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Research Article

Existence of Nontrivial Solutions and Sign-Changing Solutions for Nonlinear Dynamic Equations on Time Scales

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By using the topological degree theory and the fixed point index theory, the existence of nontrivial solutions and sign-changing solutions for a class of boundary value problem on time scales is obtained. We should point out that the integral operator corresponding to above boundary value problem is not assumed to be a cone mapping.

1. Introduction

Let $\mathbb{T}$ be a time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. For each interval I of $\mathbb{R}$, we define $I_{\mathbb{T}} = I \cap \mathbb{T}$.

In this paper, we study the existence of nontrivial solutions and sign-changing solutions of the following problem on time scales:

$$\begin{gathered} -\left[r(t)u^{\Delta}(t)\right]^{\Delta} = f(t, u(\sigma(t))), \quad t \in [a,b]_{\mathbb{T}}, \\ \alpha u(a) - \beta u^{\Delta}(a) = 0, \qquad \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) = 0, \end{gathered} \tag{1.1}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$, $(\alpha^2+\beta^2)(\gamma^2+\delta^2) \neq 0$, $r(t) > 0$, $t \in [a, \sigma(b)]_{\mathbb{T}}$, and r^{Δ} exists. Some basic definitions on time scales can be found in [1–3].

In an ordered Banach space, much interest has developed regarding the computation of the fixed-point index about cone mappings. Based on it, the existence of positive solutions of various dynamic equations has been studied extensively by numerous researchers. The reader is referred to [4–13] for some recent works on second-order boundary value problem on time scales.

Dynamic equations on time scales have attracted considerable interest because of their ability to model economic phenomena, see [14–16] and references therein. The existence of mathematics framework to describe plays an important role in the advancement in all the social sciences. Time scales calculus is likely to be applied in many fields of economics, which can combine the standard discrete and continuous models. Historically, two separate approaches have dominated mathematics modeling: differential equation, termed continuous dynamic modeling, and difference equation, termed discrete dynamic modeling. Certain economically important phenomena do not solely possess elements of the continuous or modeling elements of the discrete. For example, in a discrete model, an enterprise obtains some net income in a period of time and decides how many dividends to extend and how many to leave as retained earnings during the same period. Thus, all decisions are assumed to be made at evenly spaced intervals. The time scales to this optimization problem are much more flexible and realistic. For example, an enterprise obtains net income at one point in time, dividend-extending decisions are made at a different point in time, and keeping retained earnings takes place at yet another point in time. Moreover, dividends-extending and retained earning decisions can be modeled to occur with arbitrary, time-varying frequency. It is hard to overestimate the advantages of such an approach over the discrete or continuous models used in economics. In recent years, there is much attention paid to the existence of positive solution for second-order boundary value problems on time scales; for details, see [4–11] and references therein.

In [5], Erbe and Peterson considered the existence of positive solutions to the following problem:

$$\begin{gathered} -x^{\Delta\Delta}(t) = f(t, x(\sigma(t))), \quad t \in [a, b]_{\mathbb{T}}, \\ \alpha x(a) - \beta x^{\Delta}(a) = 0, \qquad \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0, \end{gathered} \tag{1.2}$$

where $f : [a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. The following conditions are imposed on f:

$$f_0 := \lim_{u \to 0^+} \frac{f(t,u)}{u} = 0, \quad f_\infty := \lim_{u \to +\infty} \frac{f(t,u)}{u} = +\infty, \quad \text{uniformly on } t \in [a, \sigma(b)]_{\mathbb{T}} \tag{1.3}$$

or

$$f_0 := \lim_{u \to 0^+} \frac{f(t,u)}{u} = +\infty, \quad f_\infty := \lim_{u \to +\infty} \frac{f(t,u)}{u} = 0, \quad \text{uniformly on } t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{1.4}$$

The authors obtained the existence of positive solutions for problem (1.2) by means of the cone expansion and compression fixed-point theorem.

In [6], Chyan and Henderson were concerned with determining values of λ for which there exist positive solutions of the following nonlinear dynamic equation:

$$x^{\Delta\Delta}(t) + \lambda a(t) f(x(\sigma(t))) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \tag{1.5}$$

satisfying either the conjugate boundary conditions

$$x(0) = x(\sigma(1)) = 0 \tag{1.6}$$

or the right focal boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0. \tag{1.7}$$

Their analysis still relied on the cone expansion and compression fixed-point theorem.

Furthermore, Hong and Yeh [9] generalized the main theorems in [6] to the following problem

$$\begin{gathered} x^{\Delta\Delta}(t) + \lambda f(t, x(\sigma(t))) = 0, \quad t \in [0,1]_{\mathbb{T}}, \\ \alpha u(0) - \beta x^{\Delta}(0) = 0, \qquad \gamma x(\sigma(1)) + \delta x^{\Delta}(\sigma(1)) = 0. \end{gathered} \tag{1.8}$$

They also applied the above-mentioned fixed-point theorem in a cone to yield positive solutions of (1.8) for λ belonging to a suitable open interval.

On the other hand, there is much attention paid to the study of spectral theory of linear problems on time scales, see [12, 13] and references therein. Agarwal et al. [12] obtained a fundamental result on the existence of eigenvalues and the number of the generalized zeros of associated eigenfunctions. This result provides a foundation for the further study of the behavior of eigenvalues of the Sturm-Liouville problems on time scales. In [13], Kong first extended the existence result in [12] to the Sturm-Liouville problems in a more general form and with normalized separated boundary conditions and then explored the dependence of the eigenvalues on the boundary condition. The author showed that the nth eigenvalue λ_n depends continuously on the boundary condition except at the generalized "Dirichlet" boundary conditions, where certain jump discontinuities may occur. Furthermore, λ_n as a function of the boundary condition angles is continuously differentiable wherever it is continuous. Formulas for such derivatives were obtained which reveal the monotone properties of λ_n in terms of the boundary condition angles.

We note that Li et al. [10] were concerned with the existence of positive solutions for problem (1.1) under some conditions concerning the first eigenvalue corresponding to the relevant linear operator. Their main results improved and generalized ones in [5–9, 11].

We should point out that the nonlinear item that appeared in the above dynamic equations is required to be nonnegative; moreover, the positive solutions can be obtained only by using these tools.

However, some existing nonlinear problems cannot be attributed to the cone mappings; thus, the cone mapping theory fails to solve these problems. At the same time, the existence of the sign-changing solutions of the above dynamic equations is being raised by an ever-increasing number of researchers; it is also difficult to deal with sign-changing solution problems by the cone mappings theory. Stimulated by these works, in [17], Sun and Liu used the lattice structure to present some methods of computation of the topological degree for the operator that is quasiadditive on lattice. The operator is not assumed to be a cone mapping. Furthermore, in [18], Liu and Sun established some methods of computation of the fixed-point index and the topological degree for the unilaterally asymptotically linear operators that are not assumed to be cone mappings. Very recently, in [19], Sun and Liu obtained some theorems for fixed-point index about a class of nonlinear operators which are not cone mappings by means of the theory of cone.

The main purpose of this paper is to establish some existence theorems of nontrivial solutions and sign-changing solutions of (1.1). Our results generalize and complement ones in [5–11].

This paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we presented some results obtained in [19]; then, we apply these abstract results to problem (1.1), the existence of nontrivial solutions and sign-changing solutions for sublinear problem (1.1) is obtained. In Section 4, we are concerned with the existence of sign-changing solutions and nontrivial solutions for superlinear problem (1.1). Our main tool relies on the computation of the topological degree for superlinear operators, which are obtained in [17]. In Section 5, by the use of the abstract results obtained in [18], some existence theorems of sign-changing solutions for unilaterally asymptotically linear problem (1.1) are established.

2. Preliminaries and Some Lemmas

Let E be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subset E$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. P is called solid if it contains interior points, that is, $\operatorname{int} P \neq \emptyset$. If $x \leq y$ and $x \neq y$, we write $x < y$; if cone P is solid and $y - x \in \operatorname{int} P$, we write $x \ll y$. Operator A is said to be strongly increasing if $y < x$ implies $Ay \ll Ax$. P is called total if $E = \overline{P - P}$. If P is solid, then P is total. A fixed-point u of operator A is said to be a sign-changing fixed-point if $u \notin P \cup (-P)$. Let $B : E \to E$ be a bounded linear operator. B is said to be positive if $B(P) \subset P$. For the concepts and the properties about the cones, we refer to [20–23].

For $w \in E$, let $P_w = P + w = \{x \in E \mid x \geq w\}$ and $P^w = w - P = \{x \in E \mid x \leq w\}$.

We call E a lattice under the partial ordering $\leq$ if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for arbitrary $x, y \in E$. For $x \in E$, let

$$x^+ = \sup\{x, \theta\}, \qquad x^- = \sup\{-x, \theta\}. \tag{2.1}$$

x^+ and x^- are called positive part and negative part of x, respectively. Taking $|x| = x^+ + x^-$, then $|x| \in P$, and $|x|$ is called the module of x. One can see [24] for the definition and the properties of the lattice.

For convenience, we use the following notations:

$$x_+ = x^+, \qquad x_+ = -x^-, \tag{2.2}$$

and then

$$x_+ \in P, \quad x_- \in (-P), \quad x = x_+ + x_-. \tag{2.3}$$

In the following, we always assume that E is a Banach space, P is a total cone in E, and the partial ordering $\leq$ in E is induced by P. We also suppose that E is a lattice in the partial ordering $\leq$.

Let $B : E \to E$ be a positive completely continuous linear operator, let $r(B)$ be a spectral radius of B, let B^* be the conjugated operator of B, and let P^* be the conjugated cone of P. Since $P \subset E$ is a total cone, according to the famous Krein-Rutman theorem (see [25]), we infer that if $r(B) \neq 0$, then there exist $\overline{\varphi} \in P \setminus \{\theta\}$ and $g^* \in P^* \setminus \{\theta\}$, such that

$$B\overline{\varphi} = r(B)\overline{\varphi}, \qquad B^* g^* = r(B) g^*. \tag{2.4}$$

Fixed $\overline{\varphi} \in P \setminus \{\theta\}$, $g^* \in P^* \setminus \{\theta\}$ such that (2.4) holds. For $\delta > 0$, let

$$P(g^*, \delta) = \{x \in P,\ g^*(x) \geq \delta \|x\|\}. \tag{2.5}$$

Then, $P(g^*, \delta)$ is also a cone in E.

Definition 2.1 (see [23]). Let B be a positive linear operator. The operator B is said to satisfy **H** condition if there exist $\overline{\varphi} \in P \setminus \{\theta\}$, $g^* \in P^* \setminus \{\theta\}$, and $\delta > 0$ such that (2.4) holds, and B maps P into $P(g^*, \delta)$.

Definition 2.2 (see [23]). Let $D \subset E$ and $A : D \to E$ be a nonlinear operator. A is said to be quasiadditive on lattice if there exists $v^* \in E$ such that

$$Ax = Ax_+ + Ax_- + v^*, \quad \forall x \in D, \tag{2.6}$$

where x_+ and x_- are defined by (2.2).

Definition 2.3 (see [18]). Let E be a Banach space with a cone P, and let $A : E \to E$ be a nonlinear operator. We call A a unilaterally asymptotically linear operator along P_w if there exists a bounded linear operator B such that

$$\lim_{x \in P_w, \|x\| \to \infty} \frac{\|Ax - Bx\|}{\|x\|} = 0. \tag{2.7}$$

B is said to be the derived operator of A along P_w and will be denoted by A'_{P_w}.

Similarly, we can also define a unilaterally asymptotically linear operator along P^w.

For convenience, we list the following definitions on time scales which can be found in [1–3].

Definition 2.4. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \\ \rho(r) &= \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}. \end{aligned} \tag{2.8}$$

For all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right-scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise, set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left-scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise, set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.5. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \tag{2.9}$$

for all $s \in U$.

Definition 2.6. If $G^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a). \tag{2.10}$$

Lemma 2.7 (see [17]). *Suppose that $B : E \to E$ is a positive bounded linear operator. If the spectral radius $r(B) < 1$, then $(I - B)^{-1}$ exists and is a positive bounded linear operator.*

Lemma 2.8 (see [5]). *Set*

$$d = \frac{\gamma\beta}{r(a)} + \frac{\alpha\delta}{r(\sigma(b))} + \alpha\gamma \int_a^{\sigma(b)} \frac{1}{r(\tau)} \Delta\tau \neq 0. \tag{2.11}$$

Let

$$G(t, s) = \begin{cases} \dfrac{1}{d} x(t) y(\sigma(s)), & t \leq s, \\ \dfrac{1}{d} x(\sigma(s)) y(t), & \sigma(s) \leq t, \end{cases} \tag{2.12}$$

where

$$x(t) = \alpha \int_a^t \frac{1}{r(\tau)} \Delta\tau + \frac{\beta}{r(\sigma(b))}, \qquad y(t) = \gamma \int_t^{\sigma(b)} \frac{1}{r(\tau)} \Delta\tau + \frac{\delta}{r(\sigma(b))}. \tag{2.13}$$

Then $G(t, s)$ is the Green function of the following linear boundary value problem:

$$\begin{gathered} -\left[r(t) u^\Delta(t)\right]^\Delta = 0, \quad t \in [a, b]_{\mathbb{T}}, \\ \alpha u(a) - \beta u^\Delta(a) = 0, \qquad \gamma u(\sigma(b)) + \delta u^\Delta(\sigma(b)) = 0. \end{gathered} \tag{2.14}$$

According to Lemma 2.8, problem (1.1) can be converted into the equivalent nonlinear integral equation

$$u(t) = \int_a^{\sigma(b)} G(t, s) f(s, u(\sigma(s))) \Delta s, \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.15}$$

Define the operators

$$Au(t) = \int_a^{\sigma(b)} G(t,s) f(s, u(\sigma(s)))\Delta s, \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \tag{2.16}$$

$$Ku(t) = \int_a^{\sigma(b)} G(t,s) u(\sigma(s))\Delta s, \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.17}$$

By in [13, Theorem 2.1], we can know that K has the sequence of eigenvalues:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots, \quad n \in \mathbb{N}_0^k, \tag{2.18}$$

where

$$\mathbb{N}_0^k := \begin{cases} \{1,2,\ldots\}, & k = \infty, \\ \{1,2,\ldots,k\}, & k < \infty, \end{cases} \tag{2.19}$$

the algebraic multiplicities of every eigenvalue is simple, and the spectral radius $r(K) = \lambda_1^{-1}$.

Let

$$\begin{aligned} E = \Big\{ u : [a, \sigma^2(b)]_{\mathbb{T}} \longrightarrow \mathbb{R}, u^{\Delta}(t) \text{ is continuous on } [a, \sigma^2(b)]_{\mathbb{T}}, \\ \left[r(t)u^{\Delta}(t)\right]^{\Delta} \text{ is right dense continuous on } [a,b)_{\mathbb{T}} \Big\}. \end{aligned} \tag{2.20}$$

Then E is an ordered Banach space with the supremum norm $\|u\| = \sup_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} |u(t)|$. Define a cone $P \subset E$ by

$$P = \left\{ u \in E \mid u(t) \geq 0,\ t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}} \right\}. \tag{2.21}$$

It is clear that P is a normal solid cone and E becomes a lattice under the natural ordering $\leq$.

Lemma 2.9 (see [10]). *Assume that there exists $h(t) \in P \setminus \{\theta\}$, such that*

$$G(t,s) \geq h(t) G(\tau, s), \quad t,\ \tau \in [a, \sigma(b)]_{\mathbb{T}},\ s \in [a,b]_{\mathbb{T}}. \tag{2.22}$$

Suppose that there exists $\psi^* \in P^* \setminus \{\theta\}$, such that $\psi^* = r^{-1}(K)K^*\psi^*$, $\psi^*(h(t)) \not\equiv 0$. Then operator K satisfies **H** *condition, where K is defined in* (2.17).

3. Topological Degree for Sublinear Problem (1.1)

For $R > 0$, define $T_R = \{x \in E \mid \|x\| < R\}$.

In [17], Sun and Liu obtained the following abstract results.

Lemma 3.1 (see [17]). *Let E be a Banach space, and let P be a normal solid cone in E, $w_0 \in E$, and let $A : P_{w_0} \to P_{w_0}$ be a completely continuous operator. Suppose that there exists a positive bounded linear operator $B : E \to E$ with $r(B) < 1$ and $u_0 \in P$ such that*

$$Ax \leq Bx + u_0, \quad \forall x \in P_{w_0}. \tag{3.1}$$

Then there exists $R_0 > 0$, such that the fixed-point index $i(A, T_R \cap P_{w_0}, P_{w_0}) = 1$ for all $R > R_0$.

Lemma 3.2 (see [17]). *Let E be a Banach space, and let P be a normal solid cone in E, $w_0 \in E$, and let $A : P_{w_0} \to P_{w_0}$ be a completely continuous operator. Suppose that there exists a positive bounded linear operator B and a positive completely continuous operator B_1, $u_0 \in P$, $r \in \mathbb{R}$, $r > 0$, such that*

$$Ax \leq Bx + u_0, \quad \forall x \in P, \qquad Ax \geq B_1 x, \quad \forall x \in \partial T_r \cap P. \tag{3.2}$$

If $r(B) < 1$ and $r(B_1) \geq 1$, then A has at least one nonzero positive fixed-point in P.

Lemma 3.3 (see [17]). *Let E be a Banach space, let P be a normal solid cone in E, $w_0 \in \operatorname{int}(-P)$, and let $A : P_{w_0} \to P_{w_0}$ be a completely continuous operator. Suppose that*

(i) *there exists a positive bounded linear operator B with $r(B) < 1$ and $u_0 \in P$ such that*

$$Ax \leq Bx + u_0, \quad \forall x \in P_{w_0}, \tag{3.3}$$

(ii) *$A\theta = \theta$, the Fréchet derivative A'_θ of A at θ exists, and 1 is not an eigenvalue of A'_θ.*

Then

(1) *if the sum of the algebraic multiplicities for all eigenvalues of A'_θ in $(0,1)$ is an odd number, A has at least one nonzero fixed-point;*

(2) *if $r(A'_\theta) > 1$ and $A(P) \subset P$, $A(-P) \subset (-P)$, A has at least two nonzero fixed-points, one of which is positive, the other is negative;*

(3) *if $r(A'_\theta) > 1$, the sum of the algebraic multiplicities for all eigenvalues of A'_θ in $(0,1)$ is an even number, and*

$$A(P \setminus \{\theta\}) \subset \operatorname{int}(P), \qquad A((-P) \setminus \{\theta\}) \subset \operatorname{int}(-P), \tag{3.4}$$

then A has at least three nonzero fixed-points, one of which is positive another is negative, and the third fixed-point is sign-changing.

By means of the above abstract results, we have the following theorems.

Theorem 3.4. *Suppose that there exist $M > 0$ and $\eta > 0$ such that*

$$f(t,u) \geq -M, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}},\ u \geq 0, \tag{3.5}$$

$$\lim_{|u|\to\infty} \sup \frac{f(t,u)}{u} \leq \lambda_1 - \eta, \quad \text{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.6}$$

Then problem (1.1) *has at least one solution.*

Proof. It is easy to check that $A : E \to E$ is a completely continuous operator. Choose $0 < \varepsilon < \eta$. Denote $c = \lambda_1 - \eta + \varepsilon < \lambda_1$. According to (3.6), we can know that there exists $\overline{R} > 0$ such that

$$\frac{f(t,u)}{u} \leq c, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ |u| \geq \overline{R}. \tag{3.7}$$

This implies that there exists constant number $M_0 > 0$ such that

$$f(t,u) \geq cu - M_0, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \leq 0, \tag{3.8}$$

$$f(t,u) \leq cu + M_0, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \geq 0. \tag{3.9}$$

Denote $B = cK$, where K is defined as (2.17). It is obvious that $B : E \to E$ is a positive bounded linear operator, and $r(B) = cr(K) < \lambda_1 r(K) = 1$. By Lemma 2.7, we have that $(I - B)^{-1}$ exists and is a positive bounded linear operator.

Set $b > 0$, and let

$$\begin{gathered} b(t) \equiv b, \qquad l(t) \equiv M_0 + M, \\ w_0(t) = -\left[(I - B)^{-1}(Kl + b)\right](t), \quad t \in [a,\sigma(b)]_{\mathbb{T}}. \end{gathered} \tag{3.10}$$

It is obvious that $w_0 \in (-P)$, and

$$w_0(t) - (Bw_0)(t) = -(Kl)(t) - b(t), \quad t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.11}$$

For any $u_0 \leq 0$, it follows from (3.5) and (3.8) that

$$f(t,u) \geq cu_0 - M - M_0, \quad u \geq u_0, \tag{3.12}$$

which together with $w_0 \leq 0$ implies that

$$f(t,u(t)) \geq cw_0(t) - l(t), \quad u(t) \geq w_0(t). \tag{3.13}$$

Note that $K(P) \subset P$. Combining (3.11) with (3.13) and (2.16), we have

$$\begin{aligned} Au(t) &= \int_a^{\sigma(b)} G(t,s) f(s, u(\sigma(s)))\Delta s \\ &\geq \int_a^{\sigma(b)} G(t,s)[cw_0(\sigma(s)) - l(\sigma(s))]\Delta s \\ &\geq \int_a^{\sigma(b)} G(t,s)[cw_0(\sigma(s)) - l(\sigma(s))]\Delta s - b(t) \\ &= w_0(t), \quad \forall u(t) \geq w_0(t),\ t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \end{aligned} \tag{3.14}$$

This implies that $A(P_{w_0}) \subset P_{w_0}$.

Let

$$\begin{gathered} \vartheta = \sup\{|w_0(t)| : t \in [a, \sigma(b)]_{\mathbb{T}}\}, \\ \kappa = \sup\{\left|f(t,u)\right| : t \in [a, \sigma(b)]_{\mathbb{T}}, -\vartheta \leq u \leq 0\}. \end{gathered} \tag{3.15}$$

It is easy to know from (3.9) that

$$f(t,u) \leq cu + M_0 + \kappa + c\vartheta, \quad t \in [a, \sigma(b)]_{\mathbb{T}},\ u \geq -\vartheta. \tag{3.16}$$

Thus,

$$f(t, u(t)) \leq cu(t) + M_0 + \kappa + c\vartheta, \quad \forall u(t) \geq w_0(t),\ t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{3.17}$$

Let $e_0(t) \equiv M_0 + \kappa + c\vartheta$, $u_0(t) = (Ke_0)(t)$, $t \in [a, \sigma(b)]_{\mathbb{T}}$. It is clear that $u_0 \in P$. It follows from (3.17) that

$$Au(t) \leq Bu(t) + u_0(t), \quad \forall u \in P_{w_0}. \tag{3.18}$$

Thus, all conditions in Lemma 3.1 are satisfied; the conclusion then follows from Lemma 3.1. □

Theorem 3.5. *Suppose that*

$$f(t,u) \geq 0, \quad \forall t \in [a, \sigma(b)]_{\mathbb{T}},\ u \geq 0, \tag{3.19}$$

and there exists $\eta > 0$ *such that*

$$\limsup_{u \to \infty} \frac{f(t,u)}{u} \leq \lambda_1 - \eta, \quad \liminf_{u \to 0} \frac{f(t,u)}{u} \geq \lambda_1 + \eta, \quad \text{uniformly on } t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{3.20}$$

Then problem (1.1) *has at least one positive solution.*

Proof. It is clear that $A : P \to P$ is a completely continuous operator. Choose $0 < \varepsilon < \eta$. Denote

$$c = \lambda_1 - \eta + \varepsilon, \qquad c_1 = \lambda_1 + \eta - \varepsilon. \tag{3.21}$$

then $c < \lambda_1$, $c_1 > \lambda_1$. By (3.20), we obtain that there exists $D > d > 0$ such that

$$f(t,u) \le cu, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \ge D, \tag{3.22}$$

$$f(t,u) \ge c_1 u, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ 0 \le u \le d. \tag{3.23}$$

According to (3.22), we have

$$f(t,u) \le cu + \overline{M}, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \ge 0, \tag{3.24}$$

where

$$\overline{M} = \sup_{t\in[a,\sigma(b)]_{\mathbb{T}},\ |u|\le D} \left|f(t,u)\right|. \tag{3.25}$$

Denote $B = cK$, $B_1 = c_1 K$, where K is still defined as (2.17); then, $B,\ B_1 : E \to E$ are both completely continuous positive linear operators, and

$$r(B) = cr(K) < \lambda_1 r(K) = 1, \qquad r(B_1) = c_1 r(K) > \lambda_1 r(K) = 1. \tag{3.26}$$

Let $v_0 = K(\overline{M})$. Then, $v_0 \in P$. It follows from (2.16), (3.19), (3.23), and (3.24) that

$$\begin{aligned} Au(t) &\le \int_a^{\sigma(b)} G(t,s)\left[cu(\sigma(s)) + \overline{M}\right]\Delta s \\ &= Bu(t) + v_0(t), \quad \forall u \in P,\ t \in [a,\sigma(b)]_{\mathbb{T}}, \\ Au(t) &\ge \int_a^{\sigma(b)} G(t,s) m_1 u(\sigma(s))\Delta s \\ &= B_1 u(t), \quad \forall u \in P,\ t \in [a,\sigma(b)]_{\mathbb{T}},\ \|u\| \le d. \end{aligned} \tag{3.27}$$

By Lemma 3.2, problem (1.1) has at least one positive solution. □

Remark 3.6. Condition (3.20) implies that

$$\lim_{u\to+\infty} \sup \frac{f(t,u)}{u} < \lambda_1, \quad \lim_{u\to 0^+} \inf \frac{f(t,u)}{u} > \lambda_1, \quad \text{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.28}$$

Hence, Theorem 3.5 becomes Theorem 2.1 in [10]. Therefore, our main result generalizes Theorem 2.1 in [10].

Theorem 3.7. *Assume that* (3.5) *and* (3.6) *hold. In addition, suppose that*

$$f(t,0) \equiv 0, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}}, \tag{3.29}$$

$$\lim_{u\to 0}\frac{f(t,u)}{u} = \lambda, \quad \textit{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.30}$$

Then

(i) *if* $\lambda_n < \lambda < \lambda_{n+1}$*, where* n *is an odd number, then problem* (1.1) *has at least one nontrivial solution;*

(ii) *if* $\lambda > \lambda_1$ *and*

$$f(t,u)u \geq 0, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}},\ u \in \mathbb{R}, \tag{3.31}$$

then problem (1.1) *has at least two nontrivial solutions, one of which is positive and the other is negative.*

Proof. It suffices to verify that all the conditions of Lemma 3.3 are satisfied.

Let w_0 be defined as (3.10). Since $Kl + b \in \operatorname{int} P$, we have $w_0 \in \operatorname{int}(-P)$. It follows from (3.5), (3.6), and the proof of Theorem 3.4 that $A : P_{w_0} \to P_{w_0}$ is completely continuous and condition (i) of Lemma 3.3 holds. Furthermore, (3.29) implies that $A\theta = \theta$; (3.30) implies that

$$\begin{aligned} A'_\theta u(t) &= \int_a^{\sigma(b)} G(t,s) f'_u(s,0) u(\sigma(s))\Delta s \\ &= \lambda\int_a^{\sigma(b)} G(t,s) u(\sigma(s))\Delta s, \quad t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}. \end{aligned} \tag{3.32}$$

This shows that $A'_\theta = \lambda K$. Let ι denote the sum of the algebraic multiplicities for all eigenvalues of A'_θ in $(0,1)$. Since $\lambda_n < \lambda < \lambda_{n+1}$, we can obtain that 1 is not an eigenvalue of A'_θ and $\iota = n$ is an odd number. Conclusion (1) of Lemma 3.3 holds.

Equqtion (3.31) implies that $A(P) \subset P,\ A(-P) \subset (-P)$. By $\lambda > \lambda_1$, we can know that 1 is not an eigenvalue of A'_θ, and $r(A'_\theta) = \lambda r(K) > \lambda_1 r(K) = 1$. This implies that the conclusion (2) of Lemma 3.3 holds. □

Theorem 3.8. *Assume that* (3.5), (3.6), (3.29), *and* (3.30) *hold. In addition, suppose that*

$$\begin{aligned} &\lambda_n < \lambda < \lambda_{n+1}, \quad \textit{where } n \textit{ is an even number}, \\ &f(t,u)u > 0, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}},\ u \neq 0,\ u \in \mathbb{R}. \end{aligned} \tag{3.33}$$

Then the problem (1.1) *has at least three nontrivial solutions, one of which is positive, another is negative, and the third solution is sign changing.*

Proof. For convenience, let us introduce another ordered Banach space.

Letting, e_1 be the first normalized eigenfunction of K corresponding to its first eigenvalue λ_1, then $\|e_1\| = 1$. It follows from [10, Lemma 1.4] that $e_1 > 0$. Let

$$\overline{E} = \{x \in E \mid \text{there exists } \nu > 0,\ -\nu e_1(t) \le x(t) \le \nu e_1(t)\}. \tag{3.34}$$

According to [20], we can obtain that $\overline{E}$ is an ordering Banach space, $\overline{P} = \overline{E} \cap P$ is a normal solid cone, and $K : E \to \overline{E}$ is a linear completely continuous operator satisfying $K(P \setminus \{\theta\}) \subset \operatorname{int}\overline{P}$, where

$$\operatorname{int}\overline{P} = \left\{x \in \overline{P} \mid \text{there exist } \varsigma_1 > 0,\ \varsigma_2 > 0,\ \varsigma_1 e_1(t) \le x(t) \le \varsigma_2 e_1(t)\right\}. \tag{3.35}$$

Letting $l(t)$ and B be as in Theorem 3.4, then $l \in P \setminus \{\theta\}$. Let $w_0 = -(I - B)^{-1}(Kl)$. Since $K(P) \subset \overline{P}$, we have $Kl \in \operatorname{int}\overline{P}$. Furthermore, we obtain $w_0 \in \operatorname{int}(-\overline{P})$. Noting that $K(E) \subset \overline{E}$, we have $A(E) \subset \overline{E}$.

Similarly to the proof of Theorem 3.4, we have

$$A(P_{w_0}) \subset P_{w_0}. \tag{3.36}$$

Furthermore, we can know that

$$A\left(\overline{P}_{w_0}\right) \subset \overline{P}_{w_0}. \tag{3.37}$$

Since $K : E \to \overline{E}$ is completely continuous, we have that $A : \overline{P}_{w_0} \to \overline{P}_{w_0}$ is complety continuous. According to Theorem 3.7, it is easy to show that conditions (i) and (ii) of Lemma 3.3 are satisfied, and $A'_\theta = \lambda K$.

Since $\lambda_n < \lambda < \lambda_{n+1}$, we have that 1 is not an eigenvalue of A'_θ and $\lambda > \lambda_1$. It follows that

$$r\left(A'_\theta\right) = \lambda r(K) > \lambda_1 r(K) = 1. \tag{3.38}$$

Moreover, we can know that the sum of the algebraic multiplicities for all eigenvalues of A'_θ in $(0,1)$ is n; this is an even number.

In the following, we check the condition (3.4).

By (3.33), we have

$$f(t,u) > 0, \quad \forall u > 0, \qquad f(t,u) < 0, \quad \forall u < 0,\ \forall t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.39}$$

Notice that

$$K(P \setminus \{\theta\}) \subset \operatorname{int}\overline{P}. \tag{3.40}$$

It follows that

$$A\left(\overline{P} \setminus \{\theta\}\right) \subset \operatorname{int} \overline{P}, \qquad A\left(\left(-\overline{P}\right) \setminus \{\theta\}\right) \subset \operatorname{int}\left(-\overline{P}\right). \tag{3.41}$$

Therefore, condition (3.4) holds. The conclusion follows from Lemma 3.3 (3). □

4. Topological Degree for Superlinear Problem (1.1)

In [17], Sun and Liu discussed computation for the topological degree about superlinear operators which are not cone mappings. The main tools are the partial ordering relation and the lattice structure. Their main results are the following theorems.

Lemma 4.1 (see [17]). *Let E be a Banach space, and let $A : E \to E$ be a completely continuous operator satisfying $A = BF$, where F is quasiadditive on lattice and B is a positive bounded linear operator satisfying* **H** *condition. Moreover, suppose that P is a solid cone in E and*

(i) *there exist $a_1 > r^{-1}(B)$ and $y_1 \in P$, such that*

$$Fx \geq a_1 x - y_1, \quad \forall x \in P, \tag{4.1}$$

(ii) *there exist $0 < a_2 < r^{-1}(B)$ and $y_2 \in P$, such that*

$$Fx \geq a_2 x - y_2, \quad \forall x \in (-P). \tag{4.2}$$

In addition, there exist a bounded open set Ω, which contains θ, and a positive bounded linear operator B_0 with $r(B_0) \leq 1$, such that

$$|Ax| \leq B_0 |x|, \quad \forall x \in \partial\Omega. \tag{4.3}$$

Then A has at least one nonzero fixed-point.

Lemma 4.2 (see [17]). *Let E be a Banach space, and let $A : E \to E$ be a completely continuous operator satisfying $A = BF$, where F is quasiadditive on lattice and B is a positive bounded linear operator satisfying* **H** *condition. Moreover, suppose that P is a solid cone in E and conditions (i) and (ii) of Theorem 4.1 are satisfied. In addition, assume that $A\theta = \theta$, the Fréchet derivative A'_θ of A at θ exists, and 1 is not an eigenvalue of A'_θ. Then A has at least one nonzero fixed-point.*

In [23], Sun obtained the following further result concerning sign-changing fixed-point.

Lemma 4.3 (see [23]). *Let the conditions of Theorem 4.2 be satisfied. In addition, suppose that the sum of the algebraic multiplicities for all eigenvalues of A'_θ in the interval $(0, 1)$ is an even number, and*

$$A(P \setminus \{\theta\}) \subset \operatorname{int} P, \qquad A((-P) \setminus \{\theta\}) \subset -\operatorname{int} P. \tag{4.4}$$

Then A has at least three nonzero fixed-points, one of which is positive, another is negative, the third one is a sign-changing fixed-point.

In the following, we will apply the above three abstract theorems to problem (1.1); our main results are as follows.

Theorem 4.4. *Assume that $f : [a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that there exists $\eta > 0$ such that*

$$\lim_{u\to-\infty} \sup \frac{f(t,u)}{u} \leq \lambda_1 - \eta, \quad \text{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}, \tag{4.5}$$

$$\lim_{u\to+\infty} \inf \frac{f(t,u)}{u} \geq \lambda_1 + \eta, \quad \text{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}, \tag{4.6}$$

$$\lim_{u\to 0} \sup \left|\frac{f(t,u)}{u}\right| \leq \lambda_1 - \eta, \quad \text{uniformly on } t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{4.7}$$

Then problem (1.1) *has at least one nontrivial solution.*

Proof. It is clear that $A : E \to E$ is completely continuous. According to (4.6) and (4.5), we have that there exists $R_1 > 0$, such that

$$\begin{aligned} f(t,u) &\geq \left(\lambda_1 + \frac{1}{2}\eta\right)u, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \geq R_1, \\ f(t,u) &\geq \left(\lambda_1 - \frac{1}{2}\eta\right)u, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \leq -R_1. \end{aligned} \tag{4.8}$$

Hence, there exists $\overline{M}_0 > 0$ such that

$$\begin{aligned} f(t,u) &\geq \left(\lambda_1 + \frac{1}{2}\eta\right)u - \overline{M}_0, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \geq 0, \\ f(t,u) &\geq \left(\lambda_1 - \frac{1}{2}\eta\right)u - \overline{M}_0, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ u \leq 0. \end{aligned} \tag{4.9}$$

According to the proof of Theorem 3.1 in [10], we have

$$G(t,s) \geq \frac{x(t)y(t)}{\max_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} x(t) \max_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} y(t)} G(\tau,s), \quad t,\ \tau \in \left[a,\sigma^2(b)\right]_{\mathbb{T}},\ s \in [a,\sigma(b)]_{\mathbb{T}}. \tag{4.10}$$

By Lemma 2.9, we can know that linear operator K satisfies the H condition.

Let $Fu(t) = f(t,u(t))$ for $u \in E$. It is easy to check that F satisfies (2.6) and $A = KF$. It follows from (4.9) that (4.1) and (4.2) hold, where $a_1 = \lambda_1 + (1/2)\eta > \lambda_1 = r^{-1}(K)$ and $a_2 = \lambda_1 - (1/2)\eta < \lambda_1 = r^{-1}(K)$.

By (4.7), we obtain that there exists $r_0 > 0$ such that

$$\left|f(t,u)\right| \leq \left(\lambda_1 - \frac{1}{2}\eta\right)|u|, \quad t \in [a,\sigma(b)]_{\mathbb{T}},\ |u| \leq r_0. \tag{4.11}$$

Therefore,

$$|Au| \leq \left(\lambda_1 - \frac{1}{2}\eta\right)K|u|, \quad \forall u \in E, \ \|u\| = r_0. \tag{4.12}$$

Thus, (4.3) holds. An application of Lemma 4.1 shows that problem (1.1) has at least one nontrivial solution. □

Remark 4.5. If we assume that $f : [a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$, condition (4.5), (4.6) will be removed. Therefore, Theorem 4.4 generalizes and extends Theorem 3.4 in [10].

Theorem 4.6. *Assume that $f : [a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and* (4.5) *holds. In addition, suppose that*

$$\begin{gathered} f(t,0) \equiv 0, \quad \forall t \in [a, \sigma(b)]_{\mathbb{T}}, \\ \lim_{u \to 0} \frac{f(t,u)}{u} = \lambda, \quad \text{uniformly on } t \in [a, \sigma(b)]_{\mathbb{T}}, \end{gathered} \tag{4.13}$$

and $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$. Then problem (1.1) *has at least one nontrivial solution.*

Proof. By Theorem 4.4, we only need to verify the condition "moreover" of Lemma 4.2. According to (4.13), we can know that $A\theta = \theta$ and $A'_\theta = \lambda K$. Since K has the sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, it is easy to see that $\{\lambda_n/\lambda\}_{n=1}^{\infty}$ is the sequence of all eigenvalues of A'_θ. Therefore, 1 is not an eigenvalue of A'_θ. The conclusion of Theorem 4.6 follows from Lemma 4.2. □

Theorem 4.7. *Assume that $f : [a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and* (4.5), (4.6), *and* (4.13) *hold. If*

$$\begin{gathered} \lambda_n < \lambda < \lambda_{n+1}, \quad \text{where } n \text{ is an even number}, \\ f(t,u)u > 0, \quad \forall t \in [a, \sigma(b)]_{\mathbb{T}}, \ u \neq 0, \ u \in \mathbb{R}. \end{gathered} \tag{4.14}$$

Then the problem (1.1) *has at least three nontrivial solutions, one of which is positive, another is negative, and the third solution is sign changing.*

Proof. From the proof of Theorem 4.6, it is easy to show that all conditions of Lemma 4.2 are satisfied and A'_θ has the sequence of eigenvalues $\{\lambda_n/\lambda\}_{n=1}^{\infty}$.

Since $\lambda_n < \lambda < \lambda_{n+1}$ and n is an even number, we can know that the sum of the algebraic multiplicities for all eigenvalues of A'_θ in $(0,1)$ is an even number. Condition (4.14) implies that

$$A((-P) \setminus \{\theta\}) \subset -\operatorname{int} P, \qquad A(P \setminus \{\theta\}) \subset \operatorname{int} P. \tag{4.15}$$

An application of Lemma 4.3 shows that problem (1.1) has at least three nontrivial solutions, one of which is positive, another is negative, and the third solution is sign changing.

Now let us end this section by the following two examples. □

Example 4.8. Let $\mathbb{T} = \mathbb{Z}$. Considering the following BVP:

$$\begin{gathered}\Delta^2 u(t-1) - 0.381966u + 4u^2 + u^4 = 0, \quad t \in [1,4], \\ u(0) = u(5) = 0,\end{gathered} \tag{4.16}$$

where $[1,4]$ is the discrete interval $\{1,2,3,4\}$, $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$.

By [26, Remark 2.3 and Lemma 2.9], we can know that $\lambda_k = 4\sin^2(k\pi/10)$, $k = 1,2,3,4$. In addition, the algebraic multiplicity of each eigenvalue λ_k is equal to 1.

Setting $\delta = 0.00000001$, it is easy to check that

$$\begin{aligned}\limsup_{u\to 0}\left|\frac{-0.381966u + 4u^2 + u^4}{u}\right| &= 0.381966 \\ &= 0.38196601 - 0.00000001 \\ &= \lambda_1 - \delta, \\ \limsup_{u\to -\infty}\frac{-0.381966u + 4u^2 + u^4}{u} &= \limsup_{u\to -\infty}\left(-0.381966 + 4u + u^3\right) \\ &= -\infty, \\ \limsup_{u\to +\infty}\left(-0.381966 + 4u + u^3\right) &= +\infty.\end{aligned} \tag{4.17}$$

Therefore, all conditions in Theorem 4.4 are satisfied. By Theorem 4.4, BVP (4.16) has at least one nontrivial solution.

Example 4.9. Let $\mathbb{T} = \mathbb{Z}$. Consider the following BVP:

$$\begin{gathered}\Delta^2 u(t-1) + u(0.39 + 0.007\arctan u) = 0, \quad t \in [1,4], \\ u(0) = u(5) = 0.\end{gathered} \tag{4.18}$$

In virtue of Example 4.8, we can know that $\lambda_1 = 0.38196601$, $\lambda_2 = 1.381966011$, $\lambda_3 = 3.553057584$, and $\lambda_4 = 6.316546812$.

Setting $\delta = 0.00000001$, by direct calculation, we have

$$\begin{aligned}\limsup_{u\to -\infty}\frac{u(0.39 + 0.007\arctan u)}{u} &= 0.39 - 0.007\frac{\pi}{2} = 0.39 - 0.010995574 \\ &\le 0.38196601 - 0.00000001 \\ &= \lambda_1 - \delta,\end{aligned}$$

$$
\begin{aligned}
\limsup_{u\to+\infty}\frac{u(0.39+0.007\arctan u)}{u} &= 0.39+0.007\times\frac{\pi}{2}\\
&= 0.400995574\\
&\geq 0.38196601+0.00000001\\
&= \lambda_1-\delta,\\
\lim_{u\to 0}(0.39+0.007\arctan u) &= 0.39\notin\{0.38196601, 1.381966011, 3.553057584, 6.316546812\}.
\end{aligned}
\tag{4.19}
$$

Therefore, all conditions in Theorem 4.6 are satisfied. By Theorem 4.6, BVP (4.18) has at least one nontrivial solution.

5. Topological Degree for Unilaterally Asymptotically Linear Problem (1.1)

In [18], Liu and Sun presented some methods of computing the topological degree for unilaterally asymptotically linear operators by using the lattice structure. Their main results are as follows.

Lemma 5.1 (see [18]). *Let P be a normal cone in E, and let $A : E \to E$ be completely continuous and quasiadditive on lattice. Suppose that there exist $u_*, u_1 \in P$ and a positive bounded linear operator $L_0 : E \to E$ with $r(L_0) < 1$, such that*

$$
Ax \geq -u_*, \quad \forall x\in P, \qquad Ax \geq L_0x-u_1, \quad \forall x\in(-P). \tag{5.1}
$$

In addition, suppose that A'_P exists, $A\theta = \theta$, the Fréchet derivative A'_θ of A at θ exists, and 1 is not an eigenvalue of A'_θ. Then A has at least one nontrivial fixed-point, provide that one of the following is satisfied:

(i) $r(A'_P) > 1$ *and 1 is not an eigenvalue of A'_P corresponding to a positive eigenvector;*

(ii) $r(A'_P) < 1$ *and the sum of the algebraic multiplicities for all eigenvalues of A'_θ, lying in the interval $(0, 1)$, is an odd number.*

Lemma 5.2 (see [18]). *Suppose that*

(i) *A is strongly increasing on P and $-P$;*

(ii) *both A'_P and $A'_{(-P)}$ exist with $r(A'_P) > 1$ and $r(A'_{(-P)}) > 1$; 1 is not an eigenvalue of A'_P or $A'_{(-P)}$ corresponding to a positive eigenvector;*

(iii) *$A\theta = \theta$; the Fréchet derivative A'_θ of A at θ is strongly positive, and $r(A'_\theta) < 1$;*

(iv) *the Fréchet derivative A'_∞ of A at ∞ exists; 1 is not an eigenvalue of A'_∞; the sum of the algebraic multiplicities for all eigenvalues of A'_∞, lying in the interval $(0, 1)$ is an even number.*

Then A has at least three nontrivial fixed-points, containing one sign-changing fixed-point.

Through this section, the following hypotheses are needed:

(E_1) $\lim_{u\to+\infty}(f(t,u)/u) = p$ uniformly on $t \in [a,\sigma(b)]_{\mathbb{T}}$;

(E_2) $\lim_{u\to-\infty}(f(t,u)/u) = q$ uniformly on $t \in [a,\sigma(b)]_{\mathbb{T}}$;

(E_3) $f(t,0) \equiv 0$, $\lim_{u\to 0}(f(t,u)/u) = \lambda$ uniformly on $t \in [a,\sigma(b)]_{\mathbb{T}}$.

Our main results are as follows.

Theorem 5.3. *Suppose that f satisfies (E_1)–(E_3). Then problem* (1.1) *has at least one nontrivial solution provided one of the following conditions is satisfied:*

(i) $p > \lambda_1$, $0 < q < \lambda_1$,p, $\lambda \neq \lambda_k$, $k = 1,2,\ldots$;

(ii) $0 < p < \lambda_1$, $q > \lambda_1$, q, $\lambda \neq \lambda_k$, $k = 1,2,\ldots$;

(iii) $0 < p < \lambda_1$, $0 < q < \lambda_1$, $\lambda_n < \lambda < \lambda_{n+1}$, *$n$ is an odd number.*

Proof. It follows from Definition 2.3 that

$$\begin{aligned} A'_P u(t) &= pKu(t), \quad \forall t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}, \\ A'_{(-P)} u(t) &= qKu(t), \quad \forall t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}. \end{aligned} \tag{5.2}$$

Furthermore, we have

$$\begin{aligned} A'_\theta u(t) &= \int_a^{\sigma(b)} G(t,s) f'_u(s,0) u(\sigma(s))\Delta s \\ &= \lambda \int_a^{\sigma(b)} G(t,s) u(\sigma(s))\Delta s \\ &= \lambda K u(t), \quad \forall t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}. \end{aligned} \tag{5.3}$$

Assume that condition (i) is satisfied. By $p > 0$, we can obtain that there exists $\overline{M}_1 > 0$ such that

$$f(t,u) \geq -\overline{M}_1, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}},\ u \geq 0. \tag{5.4}$$

So, we have

$$Au(t) \geq -K\overline{M}_1, \quad \forall u \geq 0. \tag{5.5}$$

Since $q < \lambda_1$, we can obtain that $q + \overline{\eta} < \lambda_1$ for some $\overline{\eta} > 0$. (E_2) implies that there exists $\mu_0 > 0$ such that

$$f(t,u) \geq (q + \overline{\eta})u - \mu_0, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}},\ u \leq 0. \tag{5.6}$$

Consequently,

$$
\begin{aligned}
Au(t) &= \int_a^{\sigma(b)} G(t,s) f(t, u(\sigma(s)))\Delta s \\
&\geq \int_a^{\sigma(b)} G(t,s)\left[(q+\overline{\eta})u(\sigma(s)) - \mu_0\right]\Delta s \\
&= (q+\overline{\eta})Ku - K\mu_0, \quad \forall u \in (-P).
\end{aligned}
\tag{5.7}
$$

Let $D_0 = (q+\overline{\eta})K$, then

$$
r(D_0) = (q+\overline{\eta}) r(K) < \lambda_1 r(K) = 1. \tag{5.8}
$$

Equation (5.5) and (5.7) imply that (5.1) holds.

Since the sequences of all eigenvalues of A'_θ, A'_P, and $A'_{(-P)}$ are $\{\lambda_n/\lambda\}_{n=1}^\infty$, $\{\lambda_n/p\}_{n=1}^\infty$, and $\{\lambda_n/q\}_{n=1}^\infty$, respectively. It is easy to see from p, $\lambda \neq \lambda_k$, $k = 1, 2, \ldots$ that 1 is not an eigenvalue of A'_θ or A'_P. Lemma 5.1(i) assures that A has at least one nontrivial fixed-point, and hence problem (1.1) has at least one nontrivial solution. Similarly, we can prove that the conclusion holds in the case that condition (ii) or (iii) is satisfied. □

Theorem 5.4. *Letting f satisfy (E_1)–(E_3) and $p = q =: \chi$. In addition, suppose that*

(i) *$f(t,u)$ is strictly increasing on u for fixed $t \in [a, \sigma(b)]_{\mathbb{T}}$;*

(ii) *$\lambda_n < \chi < \lambda_{n+1}$ and n is an even number;*

(iii) $0 < \lambda < \lambda_1$.

Then problem (1.1) *has at least three nontrivial solutions, containing a sign-changing solution.*

Proof. Similarly to the proof of Theorem 3.8, we still need to use another ordered Banach space $\overline{E}$.

From the proof of Theorem 5.3 and Theorem 3.8, we know that $A'_\theta = \lambda K$ and $A'_P = A'_{(-P)} = \chi K$. $\overline{P} \subset P$ gives that $A'_{\overline{P}} = A'_{(-\overline{P})} = \chi K$, where $\overline{P}$ is as in Theorem 3.8. It is easy to see that $A'_\infty = \chi K$. Evidently, $e_1 \in \operatorname{int} \overline{P}$. It follows from the condition (i) and $K(P \setminus \{\theta\}) \subset \operatorname{int} \overline{P}$ that A is strongly increasing.

Condition (ii) implies that 1 is not an eigenvalue of A'_P or $A'_{(-P)}$, and $r(A'_P) = rA'_{(-P)} = \chi r(K) > \lambda_1 r(K) = 1$. By the same method, we can show that A'_θ is strongly increasing. Moreover, by (iii), we have $r(A'_\theta) = \lambda r(K) < \lambda_1 r(K) = 1$. At last, from the proof of Theorem 5.3, we obtain that the sequence of all eigenvalues of A'_∞ is $\{\lambda_n/\chi\}_{n=1}^\infty$. According to (ii), we can know that the sum of the algebraic multiplicities for all eigenvalues of A'_∞ in $(0,1)$ is an even number. Hence, all the conditions of Lemma 5.2 are satisfied. The conclusion follows from Lemma 5.2. □

6. Conclusion

In this paper, some existence results of sign-changing solutions for dynamic equations on time scales are established. As far as we know, there were few papers that studied this topic.

Therefore, our results of this paper are new. It is natural for us to apply the method of this paper to more general form of dynamic equations. However, we note that the study of sign-changing solutions for time scales boundary value problems relies on the spectrum structure of linear dynamic equations on time scales. The reader is referred to [27, 28] for further works.

In most papers on time scales boundary value problem, in order to obtain the existence of positive solutions by using fixed-point theorems on a cone, the nonlinear term, which appears in the right-hand side of the equation, is required to be nonnegative. For our main results, the nonlinear term may be a sign-changing function, thus, the integral operator is not necessary to be a cone mapping. We should point out that Sun and Zhang [29] have studied nontrivial solutions of the singular superlinear Sturm-Liouville problems of ordinary differential equations; the ideas of this paper come from [17–19, 29].

In [30], Luo focused on the application of spectral theory of linear dynamic problems to the existence of positive solutions for nonlinear weighted eigenvalue problem on time scales. The author also discussed the existence of nodal solutions and the global structure of the solution for nonlinear eigenvalue problem on time scales by using the eigenvalues of the corresponding linear problem and global bifurcation theory. Up to now, no results on the spectrum structure of linear dynamic equations have been established for multipoint boundary value problems. This is an open problem.

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References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[2] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.
[3] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, vol. 370 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
[4] L. Erbe and A. Peterson, "Green's functions and comparison theorems for differential equations on measure chains," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 6, no. 1, pp. 121–137, 1999.
[5] L. Erbe and A. Peterson, "Positive solutions for a nonlinear differential equation on a measure chain," *Mathematical and Computer Modelling*, vol. 32, no. 5-6, pp. 571–585, 2000.
[6] C. J. Chyan and J. Henderson, "Eigenvalue problems for nonlinear differential equations on a measure chain," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 2, pp. 547–559, 2000.
[7] R. P. Agarwal and D. O'Regan, "Nonlinear boundary value problems on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 4, pp. 527–535, 2001.
[8] D. R. Anderson, "Eigenvalue intervals for a two-point boundary value problem on a measure chain," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 57–64, 2002.
[9] C.-H. Hong and C.-C. Yeh, "Positive solutions for eigenvalue problems on a measure chain," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 51, no. 3, pp. 499–507, 2002.
[10] H. Y. Li, J. X. Sun, and Y. J. Cui, "Positive solutions of nonlinear differential equations on a measure chain," *Chinese Annals of Mathematics. Series A.*, vol. 30, no. 1, pp. 97–106, 2009 (Chinese).
[11] Z.-C. Hao, J. Liang, and T.-J. Xiao, "Existence results for time scale boundary value problem," *Journal of Computational and Applied Mathematics*, vol. 197, no. 1, pp. 156–168, 2006.
[12] R. P. Agarwal, M. Bohner, and P. J. Y. Wong, "Sturm-Liouville eigenvalue problems on time scales," *Applied Mathematics and Computation*, vol. 99, no. 2-3, pp. 153–166, 1999.

[13] Q. Kong, "Sturm-Liouville problems on time scales with separated boundary conditions," *Results in Mathematics*, vol. 52, no. 1-2, pp. 111–121, 2008.

[14] F. M. Atici, D. C. Biles, and A. Lebedinsky, "An application of time scales to economics," *Mathematical and Computer Modelling*, vol. 43, no. 7-8, pp. 718–726, 2006.

[15] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[16] C. C. Tisdell and A. Zaidi, "Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 11, pp. 3504–3524, 2008.

[17] J. Sun and X. Liu, "Computation of topological degree in ordered Banach spaces with lattice structure and its application to superlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 348, no. 2, pp. 927–937, 2008.

[18] X. Liu and J. Sun, "Computation of topological degree of unilaterally asymptotically linear operators and its applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 96–106, 2009.

[19] J. X. Sun and X. Y. Liu, "Computation of the fixed point index for non-cone mappings and its applications," *Acta Mathematica Sinica. Chinese Series*, vol. 53, no. 3, pp. 417–428, 2010 (Chinese).

[20] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, no. 4, pp. 620–709, 1976.

[21] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[22] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

[23] J. X. Sun, *Nonlinear Functional Analysis and Applications*, Science Press, Beijing, China, 2007.

[24] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces. Vol. I*, North-Holland Mathematical Library, North-Holland, Amsterdam, The Netherlands, 1971.

[25] M. G. Kreĭn and M. A. Rutman, "Linear operators leaving invariant a cone in a Banach space," *American Mathematical Society Translations*, vol. 1950, no. 26, p. 128, 1950.

[26] T. He, W. Yang, and F. Yang, "Sign-changing solutions for discrete second-order three-point boundary value problems," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 705387, 14 pages, 2010.

[27] F. A. Davidson and B. P. Rynne, "Global bifurcation on time scales," *Journal of Mathematical Analysis and Applications*, vol. 267, no. 1, pp. 345–360, 2002.

[28] F. A. Davidson and B. P. Rynne, "Curves of positive solutions of boundary value problems on time-scales," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 491–504, 2004.

[29] J. Sun and G. Zhang, "Nontrivial solutions of singular superlinear Sturm-Liouville problems," *Journal of Mathematical Analysis and Applications*, vol. 313, no. 2, pp. 518–536, 2006.

[30] H. Luo, *Boundary value problem of nonlinear dynamic equations on time scales*, Doctoral thesis, Northwest Normal University, Lanzhou, China, 2007.


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