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Research Article

On the Values of the Weighted q-Zeta and L-Functions

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Recently, the modified q-Bernoulli numbers and polynomials are introduced in (D. V. Dolgy et al., in press). These numbers are valuable to study the weighted q-zeta and L-functions. In this paper, we study the weighted q-zeta functions and weighted L-functions from the modified q-Bernoulli numbers and polynomials with weight α.

1. Introduction

Let $q \in \mathbb{C}$ with $|q| < 1$. The modified q-Bernoulli numbers and polynomials with weight α are defined by

$$\widetilde{B}_{0,q}^{(\alpha)} = \alpha \frac{q-1}{\log q}, \qquad \left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)} + 1\right)^{n} - \widetilde{B}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.1}$$

with the usual convention about replacing $(\widetilde{B}_{q}^{(\alpha)})^{n}$ by $\widetilde{B}_{n,q}^{(\alpha)}$ (see [1, 2]).

Throughout this paper, we use the notation of q-number as

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.2}$$

(see [1–14]).

From (1.1), we note that

$$\widetilde{B}_{n,q}^{(\alpha)} = \frac{1}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{\alpha l}{[\alpha l]_q} = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{\alpha l}{[\alpha l]_q}. \tag{1.3}$$

Let $\widetilde{F}_q^{(\alpha)}(t) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}t^n/n!$, Then, by (1.3), we get

$$\widetilde{F}_q^{(\alpha)}(t) = \alpha\frac{q-1}{\log q}e^{(1/(1-q^{\alpha}))t} - \frac{\alpha t}{[\alpha]_q}\sum_{m=0}^{\infty}q^{\alpha m}e^{[m]_{q^{\alpha}}t}. \tag{1.4}$$

Let us define the modified q-Bernoulli polynomials with weight α as follows:

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-l}q^{\alpha l x}\widetilde{B}_{l,q}^{(\alpha)} = \left([x]_{q^{\alpha}} + q^{x\alpha}\widetilde{B}_q^{(\alpha)}\right)^n, \tag{1.5}$$

with the usual convention about replacing $(\widetilde{B}_q^{(\alpha)})^n$ by $\widetilde{B}_{n,q}^{(\alpha)}$ (see [1–13]).

From (1.5), we can derive the following equation:

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \frac{1}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\alpha l}{[\alpha l]_q} = \frac{1}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{\alpha l}{[\alpha l]_q}, \tag{1.6}$$

(see [2]).

Let $\widetilde{F}_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}(x)t^n/n!$, then, by (1.6), we get

$$\widetilde{F}_q^{(\alpha)}(t,x) = \alpha\frac{q-1}{\log q}e^{(1/(1-q^{\alpha}))t} - t\frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}q^{\alpha(m+x)}e^{[m+x]_{q^{\alpha}}t}. \tag{1.7}$$

In this paper, we consider the generalized q-Bernoulli numbers with weight α, and we study the weighted q-zeta function and q-analogue of L-function with weight α from the modified q-Bernoulli numbers and polynomials with weight α.

2. Weighted q-Zeta Function and Weighted q-L-Function

From (1.7), we note that

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \frac{\alpha}{(1-q)^n[\alpha]_q^n}\left(\frac{q-1}{\log q}\right) - \frac{n\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}q^{\alpha(m+x)}[m+x]_{q^{\alpha}}^{n-1}. \tag{2.1}$$

For $n \in \mathbb{N}$, we have

$$-\frac{\widetilde{B}_{n,q}^{(\alpha)}(x)}{n} = \left(\frac{\alpha}{[\alpha]_q}\right)\left(\frac{1}{1-q^{\alpha}}\right)^{n-1}\left(\frac{1}{\log q}\right) + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha(m+x)}[m+x]_{q^{\alpha}}^{n-1}. \tag{2.2}$$

Let $\Gamma(s)$ be the gamma function, then we consider the following complex integral. For $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)}\int_0^{\infty} \widetilde{F}_q^{(\alpha)}(-t,x)t^{s-2}dt = \frac{\alpha}{s-1}\frac{q-1}{\log q}(1-q^{\alpha})^{s-1} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}}, \tag{2.3}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Now, we define the twisted Hurwitz's type q-zeta function as follows.

For $s \in \mathbb{C}$, define

$$\widetilde{\zeta}_q^{(\alpha)}(s,x) = \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{(1-q^{\alpha})^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}}, \tag{2.4}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Note that $\widetilde{\zeta}_q^{(\alpha)}(s,x)$ is meromorphic function whole in complex s-plane except for $s = 1$. From (2.3) and (2.4), we can derive the following equation:

$$\widetilde{\zeta}_q^{(\alpha)}(s,x) = \frac{1}{\Gamma(s)}\int_0^{\infty} \widetilde{F}_q^{(\alpha)}(-t,x)t^{s-2}dt. \tag{2.5}$$

By (1.7), (2.3), (2.4), (2.5), and Laurent series, we get

$$\widetilde{\zeta}_q^{(\alpha)}(1-k,x) = -\frac{\widetilde{B}_{k,q}^{(\alpha)}(x)}{k}, \tag{2.6}$$

where $k \in \mathbb{N}$.

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *For $k \in \mathbb{N}$, one has*

$$\widetilde{\zeta}_q^{(\alpha)}(1-k,x) = -\frac{\widetilde{B}_{k,q}^{(\alpha)}(x)}{k}. \tag{2.7}$$

From (2.4), *one notes that*

$$\begin{aligned}\widetilde{\zeta}_q^{(\alpha)}(s,1) &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{(1-q^{\alpha})^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+1)}}{[m+1]_{q^{\alpha}}^{s}} \\ &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{(1-q^{\alpha})^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=1}^{\infty}\frac{q^{\alpha m}}{[m]_{q^{\alpha}}^{s}}.\end{aligned} \tag{2.8}$$

Now, by (2.8)*, one defines the weighted q-zeta function as follows:*

$$\begin{aligned}\tilde{\zeta}_q^{(\alpha)}(s) &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{\left(1-q^{\alpha}\right)^s}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=1}^{\infty}\frac{q^{\alpha m}}{[m]_{q^{\alpha}}^s}. \\ &= \tilde{\zeta}_q^{(\alpha)}(s,1).\end{aligned} \tag{2.9}$$

For $k \in \mathbb{N}$, by (1.1) *and* (1.5)*, one gets*

$$\begin{aligned}\tilde{\zeta}_q^{(\alpha)}(1-k) = \tilde{\zeta}_q^{(\alpha)}(1-k,1) &= -\frac{\tilde{B}_{k,q}^{(\alpha)}(1)}{k} \\ &= \begin{cases} -\left(\dfrac{\alpha}{[\alpha]_q} + \tilde{B}_{1,q}^{(\alpha)}\right) & \text{if } k = 1, \\ -\dfrac{\tilde{B}_{k,q}^{(\alpha)}}{k} & \text{if } k > 1. \end{cases}\end{aligned} \tag{2.10}$$

Therefore, by (2.10)*, one obtains the following corollary.*

Corollary 2.2. *For $k \in \mathbb{N}$, one has*

$$\tilde{\zeta}_q^{(\alpha)}(1-k) = \begin{cases} -\left(\dfrac{\alpha}{[\alpha]_q} + \tilde{B}_{1,q}^{(\alpha)}\right) & \text{if } k = 1, \\ -\dfrac{\tilde{B}_{k,q}^{(\alpha)}}{k} & \text{if } k > 1. \end{cases} \tag{2.11}$$

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Let us consider the generalized q-Bernoulli polynomials with weight α as follows:

$$\begin{aligned}\tilde{F}_{q,\chi}^{(\alpha)}(t,x) &= \frac{\alpha}{[\alpha]_q}t\sum_{m=0}^{\infty}\chi(m)q^{\alpha(m+x)}e^{[m+x]_{q^{\alpha}}t} \\ &= \sum_{n=0}^{\infty}\tilde{B}_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

The sequence $\tilde{B}_{n,\chi,q}^{(\alpha)}(x)$ will be called the nth generalized q-Bernoulli polynomials with weight α attached to χ.

In the special case, $x = 0$, $\tilde{B}_{n,\chi,q}^{(\alpha)}(0) = \tilde{B}_{n,\chi,q}^{(\alpha)}$ are called the nth generalized q-Bernoulli numbers with weight α attached to χ.

From (1.7) *and* (2.12)*, one notes that*

$$\tilde{F}_{q,\chi}^{(\alpha)}(t,x) = \frac{1}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\tilde{F}_{q^d}^{(\alpha)}\left([d]_{q^{\alpha}}t, \frac{x+a}{d}\right). \tag{2.13}$$

Thus, by (2.13)*, one gets*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.14}$$

Therefore, by (2.14)*, one obtains the following theorem.*

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.15}$$

In the special case, $x = 0$, one obtains the following corollary.

Corollary 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)} = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \tag{2.16}$$

Let

$$\begin{aligned} \widetilde{F}_{q,\chi}^{(\alpha)}(t) &= \frac{\alpha}{[\alpha]_q} t \sum_{m=0}^{\infty} \chi(m) q^{\alpha m} e^{[m]_{q^\alpha} t} \\ &= \sum_{n=0}^{\infty} \widetilde{B}_{n,\chi,q}^{(\alpha)} \frac{t^n}{n!}, \end{aligned} \tag{2.17}$$

then, by (2.12) *and* (2.17)*, one easily gets*

$$\frac{\widetilde{B}_{n,\chi,q}^{(\alpha)}(d) - \widetilde{B}_{n,\chi,q}^{(\alpha)}}{n} = \frac{\alpha}{[\alpha]_q} \sum_{l=0}^{d-1} \chi(l) q^{\alpha l} [l]_{q^\alpha}^{n-1}. \tag{2.18}$$

For $s \in \mathbb{C}$, consider

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty \widetilde{F}_{q,\chi}^{(\alpha)}(-t,x) t^{s-2} dt &= \frac{\alpha}{[\alpha]_q} \frac{1}{\Gamma(s)} \int_0^\infty \sum_{m=0}^{\infty} \chi(m) q^{\alpha(m+x)} e^{-[m+x]_{q^\alpha} t} t^{s-1} dt \\ &= \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \frac{\chi(m) q^{\alpha(m+x)}}{[m+x]_{q^\alpha}^s} \frac{1}{\Gamma(s)} \int_0^\infty e^{-y} y^{s-1} dy \\ &= \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \frac{\chi(m) q^{\alpha(m+x)}}{[m+x]_{q^\alpha}^s}, \end{aligned} \tag{2.19}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Now, one defines Hurwitz's type q-L-function with weight α as follows. For $s \in \mathbb{C}$,

$$\widetilde{L}_q^{(\alpha)}(s,\chi \mid x)(-t,x) = \frac{\alpha}{[\alpha]_q}\sum_{n=0}^{\infty}\frac{\chi(n)q^{(n+x)\alpha}}{[n+x]_{q^\alpha}^{s}}, \tag{2.20}$$

where $x \neq 0,-1,-2,-3,\ldots$.

From (2.19) *and* (2.20), *one notes that*

$$\widetilde{L}_q^{(\alpha)}(s,\chi \mid x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\widetilde{F}_{q,\chi}^{(\alpha)}(-t,x)t^{s-2}dt. \tag{2.21}$$

By (1.7) *and* (2.21) *and Laurent series, one obtains the following theorem.*

Theorem 2.5. *For* $k \in \mathbb{N}$, *one has*

$$\widetilde{L}_q^{(\alpha)}(1-k,\chi \mid x) = -\frac{\widetilde{B}_{k,\chi,q}^{(\alpha)}(x)}{k}. \tag{2.22}$$

In the special case, $x = 0$, $\widetilde{L}_q^{(\alpha)}(1-k,\chi \mid 0) = \widetilde{L}_q^{(\alpha)}(1-k,\chi)$ *are called the q-L-function with weight* α.

Let

$$\begin{aligned}\widetilde{F}_q^{(\alpha)}(s,a \mid F) &= \frac{\alpha}{[F]_q[\alpha]_q}\left(\sum_{\substack{m\equiv a(\mathrm{mod}\, F)\\ m>0}}^{\infty}\frac{q^{\alpha m}}{[m]_{q^\alpha}^{s}} + \frac{(1-q^\alpha)^s}{F(1-s)\log q}\right)\\ &= \frac{\alpha}{[F]_q[\alpha]_q}\left(\sum_{n=0}^{\infty}\frac{q^{\alpha(a+nF)}}{[a+nF]_{q^\alpha}^{s}} + \frac{(1-q^\alpha)^s}{F(1-s)\log q}\right)\\ &= \frac{[F]_{q^\alpha}}{[F]_q[F]_{q^\alpha}^{s}}\widetilde{\zeta}_{q^F}^{(\alpha)}\left(s,\frac{a}{F}\right),\end{aligned} \tag{2.23}$$

where a *and* F *are positive integers with* $0 < a < F$.

Then, by (2.23), *one gets*

$$\widetilde{H}_q^{(\alpha)}(1-n,a \mid F) = -\frac{[F]_{q^\alpha}^{n}\widetilde{B}_{n,\chi,q}^{(\alpha)}(a/F)}{[F]_q n}, \quad n \geq 1, \tag{2.24}$$

and $\widetilde{H}_q^{(\alpha)}(s,a \mid F)$ *has as simple pole as* $s = 1$ *with residue* $(\alpha/[F]_q)((q-1)/\log q^F)$.

Let χ *be the Dirichlet character with conductor* F, *then one easily sees that*

$$\widetilde{L}_q^{(\alpha)}(s,\chi) = \sum_{a=1}^{F}\chi(a)\widetilde{H}_q^{(\alpha)}(s,a \mid F). \tag{2.25}$$

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