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Research Article

Global Stability of an Eco-Epidemiological Model with Time Delay and Saturation Incidence

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We investigate a delayed eco-epidemiological model with disease in predator and saturation incidence. First, by comparison arguments, the permanence of the model is discussed. Then, we study the local stability of each equilibrium of the model by analyzing the corresponding characteristic equations and find that Hopf bifurcation occurs when the delay τ passes through a sequence of critical values. Next, by means of an iteration technique, sufficient conditions are derived for the global stability of the disease-free planar equilibrium and the positive equilibrium. Numerical examples are carried out to illustrate the analytical results.

1. Introduction

Recently, more attention has been paid to the eco-epidemiology model which considers both the ecological and epidemiological issues simultaneously due to the fact that most of the ecological populations suffer from various infectious diseases which have a significant role in regulating population sizes (see, e.g., [1–6]). Mukherjee [7] discussed a predator-prey model with disease in prey. The criteria were derived for both local stability and instability involving system parameters. In addition, considering the time required by the susceptible individuals to become infective after their interaction with the infectious individuals, Zhou et al. [8] formulated a delayed eco-epidemiology model and found that the Hopf bifurcation occurs when the delay passes through a sequence of critical values. They also gave an estimation of the length of the time delay to preserve stability. On the other hand, in the predator-prey system, the disease not only can spread in prey but also can spread in predator. Therefore, Zhang et al. [9] studied an eco-epidemiological model with disease in predator and showed that a Hopf bifurcation can occur as the delay increased. The above-mentioned works all used bilinear incidence to model disease transmission.

Note that ecologically the assumption of standard incidence instead of the former bilinear mass action incidence is meaningful for large populations and a low number of infected individuals, a very good justification behind this assumption being found in [10]. Han et al. [11] proposed four modifications of a predator-prey model with standard incidence to include an SIS or SIR parasitic infection. Thresholds were identified, and global stability results were proved. When the disease persists in the prey population and the predators have a sufficient feeding efficiency to survive, the disease also persists in the predator population. Hethcote et al. [12] considered a predator-prey model including an SIS parasitic infection in the prey with infected prey being more vulnerable to predation. Thresholds were identified which determine when the predator population survives and when the disease remains endemic.

However, there are a variety of factors that emphasize the need for a modification of the bilinear incidence and standard incidence. For example, the underlying assumption of homogeneous mixing may not always hold. Incidence rates that increase more gradually than linearly in I and S may arise from saturation effects. It has been strongly suggested by several authors that the disease transmission process may follow saturation incidence. After studying the cholera epidemic spread in Bari in 1973, Capasso and Serio [13] introduced a saturated incidence rate $g(I)S$ into epidemic models with $g(I) = \beta I/(1+\alpha I)$. A general saturation incidence rate $g(I)S = \beta I^p S/(1+\alpha I^p)$ was proposed by Liu et al. [14] and used by a number of authors; see, for example, Ruan and Wang [15] ($p = 2$), Bhattacharyya and Mukhopadhyay [16] ($p = 1$), and so forth. βI^p measures the infection force of the disease, and $1/(1+\alpha I^p)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate βSI, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

Motivated by the works of Zhang et al. [9] and Capasso and Serio [13], in this paper, we are concerned with the effect of disease in predator and saturated incidence on the dynamics of eco-epidemiological model. To this end, we consider the following delay differential equations:

$$\begin{aligned}
\dot{x}(t) &= rx(t)\left(1-\frac{x(t)}{K}\right) - ax(t)S(t),\\
\dot{S}(t) &= bx(t-\tau)S(t-\tau) - cS^2(t) - \frac{\beta S(t)I(t)}{1+\alpha I(t)},\\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{1+\alpha I(t)} - dI(t),
\end{aligned} \tag{1.1}$$

with initial conditions

$$\begin{gathered}
x(\theta) = \phi_1(\theta), \qquad S(\theta) = \phi_2(\theta), \qquad I(\theta) = \phi_3(\theta),\\
\phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0 \quad (i = 1,2,3),
\end{gathered} \tag{1.2}$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau,0]), \mathbb{R}^3_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_{+0}$, here $\mathbb{R}^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0,\ i = 1,2,3\}$.

We make the following assumptions for our model (1.1).

(A1) The prey population grows logistically with intrinsic growth rate r and environmental carrying capacity K.

(A2) There is a spread of disease in predators which are divided solely into susceptible and infectious population. a is the capturing rate of susceptible predators, b is the growth rate of susceptible predator due to predation of prey.

(A3) Susceptible predators become infected when they come in contact with infected predator, and this contact process is assumed to follow the saturation incidence rate $\beta S(t)I(t)/(1+\alpha I(t))$, with β measuring the force of infection and α the inhibition effect.

(A4) $c > 0$ models death rate due to overcrowding, and τ is the time required for the gestation of susceptible predator. d is the death rate of infected predator. All the above-mentioned parameters are assumed to be positive.

The paper is organized as follows. In the next section, the positivity of solutions and the permanence of system are discussed. By analyzing the corresponding characteristic equations, we find conditions for local stability and bifurcation results in Section 3. In Section 4, sufficient conditions are derived for the global stability of the disease-free planar equilibrium and the positive equilibrium of the system. Numerical examples are carried out to illustrate the validity of the main results. The paper ends with a conclusion in the last section.

2. Permanence

To prove the permanence of system (1.1), we need the following lemma, which is a direct application of Theorem 4.9.1 in the study by Kuang [17].

Lemma 2.1. *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t), \tag{2.1}$$

where $a, b, c, \tau > 0$ *and* $x(t) > 0$ *for all* $t \in [-\tau, 0]$.

(1) *If* $a > b$, *then* $\limsup_{t\to+\infty} x(t) = (a-b)/c$.

(2) *If* $a < b$, *then* $\limsup_{t\to+\infty} x(t) = 0$.

Theorem 2.2. *All the solutions of* (1.1) *with initial conditions* (1.2) *are all nonnegative.*

Proof. Let $(x(t), S(t), I(t))$ be the solution of system (1.1) satisfying conditions (1.2). From the first and last equations of system (1.1), we have

$$\begin{aligned} x(t) &= x(0)e^{\int_0^t [r(1-x(\xi)/K) - aS(\xi)]d\xi}, \\ I(t) &= I(0)e^{\int_0^t [\beta S(\xi)/(1+\alpha I(\xi)) - d]d\xi}. \end{aligned} \tag{2.2}$$

Hence, $x(t)$ and $I(t)$ are positive.

We now claim that $S(t) > 0$ for all $t > 0$. Otherwise, there exists a $t_1 > 0$ such that $S(t_1) = 0$ and $S(t) > 0$ for all $t \in [0, t_1)$. Then $\dot{S}(t_1) \leq 0$. From the second equation of (1.1), we have

$$\dot{S}(t_1) = bx(t_1 - \tau)S(t_1 - \tau) > 0, \tag{2.3}$$

which is a contradiction. □

Theorem 2.3. *All the solutions of* (1.1) *with initial conditions* (1.2) *are ultimately bounded.*

Proof. From the first equation of (1.1), we have

$$\dot{x}(t) \leq rx(t)\left(1 - \frac{x(t)}{K}\right). \tag{2.4}$$

Hence, we get

$$\limsup_{t \to +\infty} x(t) \leq K \doteq M_1. \tag{2.5}$$

From the second equation of system (1.1), for t sufficiently large, we have

$$\begin{aligned} \dot{S}(t) &= bx(t - \tau)S(t - \tau) - cS^2(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} \\ &\leq bKS(t - \tau) \ - cS^2(t). \end{aligned} \tag{2.6}$$

Hence, by Lemma 2.1, one can get

$$\limsup_{t \to +\infty} S(t) \leq \frac{bK}{c} \doteq M_2. \tag{2.7}$$

It follows from the third equation of (1.1) and the above inequality, that for t sufficiently large, we have

$$\begin{aligned} \dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - dI(t) \\ &\leq \frac{\beta M_2 I(t)}{1 + \alpha I(t)} - dI(t). \end{aligned} \tag{2.8}$$

Hence, one can see $\limsup_{t \to +\infty} I(t) \leq (1/(d\alpha))|\beta M_2 - d| \doteq M_3$. □

Now, we show that system (1.1) is permanent.

Theorem 2.4. *Suppose that*

(H_1)

$$\beta m_2 > d, \tag{2.9}$$

where m_2 is defined in (2.13)*, then system* (1.1) *is permanent.*

Proof. From the first equation of system (1.1), we have

$$\dot{x}(t) \geq rx(t)\left(1 - \frac{x(t)}{K} - \frac{aM_2}{r}\right). \tag{2.10}$$

It then follows that

$$\liminf_{t\to+\infty} x(t) \geq K\left[1 - \frac{aM_2}{r}\right] \doteq m_1. \tag{2.11}$$

Using the second equation of system (1.1), for t sufficiently large, we have

$$\dot{S}(t) \geq bm_1 S(t-\tau) - cS^2(t) - \frac{\beta S(t) M_3}{1+\alpha M_3}. \tag{2.12}$$

Hence, by Lemma 2.1 and (H_1), one can derive that

$$\liminf_{t\to+\infty} S(t) \geq \frac{1}{c}\left[bm_1 - \frac{\beta M_3}{1+\alpha M_3}\right] \doteq m_2. \tag{2.13}$$

From the third equation of system (1.1) and, above inequality, we have

$$\dot{I}(t) \geq \frac{\beta m_2 I(t)}{1+\alpha I(t)} - dI(t). \tag{2.14}$$

Since (H_1) holds, then

$$\liminf_{t\to+\infty} I(t) \geq \frac{1}{d\alpha}\left[\beta m_2 - d\right] \doteq m_3. \tag{2.15}$$

Therefore, the above calculations and Theorem 2.2 imply that there exist M_i, m_i $(i = 1,2,3)$ such that

$$\begin{aligned} 0 < m_1 &\leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \\ 0 < m_2 &\leq \liminf_{t\to+\infty} S(t) \leq \limsup_{t\to+\infty} S(t) \leq M_2, \\ 0 < m_3 &\leq \liminf_{t\to+\infty} I(t) \leq \limsup_{t\to+\infty} I(t) \leq M_3. \end{aligned} \tag{2.16}$$

□

3. Local Stability

System (1.1) possesses the following equilibria.

(1) The trivial equilibrium $E_0(0,0,0)$.

(2) The axial equilibrium $E_1(K,0,0)$.

(3) The disease-free planar equilibrium $E_2(x_2, S_2, 0)$, where

$$x_2 = \frac{Kcr}{Kab+rc}, \qquad S_2 = \frac{Kbr}{Kab+rc}. \tag{3.1}$$

(4) The unique positive equilibrium $E_3(x_3, S_3, I_3)$ exists if $\beta S_3 > d$, where

$$S_3 = \frac{Kb\alpha - \beta + \sqrt{(Kb\alpha - \beta)^2 + 4d\alpha(Kab/r + c)}}{2\alpha(Kab/r + c)},$$
$$I_3 = \frac{\beta S_3 - d}{d\alpha}, \qquad x_3 = K - \frac{KaS_3}{r}. \tag{3.2}$$

In the following, we discuss the local stability of each equilibrium of system (1.1) by analyzing the corresponding characteristic equations, respectively.

3.1. Stability of Equilibrium E_0

The characteristic equation of system (1.1) at the trivial equilibrium E_0 is of the form

$$\lambda(\lambda - r)(\lambda + d) = 0. \tag{3.3}$$

It is easy to see that (3.3) always has a positive root r. Hence, E_0 is always unstable.

3.2. Stability of Equilibrium E_1

The characteristic equation of system (1.1) at the axial equilibrium E_1 is of the form

$$(\lambda + K)\left(\lambda - bKe^{-\lambda\tau}\right)(\lambda + d) = 0. \tag{3.4}$$

There are two characteristic roots $\lambda_1 = -K$, $\lambda_2 = -d$, and another characteristic root is given by the root of

$$\lambda = bKe^{-\lambda\tau}. \tag{3.5}$$

It is clear that $\operatorname{Re}\lambda > 0$. Hence, E_1 is always unstable.

3.3. Stability of Equilibrium E_2

Theorem 3.1. *The disease-free planar equilibrium E_2 is locally asymptotically stable if $\beta S_2 < d$, and the equilibrium E_2 is unstable if $\beta S_2 > d$.*

Proof. The characteristic equation of system (1.1) at the disease-free planar equilibrium E_2 is of the form

$$\left(\lambda + \frac{rx_2}{K}\right)\left(\lambda + 2cS_2 - cS_2e^{-\lambda\tau}\right)(\lambda + d - \beta S_2) = 0. \tag{3.6}$$

Clearly, $\lambda_1 = -rx_2/K$ is a negative eigenvalue. The second eigenvalue is given by the root of

$$\lambda_2 = cS_2\left(e^{-\lambda_2\tau} - 2\right). \tag{3.7}$$

Suppose that $\operatorname{Re}\lambda_2 \geq 0$, then $\operatorname{Re}\lambda_2 = cS_2(e^{-\operatorname{Re}\lambda_2\tau}\cos(\tau \operatorname{Im}\lambda_2) - 2) < 0$. It is a contradiction, so $\operatorname{Re}\lambda_2 < 0$. The last eigenvalue is $\lambda_3 = \beta S_2 - d$. The equilibrium E_2 is locally asymptotically stable if $\beta S_2 < d$, and the equilibrium E_2 is unstable if $\beta S_2 > d$. □

3.4. Stability of Equilibrium E_3

The characteristic equation of system (1.1) at the positive equilibrium E_3 is of the form

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 + e^{-\lambda\tau}\left(B_1\lambda^2 + B_2\lambda + B_3\right) = 0, \tag{3.8}$$

where

$$\begin{aligned}
A_1 &= \frac{rx_3}{K} + 2cS_3 + \frac{\beta I_3}{1+\alpha I_3} + \frac{d\alpha I_3}{1+\alpha I_3},\\
A_2 &= \left(\frac{rx_3}{K} + \frac{d\alpha I_3}{1+\alpha I_3}\right) \times \left(2cS_3 + \frac{\beta I_3}{1+\alpha I_3}\right) + \frac{d\alpha I_3}{1+\alpha I_3} \times \frac{rx_3}{K} + \frac{\beta I_3}{(1+\alpha I_3)^2},\\
A_3 &= \frac{rx_3}{K} \times \left[\frac{d\beta I_3}{1+\alpha I_3} + \left(2cS_3 + \frac{\beta I_3}{1+\alpha I_3}\right) \times \frac{d\alpha I_3}{1+\alpha I_3}\right],\\
B_1 &= -bx_3,\\
B_2 &= bx_3 \times \left(-\frac{d\alpha I_3}{1+\alpha I_3} - \frac{rx_3}{K} + aS_3\right),\\
B_3 &= bx_3 \times \frac{d\alpha I_3}{1+\alpha I_3} \times \left(-\frac{rx_3}{K} + aS_3\right).
\end{aligned} \tag{3.9}$$

For $\tau = 0$, the transcendental (3.8) reduces to the following equation:

$$\lambda^3 + (A_1 + B_1)\lambda^2 + (A_2 + B_2)\lambda + A_3 + B_3 = 0. \tag{3.10}$$

We can easily get

$$\begin{gathered}
A_1 + B_1 = \frac{rx_3}{K} + cS_3 + \frac{d\alpha I_3}{1+\alpha I_3} > 0,\\
A_2 + B_2 = \left(\frac{rx_3}{K} + \frac{d\alpha I_3}{1+\alpha I_3}\right) \times cS_3 + \frac{d\alpha I_3}{1+\alpha I_3} \times \frac{rx_3}{K} + \frac{\beta I_3}{(1+\alpha I_3)^2} + bx_3aS_3 > 0,\\
A_3 + B_3 = \frac{rx_3}{K} \times \left[\frac{d\beta I_3}{1+\alpha I_3} + cS_3 \times \frac{d\alpha I_3}{1+\alpha I_3}\right] + bx_3 \times \frac{d\alpha I_3}{1+\alpha I_3} \times aS_3 > 0,\\
[A_1 + B_1] \times [A_2 + B_2] - [A_3 + B_3] > 0.
\end{gathered} \tag{3.11}$$

Therefore, the Routh-Hurwitz criterion implies that all the roots of (3.8) have negative real parts and we can conclude that the positive equilibrium E_3 is asymptotically stable in the absence of delay.

Theorem 3.2. *For system* (1.1), *if the condition* (H_2) $A_3 < B_3$ *holds, the positive equilibrium* E_3 *is conditionally stable.*

Proof. Substituting $\lambda = i\omega$ into (3.8) and separating the real and imaginary parts, one can get

$$\begin{aligned} A_1\omega^2 - A_3 &= (B_3 - B_1\omega_2)\cos(\omega\tau) + B_2\omega\sin(\omega\tau), \\ \omega^3 - A_2\omega &= B_2\omega\cos(\omega\tau) - (B_3 - B_1\omega_2)\sin(\omega\tau). \end{aligned} \tag{3.12}$$

Squaring and adding (3.12) we get

$$\omega^6 + D_1\omega^4 + D_2\omega^2 + D_3 = 0, \tag{3.13}$$

where

$$D_1 = A_1^2 - 2A_2 - B_1^2, \qquad D_2 = A_2^2 - B_2^2 - 2A_1A_3 + 2B_1B_3, \qquad D_3 = A_3^2 - B_3^2. \tag{3.14}$$

We know that $D_3 < 0$ provided that the condition (H_2) holds. There is at least a positive ω_0 satisfying (3.13), that is, the characteristic equation (3.8) has a pair of purely imaginary roots of the form $\pm i\omega_0$. From (3.12), we can get the corresponding $\tau_k > 0$ such that the characteristic (3.8) has a pair of purely imaginary roots

$$\tau_k = \frac{1}{\omega_0}\arccos\left[\frac{(A_1\omega_0^2 - A_3)(B_3 - B_1\omega_0^2) + (\omega_0^3 - A_2\omega_0)B_2\omega_0}{(B_3 - B_1\omega_0^2)^2 + (B_2\omega_0)^2}\right] + \frac{2k\pi}{\omega_0}, \quad (k = 0, 1, 2, \ldots). \tag{3.15}$$

Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ be the roots of (3.8) such that $\tau = \tau_k$ satisfying $\nu(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$. Differentiating the two sides of (3.8) with respect to τ, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda^3 + A_1\lambda^2 - A_2}{-\lambda^2(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{B_1\lambda^2 - B_3}{\lambda^2(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}. \tag{3.16}$$

Therefore,

$$\begin{aligned} \operatorname{sign}\left[\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right]_{\tau=\tau_k} &= \operatorname{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \\ &= \frac{1}{\omega_0^2}\operatorname{sign}\left[\mathrm{Re}\left(\frac{A_3 + A_1\omega_0^2 + i2\omega_0^3}{A_1\omega_0^2 - A_3 + i(\omega_0^3 - A_2\omega_0)} + \frac{B_1\omega_0^2 + B_3}{-B_1\omega_0^2 + iB_2\omega_0 + B_3}\right)\right] \\ &= \frac{1}{\omega_0^2}\operatorname{sign}\left[\frac{2\omega_0^6 + (A_1^2 - 2A_2 - B_1^2)\omega_0^4 + B_3^2 - A_3^2}{(B_3 - B_1\omega_0^2)^2 + (B_2\omega_0)^2}\right]. \end{aligned} \tag{3.17}$$

If the conditions (H_2) and (H_3) $A_1^2 - 2A_2 > B_1^2$ hold, one can see

$$\operatorname{sign}\left[\frac{\mathrm{d}\ \mathrm{Re}\ \lambda}{\mathrm{d}\tau}\right]_{\tau=\tau_k} > 0. \tag{3.18}$$

Therefore, the transversality condition holds, hence, the Hopf bifurcation occurs at $\omega = \omega_0$ and $\tau = \tau_k$. □

Theorem 3.3. *Suppose that the conditions* (H_2) *and* (H_3) *are satisfied.*

(1) *The positive equilibrium* E_3 *of system* (1.1) *is asymptotically stable for all* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$.

(2) *System* (1.1) *undergoes a Hopf Bifurcation at the positive equilibrium* E_3 *when* $\tau = \tau_k$ $(k = 0, 1, \ldots)$.

4. Global Stability

In this section, we study the global stability of equilibriums E_2 and E_3. The strategy of proofs is to use an iteration technique and comparison arguments, respectively.

Theorem 4.1. *If*

(H_4) $\beta bK < cd$, $Kab < rc$ *holds, then the disease-free planar equilibrium* E_2 *is globally asymptotically stable.*

Proof. Let $(x(t), S(t), I(t))$ be any positive solution of system (1.1) with initial conditions (1.2). Let the following hold:

$$\begin{gathered} U_1 = \limsup_{t\to+\infty} x(t), \quad U_2 = \limsup_{t\to+\infty} S(t), \quad U_3 = \limsup_{t\to+\infty} I(t), \\ V_1 = \liminf_{t\to+\infty} x(t), \quad V_2 = \liminf_{t\to+\infty} S(t), \quad V_3 = \liminf_{t\to+\infty} I(t). \end{gathered} \tag{4.1}$$

In the following we shall claim that $U_1 = V_1 = x_2$, $U_2 = V_2 = S_2$, $U_3 = V_3 = 0$.

It follows from the first equation of system (1.1) that

$$\dot{x}(t) \le rx(t)\left(1 - \frac{x(t)}{K}\right). \tag{4.2}$$

By comparison, we obtain that

$$U_1 = \limsup_{t\to+\infty} x(t) \le K + \varepsilon. \tag{4.3}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_1 \le M_1^x$, where

$$M_1^x = K. \tag{4.4}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that, if $t > T_1$, $x(t) \le M_1^x + \varepsilon$.

We, therefore, derive from the second equation of system (1.1) that, for $t > T_1 + \tau$,

$$\dot{S}(t) \leq b\left(M_1^x + \varepsilon\right)S(t-\tau) - cS^2(t). \tag{4.5}$$

Hence, by Lemma 2.1, one can get

$$U_2 = \limsup_{t\to+\infty} S(t) \leq \frac{b\left(M_1^x + \varepsilon\right)}{c} \doteq M_1^S. \tag{4.6}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > 0$ such that, if $T_2 > T_1 + \tau$, $S(t) \leq M_1^S + \varepsilon$.

It follows from the third equation of system (1.1) that, for $t > T_2$,

$$\dot{I}(t) \leq \frac{\beta\left(M_1^S + \varepsilon\right)I(t)}{1 + \alpha I(t)} - dI(t). \tag{4.7}$$

Since (H_4) holds, one can see

$$U_3 = \limsup_{t\to+\infty} I(t) \leq 0. \tag{4.8}$$

According to Theorem 2.2, we can get $\lim_{t\to+\infty} I(t) = U_3 = V_3 = 0$.

We derive from the first equation of system (1.1) that, for $t > T_2 + \tau$,

$$\dot{x}(t) \geq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a\left(M_1^S + \varepsilon\right)}{r}\right). \tag{4.9}$$

By comparison we derive that

$$V_1 = \liminf_{t\to+\infty} x(t) \geq K\left[1 - \frac{a\left(M_1^S + \varepsilon\right)}{r}\right]. \tag{4.10}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \geq N_1^x$, where

$$N_1^x = K\left[1 - \frac{aM_1^S}{r}\right]. \tag{4.11}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 > 0$ such that, if $T_3 > T_2 + \tau$, $x(t) \geq N_1^x - \varepsilon$.

We derive from the second equation of system (1.1) that, for $t > T_3$,

$$\dot{S}(t) \geq b\left(N_1^x - \varepsilon\right)S(t-\tau) - cS^2(t) - \frac{\beta S(t)\varepsilon}{1 + \alpha\varepsilon}. \tag{4.12}$$

Hence, by Lemma 2.1, one can get

$$V_2 = \liminf_{t\to+\infty} S(t) \geq \frac{1}{c}\left[b\left(N_1^x - \varepsilon\right) - \frac{\beta\varepsilon}{1+\alpha\varepsilon}\right]. \tag{4.13}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_1^S$, where

$$N_1^S = \frac{bN_1^x}{c}. \tag{4.14}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_4 > 0$ such that, if $T_4 > T_3 + \tau$, $S(t) \geq N_1^S - \varepsilon$.

Again, it follows from the first equation of system (1.2) that, for $t > T_4$,

$$\dot{x}(t) \leq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a\left(N_1^S - \varepsilon\right)}{r}\right). \tag{4.15}$$

A comparison argument yields

$$U_1 = \limsup_{t\to+\infty} x(t) \leq K\left[1 - \frac{a\left(N_1^S - \varepsilon\right)}{r}\right]. \tag{4.16}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_1 \leq M_2^x$, where

$$M_2^x = K\left[1 - \frac{aN_1^S}{r}\right]. \tag{4.17}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_5 > 0$ such that, if $T_5 > T_4 + \tau$, $x(t) \leq M_2^x + \varepsilon$.

It follows from the second equation of system (1.1) that, for $t > T_5$,

$$\dot{S}(t) \leq b\left(M_2^x + \varepsilon\right)S(t-\tau) - cS^2(t) - \frac{\beta S(t)\varepsilon}{1+\alpha\varepsilon}. \tag{4.18}$$

By Lemma 2.1, one can derive that

$$V_2 = \limsup_{t\to+\infty} S(t) \leq \frac{1}{c}\left[b\left(M_2^x + \varepsilon\right) - \frac{\beta\varepsilon}{1+\alpha\varepsilon}\right]. \tag{4.19}$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U_2 \leq M_2^S$, where

$$M_2^S = \frac{bM_2^x}{c}. \tag{4.20}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 > 0$ such that, if $T_6 > T_5 + \tau$, $S(t) \leq M_2^S + \varepsilon$.

We derive from the first equation of system (1.1) that, for $t > T_6$,

$$\dot{x}(t) \geq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a\left(M_2^S + \varepsilon\right)}{r}\right). \tag{4.21}$$

By comparison it follows that

$$V_1 = \liminf_{t \to +\infty} x(t) \geq K\left[1 - \frac{a\left(M_2^S + \varepsilon\right)}{r}\right]. \tag{4.22}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \geq N_2^x$, where

$$N_2^x = K\left[1 - \frac{aM_2^S}{r}\right]. \tag{4.23}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 > 0$ such that, if $T_7 > T_6 + \tau$, $x(t) \geq N_2^x - \varepsilon$.

We derive from the second equation of system (1.1) that, for $t > T_7$,

$$\dot{S}(t) \geq b\left(N_2^x - \varepsilon\right)S(t-\tau) - cS^2(t) - \frac{\beta S(t)\varepsilon}{1 + \alpha\varepsilon}. \tag{4.24}$$

Hence, by Lemma 2.1, one can get

$$V_2 = \liminf_{t \to +\infty} S(t) \geq \frac{1}{c}\left[b\left(N_2^x - \varepsilon\right) - \frac{\beta\varepsilon}{1 + \alpha\varepsilon}\right]. \tag{4.25}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_2^S$, where

$$N_2^S = \frac{bN_2^x}{c}. \tag{4.26}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 > 0$ such that, if $T_8 > T_7 + \tau$, $S(t) \geq N_2^S - \varepsilon$.

Continuing this process, we get four sequences $M_n^x, M_n^S, N_n^x, N_n^S$ $(n = 1, 2, \ldots)$ such that, for $n \geq 2$,

$$\begin{aligned}
M_n^x &= K\left[1 - \frac{aN_{n-1}^S}{r}\right], \\
N_n^x &= K\left[1 - \frac{aM_n^S}{r}\right], \\
M_n^S &= \frac{bM_n^x}{c}, \\
N_n^S &= \frac{bN_n^x}{c}.
\end{aligned} \tag{4.27}$$

Clearly, we have

$$N_n^x \le V_1 \le U_1 \le M_n^x, \qquad N_n^S \le V_2 \le U_2 \le M_n^S. \tag{4.28}$$

It follows from (4.27) that

$$M_{n+1}^x = K\left[1 - \frac{Kab}{rc}\right] + M_n^x \frac{K^2a^2b^2}{r^2c^2}. \tag{4.29}$$

Noting that $M_n^x \ge S_2$ and $Kab < rc$, we derive from (4.29) that

$$\begin{aligned} M_{n+1}^x &= K\left[1 - \frac{Kab}{rc}\right] + M_n^x\left[\frac{Kab}{rc} - 1\right]\left[\frac{Kab}{rc} + 1\right] \\ &\le K\left[1 - \frac{Kab}{rc}\right] + \frac{Kcr}{Kab + rc}\left[\frac{Kab}{rc} - 1\right]\left[\frac{Kab}{rc} + 1\right] \\ &= 0. \end{aligned} \tag{4.30}$$

Thus, the sequence M_n^x is monotonically nonincreasing. Therefore, it follows that $\lim_{n\to+\infty} M_n^x$ exists. Taking $n \to +\infty$, we obtain from (4.29) that

$$\lim_{n\to+\infty} M_{n+1}^x = K\left[1 - \frac{Kab}{rc}\right] + \lim_{n\to+\infty} M_n^x \frac{K^2a^2b^2}{r^2c^2}. \tag{4.31}$$

Noting that

$$\lim_{n\to+\infty} M_{n+1}^x = \lim_{n\to+\infty} M_n^x, \tag{4.32}$$

it follows from (4.31) that

$$\lim_{n\to+\infty} M_{n+1}^x = \lim_{n\to+\infty} M_n^x = x_2. \tag{4.33}$$

We derive from (4.33) and the third equation of (4.27) that

$$\lim_{n\to+\infty} M_{n+1}^S = \lim_{n\to+\infty} M_n^S = S_2. \tag{4.34}$$

Similarly, one can derive from (4.27) and (4.34) that

$$\lim_{n\to+\infty} N_n^x = x_2, \qquad \lim_{n\to+\infty} N_n^S = S_2. \tag{4.35}$$

It follows from (4.28), (4.33), and (4.35) that

$$V_1 = U_1 = x_2, \qquad V_2 = U_2 = S_2. \tag{4.36}$$

We, therefore, have

$$\lim_{t\to+\infty} x(t) = x_2, \qquad \lim_{t\to+\infty} S(t) = S_2, \qquad \lim_{t\to+\infty} I(t) = 0. \tag{4.37}$$

Hence, the disease-free planar equilibrium E_2 is globally asymptotically stable. The proof is complete. □

Theorem 4.2. *If*

(H_5) *$\beta bK > cd$ and $Kab < rc$, $\beta > Kb\alpha$ holds, then the positive equilibrium E_3 is globally asymptotically stable.*

Proof. Let $(x(t), S(t), I(t))$ be any positive solution of system (1.1) with initial conditions (1.2). Let the following hold:

$$\begin{aligned} \overline{x} &= \limsup_{t\to+\infty} x(t), \quad \overline{S} = \limsup_{t\to+\infty} S(t), \quad \overline{I} = \limsup_{t\to+\infty} I(t), \\ \underline{x} &= \liminf_{t\to+\infty} x(t), \quad \underline{S} = \liminf_{t\to+\infty} S(t), \quad \underline{I} = \liminf_{t\to+\infty} I(t). \end{aligned} \tag{4.38}$$

In the following we claim that $\overline{x} = \underline{x} = x_3$, $\overline{S} = \underline{S} = S_3$, $\overline{I} = \underline{I} = I_3$.

It follows from the first equation of system (1.1) that

$$\dot{x}(t) \le rx(t)\left(1 - \frac{x(t)}{K}\right). \tag{4.39}$$

By comparison we obtain

$$\overline{x} = \limsup_{t\to+\infty} x(t) \le K + \varepsilon. \tag{4.40}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{x} \le M_1^x$, where

$$M_1^x = K. \tag{4.41}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that, if $t > T_1$, $x(t) \le M_1^x + \varepsilon$.We obtain from the second equation of system (1.1) that, for $t > T_1 + \tau$,

$$\dot{S}(t) \le b\left(M_1^x + \varepsilon\right)S(t-\tau) - cS^2(t). \tag{4.42}$$

Hence, by Lemma 2.1, we derive that

$$\overline{S} = \limsup_{t\to+\infty} S(t) \le \frac{b\left(M_1^x + \varepsilon\right)}{c}. \tag{4.43}$$

Since it is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{S} \leq M_1^S$, where

$$M_1^S = \frac{bM_1^x}{c}. \tag{4.44}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > 0$ such that, if $T_2 > T_1 + \tau$, $S(t) \leq M_1^S + \varepsilon$.

It follows from the third equation of system (1.1) that

$$\dot{I}(t) \leq \frac{\beta(M_1^S + \varepsilon)I(t)}{1 + \alpha I(t)} - dI(t). \tag{4.45}$$

Since (H_5) holds, one can see

$$\overline{I} = \limsup_{t \to +\infty} I(t) \leq \frac{\beta(M_1^S + \varepsilon) - d}{d\alpha}. \tag{4.46}$$

Since it is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{I} \leq M_1^I$, where

$$M_1^I = \frac{\beta M_1^S - d}{d\alpha}. \tag{4.47}$$

We derive from the first equation of system (1.1) that, for $t > T_2$,

$$\dot{x}(t) \geq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a(M_1^S + \varepsilon)}{r}\right). \tag{4.48}$$

By comparison we derive that

$$\underline{x} = \liminf_{t \to +\infty} x(t) \geq K\left[1 - \frac{a(M_1^S + \varepsilon)}{r}\right]. \tag{4.49}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{x} \geq N_1^x$, where

$$N_1^x = K\left[1 - \frac{aM_1^S}{r}\right]. \tag{4.50}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 > 0$ such that, if $T_3 > T_2 + \tau$, $x(t) \geq N_1^x - \varepsilon$.

We derive from the second equation of system (1.1) that, for $t > T_3$,

$$\dot{S}(t) \geq b(N_1^x - \varepsilon)S(t - \tau) - cS^2(t) - \frac{\beta S(t)M_1^I}{1 + \alpha M_1^I}. \tag{4.51}$$

Hence, by Lemma 2.1 and (H_5), one can get

$$\underline{S} = \liminf_{t\to+\infty} S(t) \geq \frac{1}{c}\left[b\left(N_1^x - \varepsilon\right) - \frac{\beta M_1^I}{1+\alpha M_1^I}\right]. \tag{4.52}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{S} \geq N_1^S$, where

$$N_1^S = \frac{1}{c}\left[bN_1^x - \frac{\beta M_1^I}{1+\alpha M_1^I}\right]. \tag{4.53}$$

Hence, for $\varepsilon > 0$ sufficiently small, we get $S(t) \geq N_1^S - \varepsilon$.

It follows from the third equation of system (1.1) that

$$\dot{I}(t) \geq \frac{\beta\left(N_1^S - \varepsilon\right)I(t)}{1+\alpha I(t)} - dI(t). \tag{4.54}$$

Provided that $\beta N_1^S > d$, one can see

$$\underline{I} = \liminf_{t\to+\infty} I(t) \geq \frac{\beta\left(N_1^S - \varepsilon\right) - d}{d\alpha}. \tag{4.55}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{I} \geq N_1^I$, where

$$N_1^I = \frac{\beta N_1^S - d}{d\alpha}. \tag{4.56}$$

It follows from the first equation of system (1.1) that

$$\dot{x}(t) \leq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a\left(N_1^S - \varepsilon\right)}{r}\right). \tag{4.57}$$

By comparison we derive that

$$\overline{x} = \limsup_{t\to+\infty} x(t) \leq K\left[1 - \frac{a\left(N_1^S - \varepsilon\right)}{r}\right]. \tag{4.58}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{x} \leq M_2^x$, where

$$M_2^x = K\left[1 - \frac{aN_1^S}{r}\right]. \tag{4.59}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_4 > 0$ such that, if $t > T_4$, $x(t) \leq M_2^x + \varepsilon$.

We obtain from the second equation of system (1.1) that, for $t > T_4 + \tau$,

$$\dot{S}(t) \leq b\left(M_2^x + \varepsilon\right)S(t-\tau) - cS^2(t) - \frac{\beta S(t)N_1^I}{1+\alpha N_1^I}. \tag{4.60}$$

Hence, by Lemma 2.1, one can get

$$\overline{S} = \limsup_{t\to+\infty} S(t) \leq \frac{1}{c}\left[b\left(M_2^x + \varepsilon\right) - \frac{\beta N_1^I}{1+\alpha N_1^I}\right]. \tag{4.61}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{S} \leq M_2^S$, where

$$M_2^S = \frac{1}{c}\left[bM_2^x - \frac{\beta N_1^I}{1+\alpha N_1^I}\right]. \tag{4.62}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_5 > 0$ such that, if $T_5 > T_4 + \tau$, $S(t) \leq M_2^S + \varepsilon$.

It follows from the third equation of system (1.1) that

$$\dot{I}(t) \leq \frac{\beta\left(M_2^S + \varepsilon\right)I(t)}{1+\alpha I(t)} - dI(t). \tag{4.63}$$

Hence, by (H_5), one can see

$$\overline{I} = \limsup_{t\to+\infty} I(t) \leq \frac{\beta\left(M_2^S + \varepsilon\right) - d}{d\alpha}. \tag{4.64}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\overline{I} \leq M_2^I$, where

$$M_2^I = \frac{\beta M_2^S - d}{d\alpha}. \tag{4.65}$$

We derive from the first equation of system (1.1) that, for $t > T_5$,

$$\dot{x}(t) \geq rx(t)\left(1 - \frac{x(t)}{K} - \frac{a\left(M_2^S + \varepsilon\right)}{r}\right). \tag{4.66}$$

By comparison we derive that

$$\underline{x} = \liminf_{t\to+\infty} x(t) \geq K\left[1 - \frac{a\left(M_2^S + \varepsilon\right)}{r}\right]. \tag{4.67}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{x} \geq N_2^x$, where

$$N_2^x = K\left[1 - \frac{aM_2^S}{r}\right]. \tag{4.68}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 > 0$ such that, if $T_6 > T_5 + \tau$, $x(t) \geq N_2^x - \varepsilon$.

We derive from the second equation of system (1.1) that, for $t > T_6$,

$$\dot{S}(t) \geq b\left(N_2^x - \varepsilon\right)S(t-\tau) - cS^2(t) - \frac{\beta S(t)M_2^I}{1+\alpha M_2^I}. \tag{4.69}$$

By Lemma 2.1, one can get

$$\underline{S} = \liminf_{t\to+\infty} S(t) \geq \frac{1}{c}\left[b\left(N_2^x - \varepsilon\right) - \frac{\beta M_2^I}{1+\alpha M_2^I}\right]. \tag{4.70}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{S} \geq N_1^S$, where

$$N_2^S = \frac{1}{c}\left[bN_2^x - \frac{\beta M_2^I}{1+\alpha M_2^I}\right]. \tag{4.71}$$

Hence, for $\varepsilon > 0$ sufficiently small, we get $S(t) \geq N_2^S - \varepsilon$.

It follows from the third equation of system (1.1) that

$$\dot{I}(t) \geq \frac{\beta\left(N_2^S - \varepsilon\right)I(t)}{1+\alpha I(t)} - dI(t). \tag{4.72}$$

Since (H_5) holds, one can see

$$\underline{I} = \liminf_{t\to+\infty} I(t) \geq \frac{\beta\left(N_2^S - \varepsilon\right) - d}{d\alpha}. \tag{4.73}$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\underline{I} \geq N_2^I$, where

$$N_2^I = \frac{\beta N_2^S - d}{d\alpha}. \tag{4.74}$$

Continuing this process, we obtain six sequences $M_n^x, M_n^S, M_n^I, N_n^x, N_n^S, N_n^I$ $(n = 1, 2, \ldots)$ such that, for $n \geq 2$,

$$\begin{aligned}
M_n^x &= K\left[1 - \frac{aN_{n-1}^S}{r}\right], \\
M_n^S &= \frac{1}{c}\left[bM_n^x - \frac{\beta N_{n-1}^I}{1 + \alpha N_{n-1}^I}\right], \\
M_n^I &= \frac{\beta M_n^S - d}{d\alpha}, \\
N_n^x &= K\left[1 - \frac{aM_n^S}{r}\right], \\
N_n^S &= \frac{1}{c}\left[bN_n^x - \frac{\beta M_n^I}{1 + \alpha M_n^I}\right], \\
N_n^I &= \frac{\beta N_n^S - d}{d\alpha}.
\end{aligned} \tag{4.75}$$

A direct calculation shows that

$$\begin{aligned}
M_2^x - M_1^x &= -\frac{KaN_1^S}{r} < 0, \\
M_2^S - M_1^S &= \frac{1}{c}\left[b\left(M_2^x - M_1^x\right) - \frac{\beta N_1^I}{1 + \alpha N_1^I}\right] < 0, \\
M_2^I - M_1^I &= \frac{\beta\left(M_2^S - M_1^S\right)}{d\alpha} < 0, \\
N_2^x - N_1^x &= \frac{Ka\left(M_1^S - M_2^s\right)}{r} > 0, \\
N_2^S - N_1^S &= \frac{1}{c}\left[b\left(N_2^x - N_1^x\right) + \frac{\beta M_1^I}{1 + \alpha M_1^I} - \frac{\beta M_2^I}{1 + \alpha M_2^I}\right] > 0, \\
N_2^I - N_1^I &= \frac{\beta\left(N_2^S - N_1^S\right)}{d\alpha} > 0.
\end{aligned} \tag{4.76}$$

By induction, we can show that $M_{n+1}^S < M_n^S$, $N_{n+1}^S > N_n^S$. Therefore, the sequence M_n^S is decreasing, and the sequence N_n^S is increasing. Clearly, we have

$$N_n^x \leq \underline{x} \leq \overline{x} \leq M_n^x, \qquad N_n^S \leq \underline{S} \leq \overline{S} \leq M_n^S, \qquad N_n^I \leq \underline{I} \leq \overline{I} \leq M_n^I. \tag{4.77}$$

Hence, the limits of the sequences M_n^S and N_n^S exist. Denote

$$\overline{S} = \lim_{t \to +\infty} M_n^S, \qquad \underline{S} = \lim_{t \to +\infty} N_n^S. \tag{4.78}$$

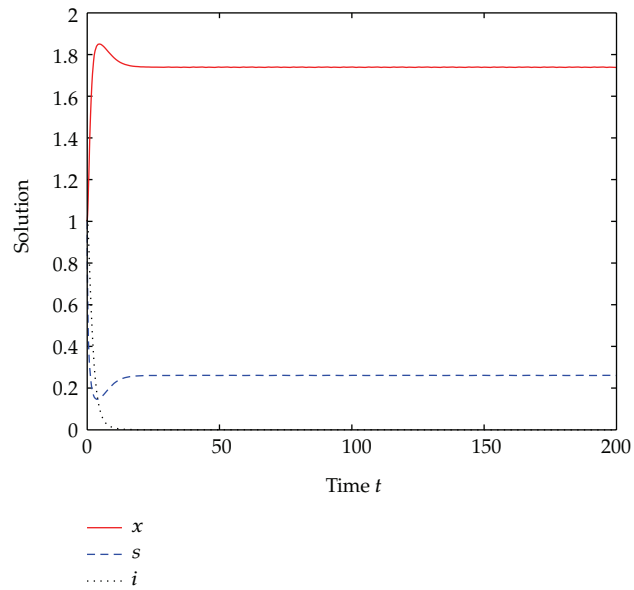

Figure 1: The temporal solution found by numerical integration of system (1.1) with $r = 2$, $K = 2$, $a = 2$, $c = 2$, $\beta = 2$, $\alpha = 1$, $d = 0.8$, $b = 0.3$, $\tau = 1$, and $(x_0, S_0, I_0) = (1, 1, 1)$.

We derive from (4.75) that

$$\left[\overline{S} - \underline{S}\right]\left[\beta - Kb\alpha + \frac{Kba}{dr}\left(\overline{S} + \underline{S}\right)\right] = 0. \tag{4.79}$$

Since (H_5) holds, $\beta - Kb\alpha + (Kba/dr)(\overline{S} + \underline{S}) > 0$. It, therefore, follows from (4.79) that $\overline{S} = \underline{S}$. Accordingly, we derive from (4.75) that

$$\overline{S} = \frac{1}{c}\left[bK\left(1 - \frac{a\overline{S}}{r}\right) - \frac{\beta}{\alpha} + \frac{d}{\overline{S}}\right]. \tag{4.80}$$

By a simple calculation, we obtain

$$\overline{S} = \underline{S} = S_3. \tag{4.81}$$

It follows from (4.75) and (4.81) that $\overline{I} = \underline{I} = I_3$, $\overline{x} = \underline{x} = x_3$. Hence, the unique positive equilibrium E_3 is globally asymptotically stable. The proof is complete. □

In the following we will present two examples to verify our results obtained earlier.

Example 4.3. In system (1.1), we let $r = 2$, $K = 2$, $a = 1$, $c = 2$, $\beta = 2$, $\alpha = 1$, $d = 0.8$, $b = 0.3$, $\tau = 1$. It is easy to show that $Kb\beta - cd = -0.4 < 0, rc - Kab = 3.4 > 0$. By Theorem 4.1 we see that the equilibrium $E_2(1.7391, 0.2609, 0)$ of system (1.1) is globally stable (see Figure 1).

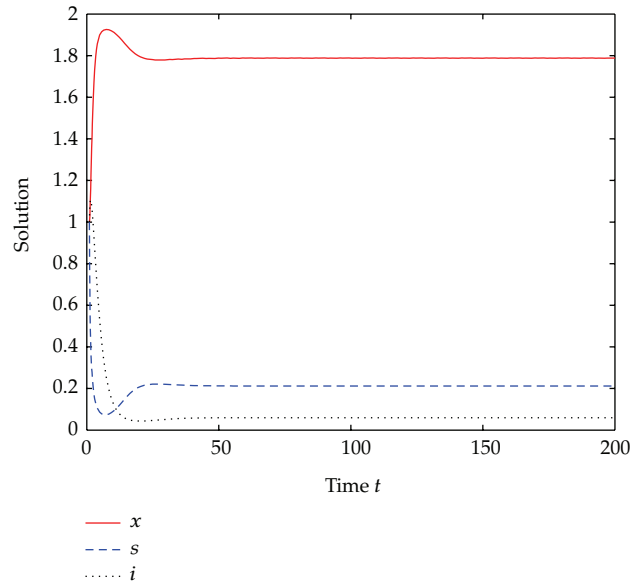

Figure 2: The temporal solution found by numerical integration of system (1.1) with $r = 2$, $K = 2$, $a = 1$, $c = 2$, $\beta = 2$, $\alpha = 1$, $d = 0.4$, $b = 0.3$, $\tau = 1$, and $(x_0, S_0, I_0) = (1, 1, 1)$.

Example 4.4. In system (1.1), we let $r = 2, K = 2, a = 1, c = 2, \beta = 2, \alpha = 1, d = 0.4, b = 0.3, \tau = 1$. It is easy to show that $Kb\beta - cd = 0.4 > 0, Kab - rc = -3.4 < 0, \beta - Kb\alpha = 1.400 > 0$. By Theorem 4.2 we see that the equilibrium $E_3(1.7881, 0.2119, 0.0596)$ of system (1.1) is globally stable, as depicted in Figure 2.

5. Conclusion

In this paper, we have incorporated the disease for the predator and the time delay into an eco-epidemiology model. A saturation incidence function was used to model the behavioral change of the susceptible predator when their number increases or due to the crowding effect of the infected predator. First, by comparison arguments, the permanence of system (1.1) was studied. Then, by analyzing the corresponding characteristic equations, sufficient conditions were derived for the local stability of each equilibrium of system (1.1). From Theorem 3.3, we showed that system (1.1) undergoes a Hopf bifurcation when the delay passes through a sequence of critical values. Next, by using the iteration technique and comparison arguments, we derived sufficient conditions for the global stability of the disease-free planer equilibrium and positive equilibrium of system (1.1). By Theorems 4.1 and 4.2, we showed that (1) if (H_4) holds, the infected predator population becomes extinct and the disease will be eliminated; that is, only sound predator and prey coexist; (2) if (H_5) holds, the prey, the sound predator and the infected predator coexist. The disease will not be eliminated, and the system is permanent.

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