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Research Article

First-Order Boundary Value Problem with Nonlinear Boundary Condition on Time Scales

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This work is concerned with the following first-order dynamic equation on time scale $\mathbb{T}$, $x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(t))$, $t \in [0, T]_{\mathbb{T}}$ with the nonlinear boundary condition $x(0) = g(x(\sigma(T)))$. By applying monotone iteration method, we not only obtain the existence of positive solutions, but also establish iterative schemes for approximating the solutions.

1. Introduction

The theory of time scales was introduced by Hilger in his Ph.D. thesis [1] in 1988 in order to unify continuous and discrete analysis. The study of dynamic equations on time scales is a fairly new subject and research in this area is rapidly growing. For some basic definitions and relevant results on time scales, see [2, 3].

Recently, first-order boundary value problems (BVPs for short) on time scales have attracted much attention from many authors. For example, for first-order periodic boundary value problem (PBVP for short) on time scales

$$\begin{gathered} u^{\Delta}(t) = f(t, u(t)), \quad t \in [a, b]_{\mathbb{T}}, \\ u(a) = u(\sigma(b)), \end{gathered} \tag{1.1}$$

Cabada [4] developed the method of lower and upper solutions coupled with the monotone iterative techniques to derive the existence of extremal solutions. In [5], Cabada and Vivero

studied the existence of solutions for the first-order dynamic equation with nonlinear functional boundary value conditions

$$\begin{gathered} u^{\Delta}(t) = f(t, u^{\sigma}(t)), \quad \text{for } \Delta - \text{a.e. } t \in [a, b]_{\mathbb{T}}, \\ B(u(a), u) = 0. \end{gathered} \tag{1.2}$$

They proved the uniqueness of solutions and developed the monotone iterative technique when $B(x, y) = x - g(y^{\sigma}(b))$ and g was a continuous and nonincreasing function. In 2007, Sun and Li [6] considered the first-order PBVP on time scales

$$\begin{gathered} x^{\Delta}(t) + p(t)x(\sigma(t)) = g(t, x(\sigma(t))), \quad t \in [0, T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)). \end{gathered} \tag{1.3}$$

Some existence criteria of at least one solution were established by using novel inequalities and the Schaefer fixed point theorem. In [7], by applying several well-known fixed point theorems, Sun and Li obtained some existence and multiplicity criteria of positive solutions for the first-order PBVP on time scales

$$\begin{gathered} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(x(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)). \end{gathered} \tag{1.4}$$

In 2010, Zhao and Sun [8] investigated the first-order PBVP on time scales

$$\begin{gathered} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)). \end{gathered} \tag{1.5}$$

Some existence criteria of positive solutions were established, and the method used was the monotone iterative technique. For other related results, one can refer to [9–14] and the references therein.

Motivated greatly by the above-mentioned works, in this paper, we are interested in the existence and iteration of positive solutions for the following first-order dynamic equation with nonlinear boundary condition on time scales:

$$\begin{gathered} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(t)), \quad t \in [0, T]_{\mathbb{T}}, \\ x(0) = g(x(\sigma(T))), \end{gathered} \tag{1.6}$$

where $\mathbb{T}$ is an arbitrary time scale, $T > 0$ is fixed, and $0, T \in \mathbb{T}$. For each interval $\mathbf{I}$ of $\mathbb{R}$, we denote by $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$. Throughout this paper, we always assume that $p : [0, T]_{\mathbb{T}} \to (0, +\infty)$ is right-dense continuous. Here, a solution x of the BVP (1.6) is said to be positive if x is nonnegative and nontrivial. By applying monotone iteration method, we not only obtain the existence of positive solutions for the BVP (1.6), but also establish iterative schemes for approximating the solutions. It is worth mentioning that the initial terms of our iterative

schemes are constant functions, which implies that the iterative schemes are significant and feasible. In our arguments, the following monotone iteration method [15] is very crucial.

Theorem 1.1. *Let K be a normal cone of a Banach space E and $v_0 \leq w_0$. Suppose that*

(a_1) *$T : [v_0, w_0] \to E$ is completely continuous,*

(a_2) *T is monotone increasing on $[v_0, w_0]$,*

(a_3) *v_0 is a lower solution of T, that is, $v_0 \leq Tv_0$,*

(a_4) *w_0 is an upper solution of T; that is, $Tw_0 \leq w_0$.*

Then, the iterative sequences

$$v_n = Tv_{n-1}, \quad w_n = Tw_{n-1} \quad (n = 1, 2, 3, \ldots) \tag{1.7}$$

satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0, \tag{1.8}$$

and converge to, respectively, v and $w \in [v_0, w_0]$, which are fixed points of T.

2. Main Results

Theorem 2.1. *Assume that $f : [0, T]_{\mathbb{T}} \times [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to [0, +\infty)$ are continuous and $\int_0^{\sigma(T)} f(s, 0)\Delta s > 0$. If there exists a constant $R > 0$ such that the following conditions are satisfied:*

(H_1) *$f(t, u) \leq f(t, v) \leq R/2\sigma(T)$, $t \in [0, T]_{\mathbb{T}}$, $0 \leq u \leq v \leq R$,*

(H_2) *$g(u) \leq g(v) \leq R/2$, $0 \leq u \leq v \leq R$,*

then the BVP (1.6) *has positive solutions.*

Proof. Let

$$E = \{x \mid x : [0, \sigma(T)]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous}\} \tag{2.1}$$

be equipped with the norm

$$\|x\| = \max_{t \in [0, \sigma(T)]_{\mathbb{T}}} |x(t)|. \tag{2.2}$$

Then, E is a Banach space. Denote

$$K = \{x \in E \mid x(t) \geq 0, \ t \in [0, \sigma(T)]_{\mathbb{T}}\}. \tag{2.3}$$

Then, K is a normal cone of E. Now, if we define an operator $\Phi : K \to K$ by

$$(\Phi x)(t) = \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0) f(s,x(s))\Delta s + g(x(\sigma(T)))\right], \quad t \in [0,\sigma(T)]_{\mathbb{T}}, \tag{2.4}$$

then it is easy to know that fixed points of Φ are nonnegative solutions of the BVP (1.6).

Let $v_0(t) \equiv 0$ and $w_0(t) \equiv R$, $t \in [0,\sigma(T)]_{\mathbb{T}}$. Now, we divide our proof into the following steps.

Step 1. We verify that $\Phi : [v_0, w_0] \to K$ is completely continuous.

First, we will show that $\Phi : [v_0, w_0] \to K$ is continuous.

Let x_n $(n = 1, 2, \ldots)$, $x \in [v_0, w_0]$ and $\lim_{n\to\infty} x_n = x$. Then,

$$0 \le x_n(t) \le R, \quad 0 \le x(t) \le R, \quad \text{for } t \in [0,\sigma(T)]_{\mathbb{T}}. \tag{2.5}$$

For any given $\epsilon > 0$, since f is uniformly continuous on $[0,T]_{\mathbb{T}} \times [0,R]$, there exists $\delta_1 > 0$ such that for any $u_1, u_2 \in [0,R]$ with $|u_1 - u_2| < \delta_1$,

$$\left|f(s,u_1) - f(s,u_2)\right| < \frac{\epsilon}{2\sigma(T)e_p(\sigma(T),0)}, \quad s \in [0,T]_{\mathbb{T}}. \tag{2.6}$$

On the other hand, since g is continuous at $x(\sigma(T))$, there exists $\delta_2 > 0$ such that for any $u \in [0,+\infty)$ with $|u - x(\sigma(T))| < \delta_2$,

$$\left|g(u) - g(x(\sigma(T)))\right| < \frac{\epsilon}{2}. \tag{2.7}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, it follows from $\lim_{n\to\infty} x_n = x$ that there exists a positive integer N such that for any $n > N$,

$$|x_n(s) - x(s)| < \delta, \quad s \in [0,\sigma(T)]_{\mathbb{T}}. \tag{2.8}$$

In view of (2.6), (2.7), and (2.8), we know that for any $n > N$,

$$\begin{aligned}
&|(\Phi x_n)(t) - (\Phi x)(t)| \\
&\quad = \frac{1}{e_p(t,0)}\left|\int_0^t e_p(s,0)\left[f(s,x_n(s)) - f(s,x(s))\right]\Delta s + \left[g(x_n(\sigma(T))) - g(x(\sigma(T)))\right]\right| \\
&\quad \le e_p(\sigma(T),0)\int_0^{\sigma(T)} \left|f(s,x_n(s)) - f(s,x(s))\right|\Delta s + \left|g(x_n(\sigma(T))) - g(x(\sigma(T)))\right| \\
&\quad < \epsilon, \quad t \in [0,\sigma(T)]_{\mathbb{T}},
\end{aligned} \tag{2.9}$$

which indicates that $\lim_{n\to\infty} \Phi x_n = \Phi x$. So, $\Phi : [v_0, w_0] \to K$ is continuous.

Next, we will show that $\Phi : [v_0, w_0] \to K$ is compact.

Let $X \subset [v_0, w_0]$ be a bounded set. Then, there exists a constant $L > 0$ such that for any $x \in X$ and $s \in [0, \sigma(T)]_{\mathbb{T}}$, $0 \le x(s) \le L$. Define $Q = \max_{s\in[0,T]_{\mathbb{T}}, u\in[0,L]} f(s,u)$ and $M = \max_{u\in[0,L]} g(u)$.

On the one hand, for any $x \in X$, we have

$$\begin{aligned}(\Phi x)(t) &= \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0) f(s,x(s))\Delta s + g(x(\sigma(T)))\right] \\ &\le e_p(\sigma(T),0)\int_0^{\sigma(T)} f(s,x(s))\Delta s + g(x(\sigma(T))) \\ &\le Q\sigma(T)e_p(\sigma(T),0) + M, \quad t \in [0,\sigma(T)]_{\mathbb{T}},\end{aligned} \tag{2.10}$$

which shows that $\Phi(X)$ is uniformly bounded.

On the other hand, for any $x \in X$ and $t_1, t_2 \in [0, \sigma(T)]_{\mathbb{T}}$ with $t_1 \le t_2$, we have

$$\begin{aligned}&|(\Phi x)(t_2) - (\Phi x)(t_1)| \\ &\quad = \left|\left[\frac{1}{e_p(t_2,0)} - \frac{1}{e_p(t_1,0)}\right]\left[\int_0^{t_1} e_p(s,0) f(s,x(s))\Delta s + g(x(\sigma(T)))\right]\right. \\ &\qquad \left. + \frac{1}{e_p(t_2,0)}\int_{t_1}^{t_2} e_p(s,0) f(s,x(s))\Delta s\right| \\ &\quad \le \left[\frac{1}{e_p(t_1,0)} - \frac{1}{e_p(t_2,0)}\right]\left[\int_0^{\sigma(T)} e_p(s,0) f(s,x(s))\Delta s + g(x(\sigma(T)))\right] \\ &\qquad + \int_{t_1}^{t_2} e_p(s,0) f(s,x(s))\Delta s \\ &\quad \le \left[\frac{1}{e_p(t_1,0)} - \frac{1}{e_p(t_2,0)}\right]\left[Q\sigma(T)e_p(\sigma(T),0) + M\right] + Qe_p(\sigma(T),0)(t_2 - t_1),\end{aligned} \tag{2.11}$$

which implies that $\Phi(X)$ is equicontinuous. Consequently, $\Phi : [v_0, w_0] \to K$ is compact.

Step 2. We assert that Φ is monotone increasing on $[v_0, w_0]$.

Suppose that $u, v \in [v_0, w_0]$ and $u \le v$. Then, $0 \le u(t) \le v(t) \le R$, $t \in [0, \sigma(T)]_{\mathbb{T}}$. By (H_1) and (H_2), we have

$$\begin{aligned}(\Phi u)(t) &= \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0) f(s,u(s))\Delta s + g(u(\sigma(T)))\right] \\ &\le \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0) f(s,v(s))\Delta s + g(v(\sigma(T)))\right] \\ &= (\Phi v)(t), \quad t \in [0,\sigma(T)]_{\mathbb{T}},\end{aligned} \tag{2.12}$$

which shows that $\Phi u \le \Phi v$.

Step 3. We prove that v_0 is a lower solution of Φ.

For any $t \in [0, \sigma(T)]_{\mathbb{T}}$, it is obvious that

$$(\Phi v_0)(t) = \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0) f(s,0)\Delta s + g(0)\right] \geq 0 = v_0(t), \tag{2.13}$$

which implies that $v_0 \leq \Phi v_0$.

Step 4. We show that w_0 is an upper solution of Φ.

In view of (H_1) and (H_2), we have

$$\begin{aligned}(\Phi w_0)(t) &= \int_0^t \frac{e_p(s,0)}{e_p(t,0)} f(s,R)\Delta s + \frac{g(R)}{e_p(t,0)} \\ &\leq \int_0^t f(s,R)\Delta s + g(R) \\ &\leq \int_0^{\sigma(T)} f(s,R)\Delta s + g(R) \\ &\leq R = w_0(t), \quad t \in [0,\sigma(T)]_{\mathbb{T}},\end{aligned} \tag{2.14}$$

which indicates that $\Phi w_0 \leq w_0$.

Step 5. We claim that the BVP (1.6) has positive solutions.

In fact, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as follows:

$$v_n = \Phi v_{n-1}, \quad w_n = \Phi w_{n-1}, \quad n = 1,2,3,\ldots, \tag{2.15}$$

then it follows from Theorem 1.1 that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0, \tag{2.16}$$

and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge to, respectively, v and $w \in [v_0, w_0]$, which are solutions of the BVP (1.6). Moreover, it follows from

$$\begin{aligned}(\Phi v_0)(\sigma(T)) &= \frac{1}{e_p(\sigma(T),0)}\left[\int_0^{\sigma(T)} e_p(s,0) f(s,0)\Delta s + g(0)\right] \\ &\geq \frac{1}{e_p(\sigma(T),0)}\int_0^{\sigma(T)} f(s,0)\Delta s \\ &> 0\end{aligned} \tag{2.17}$$

that

$$0 < (\Phi v_0)(\sigma(T)) \leq (\Phi v)(\sigma(T)) = v(\sigma(T)) \leq w(\sigma(T)), \tag{2.18}$$

which shows that v and w are positive solutions of the BVP (1.6). □

Example 2.2. Let $\mathbb{T} = [0,1] \cup [2,3]$. We consider the following BVP on $\mathbb{T}$:

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = \frac{t\sqrt{x(t)+3}}{9}, \quad t \in [0,3]_{\mathbb{T}},$$
$$x(0) = \frac{x^2(3)}{12}. \tag{2.19}$$

Since $f(t,x) = t\sqrt{x+3}/9$ and $g(x) = x^2/12$, if we choose $R = 6$, then all the conditions of Theorem 2.1 are fulfilled. So, it follows from Theorem 2.1 that the BVP (2.19) has positive solutions v and w. Furthermore, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as follows:

$$v_n = \Phi v_{n-1}, \quad w_n = \Phi w_{n-1}, \quad n = 1,2,3,\ldots, \tag{2.20}$$

where $v_0(t) \equiv 0$ and $w_0(t) \equiv 6$, then

$$\lim_{n\to\infty} v_n = v, \qquad \lim_{n\to\infty} w_n = w. \tag{2.21}$$

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