1 \leq i \leq n, \tag{2.3}$$

in which $\eta_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned} &\eta_{ij}(\theta,t) = \eta_{ij}(0,t), \quad \theta \geq 0, \\ &\eta_{ij}(\theta,t) = 0, \quad \theta \leq -r_j, \\ &\eta_{ij}(\cdot,t) \in \mathrm{BV}\left[-r_j, 0\right], \end{aligned} \tag{2.4}$$

where $\eta_{ij}(\cdot,t)$ is continuous from the left in $(-r_j, 0)$.

We make the following assumptions for (2.1).

(h0) If $\phi, \psi \in C^+$, $\phi \leq \psi$, and $\phi_i(0) = \psi_i(0)$ for some i, then $f_i(t,\phi) \leq f_i(t,\psi)$.

(h1) For all $\phi \in C_r^+$ with $\phi_i(0) = 0$, $L_i(t,\phi) \geq 0$ for $t \in \mathbb{R}$.

(h2) The matrix $A(t)$ defined by

$$A(t) = \mathrm{col}(L(t,\widehat{e}_1), L(t,\widehat{e}_2), \ldots, L(t,\widehat{e}_n)) = \left(\eta_{ij}(0,t)\right) \tag{2.5}$$

is irreducible for each $t \in \mathbb{R}$.

(h3) For each j, for which $r_j > 0$, there exists i such that for all $t \in \mathbb{R}$ and for positive constant ε sufficiently small, $\eta_{ij}(-r_j + \varepsilon, t) > 0$.

(h4) If $\phi = 0$, then $x(t,t_0,\phi) \equiv 0$ for all $t \geq t_0$.

The following result was established by Wang et al. [18].

Lemma 2.1. *Let (h1)–(h4) hold. Then, the hypothesis (h0) is valid*

(i) *If ϕ and ψ are distinct elements of C_r^+ with $\phi \le \psi$ and $[t_0, t_0+\sigma)$ with $n|r| < \sigma \le \infty$ is the intersection of the maximal intervals of existence of $x(t,t_0,\phi)$ and $x(t,t_0,\psi)$, then*

$$0 \le \begin{cases} x(t,t_0,\phi) \le x(t,t_0,\psi) & \text{for } t_0 \le t < t_0+\sigma, \\ x(t,t_0,\phi) < x(t,t_0,\psi) & \text{for } t_0+n|r| \le t < t_0+\sigma. \end{cases} \tag{2.6}$$

(ii) *If $\phi \in C_r^+$, $\phi \not\equiv 0$, $t_0 \in \mathbb{R}$ and $x(t,t_0,\phi)$ is defined on $[t_0, t_0+\sigma)$ with $\sigma > n|r|$, then*

$$0 < x(t,t_0,\phi) \quad \text{for } t_0+n|r| \le t < t_0+\sigma. \tag{2.7}$$

This lemma shows that if (h1)–(h4) hold, then the positivity of solutions of (2.1) follows.

The following definition and results are useful in proving our main result in this section.

Definition 2.2. Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix, and let $P_1, \ldots, P_n$ be distinct points of the complex plane. For each nonzero element a_{ij} of A, connect P_i to P_j with a directed line $\overline{P_iP_j}$. The resulting figure in the complex plane is a directed graph for A. We say that a directed graph is strongly connected if, for each pair of nodes P_i, P_j with $i \ne j$, there is a directed path

$$\overrightarrow{P_iP_{k_1}}, \overrightarrow{P_{k_1}P_{k_2}}, \ldots, \overrightarrow{P_{k_{r-1}}P_j}, \tag{2.8}$$

connecting P_i and P_j. Here, the path consists of r directed lines.

Lemma 2.3 (see [19]). *A square matrix is irreducible if and only if its directed graph is strongly connected.*

Lemma 2.4 (see [20]). *If* (2.1) *is cooperative and irreducible in D, where D is an open subset of C, and the solutions with positive initial data is bounded, then the trajectory of* (2.1) *tends to some single equilibrium.*

We now consider the following delay differential system:

$$\begin{aligned} \dot{u}_1(t) &= \frac{a_1u_2(t-\tau)}{1+\alpha u_2(t-\tau)} - au_1(t), \\ \dot{u}_2(t) &= ku_1(t) - \frac{a_2u_2(t)}{1+\alpha u_2(t)} - uu_2(t) \end{aligned} \tag{2.9}$$

with initial conditions

$$u_i(s) = \phi_i(s) \ge 0, \quad s \in [-\tau, 0), \quad \phi_i(0) > 0, \quad \phi_i \in C([-\tau,0), \mathbb{R}_+) \quad (i = 1,2). \tag{2.10}$$

System (2.9) always has a trivial equilibrium $A^0(0,0)$. If $ka_1 - aa_2 > au$, then system (2.9) has a unique positive equilibrium $A^*(u_1^*, u_2^*)$, where

$$u_1^* = \frac{a_1(ka_1 - aa_2 - au)}{\alpha a(ka_1 - aa_2)}, \qquad u_2^* = \frac{ka_1 - aa_2 - au}{\alpha au}. \tag{2.11}$$

The characteristic equation of system (2.9) at the positive equilibrium A^0 takes the form

$$\lambda^2 + p_1\lambda + p_0 + q_0 e^{-\lambda\tau} = 0, \tag{2.12}$$

where

$$p_0 = a(a_2 + u), \qquad p_1 = a + a_2 + u, \qquad q_0 = -ka_1. \tag{2.13}$$

Noting that

$$p_1 > 0, \qquad p_0 + q_0 = au - (ka_1 - aa_2), \tag{2.14}$$

if $ka_1 - aa_2 < au$, then the equilibrium A^0 is locally stable when $\tau = 0$. If $ka_1 - aa_2 > au$, then A^0 is unstable when $\tau = 0$.

It is easy to show that $p_1^2 - 2p_0 = a^2 + (a_2 + u)^2 > 0$. If $ka_1 - aa_2 < au$, then $p_0^2 - q_0^2 > 0$. By Kuang [21], we see that the equilibrium A^0 is locally asymptotically stable for all $\tau > 0$. If $ka_1 - aa_2 > au$, then A^0 is unstable for all $\tau > 0$.

The characteristic equation of system (2.9) at the positive equilibrium A^* is of the form

$$\lambda^2 + g_1\lambda + g_0 + h_0 e^{-\lambda\tau} = 0, \tag{2.15}$$

where

$$g_0 = a\left(u + \frac{a_2}{\left(1 + \alpha u_2^*\right)^2}\right), \qquad g_1 = a + u + \frac{a_2}{\left(1 + \alpha u_2^*\right)^2}, \qquad h_0 = -\frac{ka_1}{\left(1 + \alpha u_2^*\right)^2}. \tag{2.16}$$

Note that

$$g_1 > 0, \qquad g_0 + h_0 = \frac{au[au - (ka_1 - aa_2)]}{ka_1 - aa_2}. \tag{2.17}$$

Hence, if $ka_1 - aa_2 > au$, then the positive equilibrium A^* is locally stable when $\tau = 0$. If $ka_1 - aa_2 < au$, then A^* is unstable when $\tau = 0$.

It is easy to see that

$$g_1^2 - 2g_0 = a^2 + \left(u + \frac{a_2}{\left(1 + \alpha u_2^*\right)^2}\right)^2 > 0. \tag{2.18}$$

If $ka_1 - aa_2 > au$, then $g_0^2 - h_0^2 > 0$. By Kuang [21], we see that the positive equilibrium A^* is locally asymptotically stable for all $\tau > 0$. If $ka_1 - aa_2 < au$, then A^* is unstable for all $\tau > 0$.

Lemma 2.5. *For system* (2.9)*, one hase the following.*

(i) *If* $ka_1 - aa_2 > au$*, then the positive equilibrium* $A^*(u_1^*, u_2^*)$ *is globally stable.*

(ii) *If* $ka_1 - aa_2 < au$*, then the equilibrium* $A^0(0,0)$ *is globally stable.*

Proof. We represent the right-hand side of (2.9) by $f(t, x_t) = (f_1(t, x_t), f_2(t, x_t))$, and set

$$L(t,\cdot) = D_\phi f(t,\phi). \tag{2.19}$$

By direct calculation, we have

$$\begin{aligned} L_1(t,h) &= \frac{a_1}{\left(1+\alpha\phi_2(-\tau)\right)^2} h_2(-\tau) - ah_1(0), \\ L_2(t,h) &= kh_1(0) - \frac{a_2}{\left(1+\alpha\phi_2(0)\right)^2} h_2(0) - uh_2(0). \end{aligned} \tag{2.20}$$

We now claim that the hypotheses (h1)–(h4) hold for system (2.9). It is easily seen that (h1) and (h4) hold for system (2.9). We need only to verify that (h2) and (h3) hold.

The matrix $A(t)$ takes the form

$$\begin{pmatrix} -a & \dfrac{a_1}{\left(1+\alpha\phi_2(-\tau)\right)^2} \\ k & -\dfrac{a_2}{\left(1+\alpha\phi_2(0)\right)^2} - u \end{pmatrix}. \tag{2.21}$$

Clearly, the matrix $A(t)$ is irreducible for each $t \in \mathbb{R}$.

From the definition of $A(t)$ and η_{ij}, it is readily seen that $\eta_{12}(\theta,t) = \eta_{12}(0,t) = a_1/(1+\alpha\phi_2(-\tau))^2$, $\eta_{21}(\theta,t) = \eta_{21}(0,t) = k$ for $\theta \geq 0$, $\eta_{ij}(\theta,t) = 0$, $i \neq j$ for $\theta \leq -\tau$, and $\eta_{ij}(\cdot,t) \in \mathrm{BV}[-\tau,0]$, where η_{ij} is a positive Borel measure on $[-\tau,0]$. Therefore, $\eta_{ij}(\cdot,t) > 0$. Thus, for each j, there is $i \neq j$ such that $\eta_{ij}(-r_j + \varepsilon, t) = \eta_{ij}(-\tau + \varepsilon, t) > 0$ for all $t \in \mathbb{R}$ and for $\varepsilon > 0$ sufficiently small, $i = 1, 2$. Hence, (h3) holds.

Thus, the conditions of Lemma 2.1 are satisfied. Therefore, the positivity of solutions of system (2.9) follows. It is easy to see that system (2.9) is cooperative. By Lemma 2.3, we see that any solution starting from $D = C_\tau^+$ converges to some single equilibrium. However, system (2.9) has only two equilibria: A^0 and A^*. Note that if $ka_1 - aa_2 > au$, then the positive equilibrium A^* is locally stable, and the equilibrium A^0 is unstable. Hence, any solution starting from D converges to $A^*(u_1^*, u_2^*)$ if $ka_1 - aa_2 > au$. Using a similar argument one can show the global stability of the equilibrium A^0 when $ka_1 - aa_2 < au$. This completes the proof. □

3. Local Stability

In this section, we discuss the local stability of equilibria of system (1.4) by analyzing the corresponding characteristic equations.

System (1.4) always has an infection free equilibrium $E^0(\lambda/d, 0, 0)$.

Let

$$\mathcal{R}_0 = \frac{\lambda\beta(k-a)}{aud}. \tag{3.1}$$

$\mathcal{R}_0$ is called the basic reproduction ratio of system (1.4). It is easy to show that if $\mathcal{R}_0 > 1$, system (1.4) has a virus infected equilibrium $E^*(x^*, y^*, v^*)$, where

$$x^* = \frac{au(1+\alpha v^*)}{\beta(k-a)}, \qquad y^* = \frac{u}{k-a}v^*, \qquad v^* = \frac{\lambda\beta(k-a) - aud}{au(\alpha d + \beta)}. \tag{3.2}$$

The characteristic equation of system (1.4) at the infection free equilibrium E^0 is of the form

$$(s+d)\left(s^2 + p_1 s + p_0 + q_0 e^{-s\tau}\right) = 0, \tag{3.3}$$

where

$$p_0 = a\left(u + \frac{\lambda\beta}{d}\right), \qquad p_1 = a + u + \frac{\lambda\beta}{d}, \qquad q_0 = \frac{k\lambda\beta}{d}. \tag{3.4}$$

Obviously, (3.3) always has a negative real root $s = -d$. All other roots of (3.3) are determined by

$$s^2 + p_1 s + p_0 + q_0 e^{-s\tau} = 0. \tag{3.5}$$

It is easy to show that $p_1 > 0$, $p_0 + q_0 > 0$, then the infection free equilibrium E^0 of system (1.4) is locally asymptotically stable when $\tau = 0$.

If $i\omega(\omega > 0)$ is a solution of (3.3), by calculating, we have the following:

$$\omega^4 + \left(p_1^2 - 2p_0\right)\omega^2 + p_0^2 - q_0^2 = 0. \tag{3.6}$$

Note that

$$\begin{aligned} p_1^2 - 2p_0 &= a^2 + \left(u + \frac{\lambda\beta}{d}\right) > 0, \\ p_0^2 - q_0^2 &= \frac{1}{d^2}\left[aud + \lambda\beta(k+a)\right]\left[aud + \lambda\beta(a-k)\right]. \end{aligned} \tag{3.7}$$

If $\mathcal{R}_0 < 1$, then $p_0^2 - q_0^2 > 0$. Therefore, (3.6) has no positive roots. Accordingly, if $\mathcal{R}_0 < 1$, the infection free equilibrium E^0 of system (1.4) is locally asymptotically stable for all $\tau > 0$; if $\mathcal{R}_0 > 1$, (3.6) has at least a positive real root. Accordingly, E^0 is unstable.

The characteristic equation of system (1.4) at the virus infected equilibrium $E^*(x^*, y^*, v^*)$ takes the form

$$s^3 + g_2 s^2 + g_1 s + g_0 + (h_1 s + h_0) e^{-s\tau} = 0, \tag{3.8}$$

where

$$\begin{aligned} g_0 &= a\left(du + d\frac{\beta x^*}{(1+\alpha v^*)^2} + u\frac{\beta v^*}{1+\alpha v^*}\right), \\ g_1 &= au + du + ad + (a+d)\frac{\beta x^*}{(1+\alpha v^*)^2} + (a+u)\frac{\beta v^*}{1+\alpha v^*}, \\ g_2 &= a + u + d + \frac{\beta x^*}{(1+\alpha v^*)^2} + \frac{\beta v^*}{1+\alpha v^*}, \\ h_0 &= -\frac{kd\beta x^*}{(1+\alpha v^*)^2}, \qquad h_1 = -\frac{k\beta x^*}{(1+\alpha v^*)^2}. \end{aligned} \tag{3.9}$$

When $\tau = 0$, (3.8) becomes

$$s^3 + g_2 s^2 + (g_1 + h_1)s + g_0 + h_0 = 0. \tag{3.10}$$

By direct calculation, we have

$$\begin{aligned} g_0 + h_0 &= au(\alpha d + \beta)\frac{v^*}{1+\alpha v^*} > 0, \\ g_2(g_1 + h_1) - (g_0 + h_0) &= \frac{a^2 u\alpha v^*}{1+\alpha v^*} + \frac{a^2\lambda}{x^*} + \frac{kd\beta x^*}{(1+\alpha v^*)^2} + \left[u + \frac{\lambda}{x^*} + \frac{\beta x^*}{(1+\alpha v^*)^2}\right] \\ &\quad \times \left[\frac{au\alpha v^*}{1+\alpha v^*} + \frac{(u+a)\lambda}{x^*} + \frac{d\beta x^*}{(1+\alpha v^*)^2}\right] > 0. \end{aligned} \tag{3.11}$$

Clearly, all roots of (3.10) have only negative real parts.

If $i\omega(\omega > 0)$ is a solution of (3.8), separating real and imaginary parts, it follows that

$$\begin{aligned} \omega^3 - g_1\omega &= h_1\omega\cos\omega\tau - g_0\sin\omega\tau, \\ g_2\omega^2 - g_0 &= h_1\omega\sin\omega\tau + h_0\cos\omega\tau. \end{aligned} \tag{3.12}$$

Squaring and adding the two equations of (3.12), we derive that

$$\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3 = 0, \tag{3.13}$$

where

$$C_1 = g_2^2 - 2g_1, \qquad C_2 = g_1^2 - 2g_0 g_2 - h_1^2, \qquad C_3 = g_0^2 - h_0^2. \tag{3.14}$$

Clearly, $C_3 = g_0^2 - h_0^2 > 0$. It is easy to show that

$$\begin{aligned} C_1 &= a^2 + u^2 + d^2 + \frac{\beta x^*}{1+\alpha v^*}\left[2\left(d + \frac{u}{1+\alpha v^*}\right) + \left(1 + \frac{1}{1+\alpha v^*}\right)^2\right] > 0, \\ C_2 &= \left(a^2 + u^2\right)\left(d + \frac{\beta v^*}{1+\alpha v^*}\right)^2 + \frac{au\alpha v^*}{1+\alpha v^*}\left[au + \frac{(k+a)\beta x^*}{(1+\alpha v^*)^2}\right] \\ &\quad + 2\left(a^2 + du\right)\frac{\beta^2 x^* v^*}{(1+\alpha v^*)^3} + \frac{d^2\beta x^*}{(1+\alpha v^*)^2}\left[2u + \frac{\beta x^*}{(1+\alpha v^*)^2}\right] > 0. \end{aligned} \tag{3.15}$$

Hence, (3.13) has no positive roots. Accordingly, if $\mathcal{R}_0 > 1$, the virus-infected equilibrium E^* of system (1.4) exists and is locally asymptotically stable for all $\tau > 0$.

Based on the discussions above, we have the following result.

Theorem 3.1. *For system* (1.4)*, one has the following.*

(i) *If* $\mathcal{R}_0 < 1$*, the infection free equilibrium* $E^0(\lambda/d, 0, 0)$ *is locally asymptotically stable. If* $\mathcal{R}_0 > 1$*, then* $E^0(\lambda/d, 0, 0)$ *is unstable.*

(ii) *If* $\mathcal{R}_0 > 1$*, the virus infected equilibrium* $E^*(x^*, y^*, v^*)$ *is locally asymptotically stable.*

4. Global Stability

In this section, we discuss the global stability of the uninfected equilibrium and the virus infected equilibrium of system (1.4), respectively. The technique of proofs is to use a comparison argument and an iteration scheme [22].

Theorem 4.1. *Let* $\mathcal{R}_0 > 1$*. The virus infected equilibrium* $E^*(x^*, y^*, v^*)$ *of system* (1.4) *is globally asymptotically stable provided that*

(H1) $\alpha d(k-a) > a\beta$,

(H2) $0 < (k+a)[\lambda\alpha(k+a) - au] < 4k\lambda a\alpha$.

Proof. Let $(x(t), y(t), v(t))$ be any positive solution of system (1.4) with initial condition (1.5). Let

$$\begin{aligned} U_1 &= \limsup_{t\to+\infty} x(t), & V_1 &= \liminf_{t\to+\infty} x(t), \\ U_2 &= \limsup_{t\to+\infty} y(t), & V_2 &= \liminf_{t\to+\infty} y(t), \\ U_3 &= \limsup_{t\to+\infty} v(t), & V_3 &= \liminf_{t\to+\infty} v(t). \end{aligned} \tag{4.1}$$

Now, we claim that $U_1 = V_1 = x^*$, $U_2 = V_2 = y^*$, $U_3 = V_3 = v^*$.

It follows from the first equation of system (1.4) that

$$\dot{x}(t) \le \lambda - dx(t). \tag{4.2}$$

By comparison, we derive that

$$U_1 = \limsup_{t\to+\infty} x(t) \le \frac{\lambda}{d} := M_1^x. \tag{4.3}$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \le M_1^x + \varepsilon$. We therefore, derive from the second and the third equations of system (1.4) that for $t > T_1 + \tau$,

$$\begin{aligned} \dot{y}(t) &\le \frac{\beta\left(M_1^x + \varepsilon\right)v(t-\tau)}{1+\alpha v(t-\tau)} - ay(t), \\ \dot{v}(t) &\le ky(t) - uv(t). \end{aligned} \tag{4.4}$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{u}_1(t) &= \frac{\beta\left(M_1^x + \varepsilon\right)u_2(t-\tau)}{1+\alpha u_2(t-\tau)} - au_1(t), \\ \dot{u}_2(t) &= ku_1(t) - uu_2(t). \end{aligned} \tag{4.5}$$

Since $\mathcal{R}_0 > 1$, by Lemma 2.5, it follows from (4.5) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{k\beta\left(M_1^x + \varepsilon\right) - au}{ka\alpha}, \\ \lim_{t\to+\infty} u_2(t) &= \frac{k\beta\left(M_1^x + \varepsilon\right) - au}{au\alpha}. \end{aligned} \tag{4.6}$$

By comparison, we obtain that

$$\begin{aligned} U_2 &= \limsup_{t\to+\infty} y(t) \le \frac{k\beta\left(M_1^x + \varepsilon\right) - au}{ka\alpha}, \\ U_3 &= \limsup_{t\to+\infty} v(t) \le \frac{k\beta\left(M_1^x + \varepsilon\right) - au}{au\alpha}. \end{aligned} \tag{4.7}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$, it follows that $U_2 \le M_1^y$, $U_3 \le M_1^v$, where

$$M_1^y = \frac{k\beta M_1^x - au}{ka\alpha}, \qquad M_1^v = \frac{k\beta M_1^x - au}{au\alpha}. \tag{4.8}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 \ge T_1 + \tau$ such that if $t > T_2$, $y(t) \le M_1^y + \varepsilon$, $v(t) \le M_1^v + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (1.4) that for $t > T_2$,

$$\dot{x}(t) \ge \lambda - dx(t) - \frac{\beta x(t)\left(M_1^v + \varepsilon\right)}{1+\alpha\left(M_1^v + \varepsilon\right)}. \tag{4.9}$$

A comparison argument shows that

$$V_1 = \liminf_{t\to+\infty} x(t) \geq \frac{\lambda\left(1+\alpha\left(M_1^v+\varepsilon\right)\right)}{d+\left(\alpha d+\beta\right)\left(M_1^v+\varepsilon\right)}. \tag{4.10}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$, it follows that $V_1 \geq N_1^x$, where

$$N_1^x = \frac{\lambda\left(1+\alpha M_1^v\right)}{d+\left(\alpha d+\beta\right)M_1^v}. \tag{4.11}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 \geq T_2$ such that if $t > T_3$, $x(t) \geq N_1^x - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the second and the third equations of system (1.4) that for $t > T_3 + \tau$,

$$\begin{aligned} \dot{y}(t) &\geq \frac{\beta\left(N_1^x-\varepsilon\right)v(t-\tau)}{1+\alpha v(t-\tau)} - ay(t), \\ \dot{v}(t) &\geq ky(t) - \frac{\beta\left(M_1^x+\varepsilon\right)v(t)}{1+\alpha v(t)} - uv(t). \end{aligned} \tag{4.12}$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{u}_1(t) &= \frac{\beta\left(N_1^x-\varepsilon\right)u_2(t-\tau)}{1+\alpha u_2(t-\tau)} - au_1(t), \\ \dot{u}_2(t) &= ku_1(t) - \frac{\beta\left(M_1^x+\varepsilon\right)u_2(t)}{1+\alpha u_2(t)} - uu_2(t). \end{aligned} \tag{4.13}$$

Since (H1) holds, by Lemma 2.5, it follows from (4.13) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{\beta\left(N_1^x-\varepsilon\right)\left[k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)-au\right]}{a\alpha\left[k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)\right]}, \\ \lim_{t\to+\infty} u_2(t) &= \frac{k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)-au}{au\alpha}. \end{aligned} \tag{4.14}$$

By comparison, we derive that

$$\begin{aligned} V_2 = \liminf_{t\to+\infty} y(t) &\geq \frac{\beta\left(N_1^x-\varepsilon\right)\left[k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)-au\right]}{a\alpha\left[k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)\right]}, \\ V_3 = \liminf_{t\to+\infty} v(t) &\geq \frac{k\beta\left(N_1^x-\varepsilon\right)-a\beta\left(M_1^x+\varepsilon\right)-au}{au\alpha}. \end{aligned} \tag{4.15}$$

Since these two inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_1^y$, $V_3 \geq N_1^v$, where

$$N_1^y = \frac{\beta N_1^x \left(k\beta N_1^x - a\beta M_1^x - au\right)}{a\alpha\left(k\beta N_1^x - a\beta M_1^x\right)}, \qquad N_1^v = \frac{k\beta N_1^x - a\beta M_1^x - au}{au\alpha}. \tag{4.16}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_4 \geq T_3 + \tau$ such that if $t > T_4$, $y(t) \geq N_1^y - \varepsilon, v(t) \geq N_1^v - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (1.4) that for $t > T_4$,

$$\dot{x}(t) \leq \lambda - dx(t) - \frac{\beta x(t)\left(N_1^v - \varepsilon\right)}{1 + \alpha\left(N_1^v - \varepsilon\right)}. \tag{4.17}$$

By comparison, we derive that

$$U_1 = \limsup_{t\to+\infty} x(t) \leq \frac{\lambda\left(1 + \alpha\left(N_1^v - \varepsilon\right)\right)}{d + \left(\alpha d + \beta\right)\left(N_1^v - \varepsilon\right)}. \tag{4.18}$$

Since this is true for arbitrary $\varepsilon > 0$, it follows that $U_1 \leq M_2^x$, where

$$M_2^x = \frac{\lambda\left(1 + \alpha N_1^v\right)}{d + \left(\alpha d + \beta\right)N_1^v}. \tag{4.19}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_5 \geq T_4$ such that if $t > T_5$, $x(t) \leq M_2^x + \varepsilon$. It therefore, follows from the second and the third equations of system (1.4) that for $t > T_5 + \tau$,

$$\begin{aligned} \dot{y}(t) &\leq \frac{\beta\left(M_2^x + \varepsilon\right)v(t-\tau)}{1 + \alpha v(t-\tau)} - ay(t), \\ \dot{v}(t) &\leq ky(t) - \frac{\beta\left(N_2^x - \varepsilon\right)v(t-\tau)}{1 + \alpha v(t-\tau)} - uv(t). \end{aligned} \tag{4.20}$$

By Lemma 2.5 and a comparison argument, we derive from (4.20) that

$$\begin{aligned} U_2 = \limsup_{t\to+\infty} y(t) &\leq \frac{\beta\left(M_2^x + \varepsilon\right)\left[k\beta\left(M_2^x + \varepsilon\right) - a\beta\left(N_1^x - \varepsilon\right) - au\right]}{a\alpha\left[k\beta\left(M_2^x + \varepsilon\right) - a\beta\left(N_1^x - \varepsilon\right)\right]}, \\ U_3 = \limsup_{t\to+\infty} v(t) &\leq \frac{k\beta\left(M_2^x + \varepsilon\right) - a\beta\left(N_1^x - \varepsilon\right) - au}{au\alpha}. \end{aligned} \tag{4.21}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$, it follows that $U_2 \leq M_2^y$, $U_3 \leq M_2^v$, where

$$M_2^y = \frac{\beta M_2^x \left(k\beta M_2^x - a\beta N_1^x - au\right)}{a\alpha\left(k\beta M_2^x - a\beta N_1^x\right)}, \qquad M_2^v = \frac{k\beta M_2^x - a\beta N_1^x - au}{au\alpha}. \tag{4.22}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \geq T_5 + \tau$ such that if $t > T_6$, $y(t) \leq M_2^y + \varepsilon$, $v(t) \leq M_2^v + \varepsilon$.

Again, for $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (1.4) that for $t > T_6$,

$$\dot{x}(t) \geq \lambda - dx(t) - \frac{\beta x(t)\left(M_2^v + \varepsilon\right)}{1 + \alpha\left(M_2^v + \varepsilon\right)}. \tag{4.23}$$

A comparison argument shows that

$$V_1 = \liminf_{t \to +\infty} x(t) \geq \frac{\lambda\left(1 + \alpha\left(M_2^v + \varepsilon\right)\right)}{d + \left(\alpha d + \beta\right)\left(M_2^v + \varepsilon\right)}. \tag{4.24}$$

Since this is true for arbitrary $\varepsilon > 0$, it follows that $V_1 \geq N_2^x$, where

$$N_2^x = \frac{\lambda\left(1 + \alpha M_2^v\right)}{d + \left(\alpha d + \beta\right) M_2^v}. \tag{4.25}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 \geq T_6$ such that if $t > T_7$, $x(t) \geq N_2^x - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the second and the third equations of system (1.4) that for $t > T_7 + \tau$,

$$\begin{aligned} \dot{y}(t) &\geq \frac{\beta\left(N_2^x - \varepsilon\right)v(t-\tau)}{1 + \alpha v(t-\tau)} - ay(t), \\ \dot{v}(t) &\geq ky(t) - \frac{\beta\left(M_2^x + \varepsilon\right)v(t)}{1 + \alpha v(t)} - uv(t). \end{aligned} \tag{4.26}$$

Since (H1) holds, by Lemma 2.5 and a comparison argument, it follows from (4.26) that

$$\begin{aligned} V_2 = \liminf_{t \to +\infty} y(t) &\geq \frac{\beta\left(N_2^x - \varepsilon\right)\left[k\beta\left(N_2^x - \varepsilon\right) - a\beta\left(M_2^x + \varepsilon\right) - au\right]}{a\alpha\left[k\beta\left(N_2^x - \varepsilon\right) - a\beta\left(M_2^x + \varepsilon\right)\right]}, \\ V_3 = \liminf_{t \to +\infty} v(t) &\geq \frac{k\beta\left(N_2^x - \varepsilon\right) - a\beta\left(M_2^x + \varepsilon\right) - au}{au\alpha}. \end{aligned} \tag{4.27}$$

Since these two inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_2^y$, $V_3 \geq N_2^v$, where

$$N_2^y = \frac{\beta N_2^x\left(k\beta N_2^x - a\beta M_2^x - au\right)}{a\alpha\left(k\beta N_2^x - a\beta M_2^x\right)}, \qquad N_2^v = \frac{k\beta N_2^x - a\beta M_2^x - au}{au\alpha}. \tag{4.28}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_8 \geq T_7 + \tau$ such that if $t > T_8$, $y(t) \geq N_2^y - \varepsilon$, $v(t) \geq N_2^v - \varepsilon$.

Continuing this process, we derive six sequences M_n^x, M_n^y, M_n^v, N_n^x, N_n^y, N_n^v ($n = 1, 2, \ldots$) such that for $n \geq 2$,

$$\begin{aligned}
M_n^x &= \frac{\lambda\left(1+\alpha N_{n-1}^v\right)}{d+(\alpha d+\beta)N_{n-1}^v},\\
M_n^y &= \frac{\beta M_n^x\left(k\beta M_n^x - a\beta N_{n-1}^x - au\right)}{a\alpha\left(k\beta M_n^x - a\beta N_{n-1}^x\right)},\\
M_n^v &= \frac{k\beta M_n^x - a\beta N_{n-1}^x - au}{au\alpha},\\
N_n^x &= \frac{\lambda(1+\alpha M_n^v)}{d+(\alpha d+\beta)M_n^v},\\
N_n^y &= \frac{\beta N_n^x\left(k\beta N_n^x - a\beta M_n^x - au\right)}{a\alpha\left(k\beta N_n^x - a\beta M_n^x\right)},\\
N_n^v &= \frac{k\beta N_n^x - a\beta M_n^x - au}{au\alpha}.
\end{aligned} \tag{4.29}$$

Clearly, we have

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, \qquad N_n^y \leq V_2 \leq U_2 \leq M_n^y, \qquad N_n^v \leq V_3 \leq U_3 \leq M_n^v. \tag{4.30}$$

It is easy to show that the sequences M_n^x, M_n^y, and M_n^v are nonincreasing, and the sequences N_n^x, N_n^y, N_n^v are nondecreasing. Hence, the limit of each sequence in M_n^x, M_n^y, M_n^v, N_n^x, N_n^y, and N_n^v exists. Denote

$$\begin{aligned}
\overline{x} &= \lim_{n\to+\infty} M_n^x, \qquad \underline{x} = \lim_{n\to+\infty} N_n^x,\\
\overline{y} &= \lim_{n\to+\infty} M_n^y, \qquad \underline{y} = \lim_{n\to+\infty} N_n^y,\\
\overline{v} &= \lim_{n\to+\infty} M_n^v, \qquad \underline{v} = \lim_{n\to+\infty} N_n^v.
\end{aligned} \tag{4.31}$$

We, therefore, obtain from (4.29) and (4.31) that

$$a(\alpha d+\beta)\overline{x}^2 - k(\alpha d+\beta)\overline{x}\,\underline{x} = a(\lambda\alpha - u)\overline{x} - k\lambda\alpha\underline{x}, \tag{4.32}$$

$$a(\alpha d+\beta)\underline{x}^2 - k(\alpha d+\beta)\overline{x}\,\underline{x} = a(\lambda\alpha - u)\underline{x} - k\lambda\alpha\overline{x}. \tag{4.33}$$

(4.32) minus (4.33),

$$a(\alpha d+\beta)\left(\overline{x}^2 - \underline{x}^2\right) = (\lambda a\alpha + \lambda k\alpha - au)(\overline{x} - \underline{x}). \tag{4.34}$$

Assume that $\overline{x} \neq \underline{x}$. Then, we derive from (4.34) that

$$\overline{x} + \underline{x} = \frac{\lambda a\alpha + \lambda k\alpha - au}{a(\alpha d + \beta)}. \tag{4.35}$$

(4.32) plus (4.33),

$$a(\alpha d + \beta)(\overline{x} + \underline{x})^2 - 2(k + a)(\alpha d + \beta)\overline{x}\underline{x} = (\lambda a\alpha - \lambda k\alpha - au)(\overline{x} + \underline{x}). \tag{4.36}$$

On substituting (4.35) into (4.36), it follows that

$$\overline{x}\underline{x} = \frac{\lambda k\alpha}{a(k + a)(\alpha d + \beta)^2}(\lambda a\alpha + \lambda k\alpha - au). \tag{4.37}$$

Note that $\overline{x} > 0$, $\underline{x} > 0$. Let (H2) hold. It follows from (4.35) and (4.37) that

$$(\overline{x} + \underline{x})^2 - 4\overline{x}\underline{x} = \frac{(\lambda a\alpha + \lambda k\alpha - au)}{a^2(\alpha d + \beta)^2}\left[(\lambda a\alpha + \lambda k\alpha - au) - \frac{4k\lambda a\alpha}{k + a}\right]. \tag{4.38}$$

Hence, we have $(\overline{x} + \underline{x})^2 - 4\overline{x}\underline{x} < 0$. This is a contradiction. Accordingly, we have $\overline{x} = \underline{x}$. We, therefore, derive from (4.31) that $\overline{y} = \underline{y}$, $\overline{v} = \underline{v}$. Note that if (H1) and (H2) hold, by Theorem 3.1, the virus-infected equilibrium E^* is locally stable, we conclude that E^* is globally stable. The proof is complete. □

Theorem 4.2. *If $\mathcal{R}_0 < 1$ holds, the infection-free equilibrium $E^0(\lambda/d, 0, 0)$ of system (1.4) is globally asymptotically stable.*

Proof. Let $(x(t), y(t), v(t))$ be any positive solution of system (1.4) with initial condition (1.5). If $\mathcal{R}_0 < 1$, choose $\varepsilon > 0$ sufficiently small satisfying

$$k\beta\left(\frac{\lambda}{d} + \varepsilon\right) < au. \tag{4.39}$$

It follows from the first equation of system (1.4) that

$$\dot{x}(t) \leq \lambda - dx(t). \tag{4.40}$$

By comparison, we derive that

$$\limsup_{t \to +\infty} x(t) \leq \frac{\lambda}{d}. \tag{4.41}$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (4.39), there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \le (\lambda/d) + \varepsilon$. We, therefore, derive from the second and the third equations of system (1.4) that for $t > T_1 + \tau$,

$$\begin{aligned} \dot{y}(t) &\le \frac{\beta((\lambda/d) + \varepsilon)v(t-\tau)}{1 + \alpha v(t-\tau)} - ay(t), \\ \dot{v}(t) &\le ky(t) - uv(t). \end{aligned} \tag{4.42}$$

Consider the following auxiliary equation:

$$\begin{aligned} \dot{u}_1(t) &= \frac{\beta((\lambda/d) + \varepsilon)u_2(t-\tau)}{1 + \alpha u_2(t-\tau)} - au_1(t), \\ \dot{u}_2(t) &= ku_1(t) - uu_2(t). \end{aligned} \tag{4.43}$$

If $\mathcal{R}_0 < 1$, then by Lemma 2.5, it follows from (4.41) and (4.43) that

$$\lim_{t \to +\infty} u_1(t) = 0, \qquad \lim_{t \to +\infty} u_1(t) = 0. \tag{4.44}$$

By comparison, we obtain that

$$\lim_{t \to +\infty} y(t) = 0, \qquad \lim_{t \to +\infty} v(t) = 0. \tag{4.45}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1 + \tau$ such that if $t > T_2$, $y(t) < \varepsilon$, $v(t) < \varepsilon$.

It follows from the first equation of system (1.4) that for $t > T_2$,

$$\dot{x}(t) \ge \lambda - dx(t) - \frac{\beta x(t)\varepsilon}{1 + \alpha\varepsilon}. \tag{4.46}$$

By comparison, we derive that

$$\liminf_{t \to +\infty} x(t) \ge \frac{\lambda(1 + \alpha\varepsilon)}{d + (\alpha d + \beta)(1 + \alpha\varepsilon)}. \tag{4.47}$$

Letting $\varepsilon \to 0$, it follows that

$$\liminf_{t \to +\infty} x(t) \ge \frac{\lambda}{d}. \tag{4.48}$$

Noting that (4.41) holds, we conclude that

$$\lim_{t \to +\infty} x(t) = \frac{\lambda}{d}. \tag{4.49}$$

This completes the proof. □

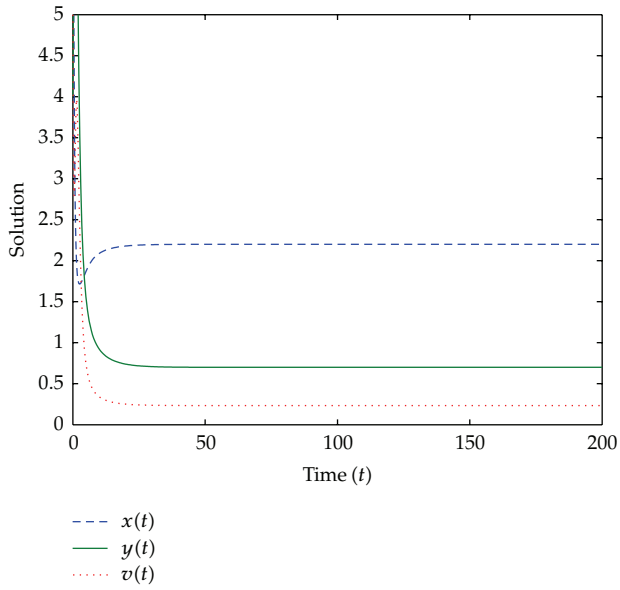

Figure 1: The numerical solution of system (1.4) when $\alpha = 2$, $\beta = 2$, $\lambda = 4$, $a = 1$, $d = 1.5$, $k = 2$, $u = 3$, $\tau = 1$, and $(\phi_1, \phi_2, \phi_3) = (10, 4, 6)$.

5. Numerical Examples

In this section, we give two examples to illustrate the main theoretical results above.

Example 5.1. In system (1.4), let $\alpha = 2$, $\beta = 2$, $\lambda = 4$, $a = 1$, $d = 1.5$, $k = 2$, $u = 3$, and $\tau = 1$. By calculation, we have $\mathcal{R}_0 = 16/9 > 1$, and system (1.4) has a virus-infected equilibrium $E^*(11/5, 7/10, 7/30)$. Clearly, (H1) and (H2) hold. By Theorem 4.1, we see that E^* is globally asymptotically stable. Numerical simulation illustrates the above result (see Figure 1).

Example 5.2. In system (1.4), let $\alpha = 1$, $\beta = 1$, $\lambda = 2.5$, $a = 2$, $d = 1$, $k = 1$, $u = 2$, and $\tau = 3$. Noting that $\mathcal{R}_0 = -5/8 < 1$, system (1.4) always has an infection-free equilibrium $E^0(5/2, 0, 0)$. By Theorem 4.2, we see that E^0 is globally asymptotically stable. Numerical simulation illustrates this fact (see Figure 2).

6. Conclusions

In this paper, we have discussed a virus infection model with time delay, absorption, and saturation incidence. The basic reproduction ratio $\mathcal{R}_0$ was found. We investigated the global asymptotic stability of the infection-free equilibrium and the virus-infected equilibrium of system (1.4), respectively. When the basic reproduction ratio is greater than unity, by using the iteration scheme, we have established sufficient conditions for the global stability of the virus-infected equilibrium of system (1.4). By Theorem 4.1, we see that when $\mathcal{R}_0 > 1$ and (H1) and (H2) hold, the virus-infected equilibrium is globally stable. Biologically, these indicate that when the death rate of infected cells and the production rate of free viruses from infected cells are sufficiently large, then the solutions of system (1.4) tend to the virus infected equilibrium which means that the virus persists in the host. On the other hand, by

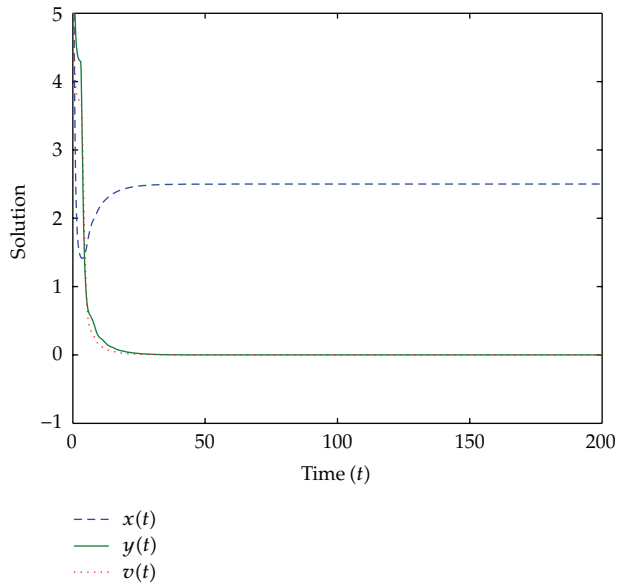

Figure 2: The numerical solution of system (1.4) when $\alpha = 1$, $\beta = 1$, $\lambda = 2.5$, $a = 2$, $d = 1$, $k = 1$, $u = 2$, $\tau = 3$, and $(\phi_1, \phi_2, \phi_3) = (10, 8, 6)$.

Theorem 4.2, we see that if the basic reproduction ratio is less than unity, the infection free equilibrium is globally asymptotically stable. Biologically, if the rate at which new uninfected cells are generated and the average number of effective contacts of one infective individual per unit time are small enough and the death rates of uninfected cells, infected cells and pathogens are large enough such that $\mathcal{R}_0 < 1$, then the virus is cleared. We would like to point out here that Theorem 4.1 has room for improvement, we leave this for future work.

Acknowledgments

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