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Review Article

Solution and Attractivity for a Rational Recursive Sequence

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This paper is concerned with the behavior of solution of the nonlinear difference equation $x_{n+1} = ax_{n-1} + bx_nx_{n-1}/(cx_n + dx_{n-2})$, $n = 0, 1, \ldots$, where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we give specific form of the solution of four special cases of this equation.

1. Introduction

In this paper we deal with the behavior of the solution of the following difference equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_nx_{n-1}}{cx_n + dx_{n-2}}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we obtain the solution of some special cases of (1.1).

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \longrightarrow I, \tag{1.2}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [1].

Definition 1.1 (equilibrium point). A point $\overline{x} \in I$ is called an equilibrium point of (1.3) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.4}$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of (1.3), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2 (stability). (i) The equilibrium point $\overline{x}$ of (1.3) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta, \tag{1.5}$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \forall n \geq -k. \tag{1.6}$$

(ii) The equilibrium point $\overline{x}$ of (1.3) is locally asymptotically stable if $\overline{x}$ is locally stable solution of (1.3) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma, \tag{1.7}$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{1.8}$$

(iii) The equilibrium point $\overline{x}$ of (1.3) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{1.9}$$

(iv) The equilibrium point $\overline{x}$ of (1.3) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of (1.3).

(v) The equilibrium point $\overline{x}$ of (1.3) is unstable if $\overline{x}$ is not locally stable. The linearized equation of (1.3) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.10}$$

Theorem A (see [2]). *Assume that $p, q \in R$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|p| + |q| < 1, \tag{1.11}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \ldots. \tag{1.12}$$

Remark 1.3. Theorem A can be easily extended to a general linear equations of the form

$$x_{\mathrm{n}+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0, 1, \ldots, \tag{1.13}$$

where $p_1, p_2, \ldots, p_k \in R$ and $k \in \{1, 2, \ldots\}$. Then (1.13) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{1.14}$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{1.15}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B (see [1]). *Let $[a, b]$ be an interval of real numbers and assume that*

$$g : [a, b]^3 \longrightarrow [a, b], \tag{1.16}$$

is a continuous function satisfying the following properties:

(a) *$g(x, y, z)$ is nondecreasing in x and y in $[a, b]$ for each $z \in [a, b]$, and is nonincreasing in $z \in [a, b]$ for each x and y in $[a, b]$;*

(b) *if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$M = g(M, M, m), \qquad m = g(m, m, M), \tag{1.17}$$

then

$$m = M. \tag{1.18}$$

Then (1.15) *has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of* (1.15) *converges to $\overline{x}$.*

Definition 1.4 (periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.5 (Fibonacci sequence). The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \ldots\}$, that is, $F_m = F_{m-1} + F_{m-2}$, $m \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$ is called Fibonacci sequence.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. Some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations.

However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. From the known work, one can see that it is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [3–23] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Many researchers have investigated the behavior of the solution of difference equations, for example, Aloqeili [24] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}. \tag{1.19}$$

Amleh et al. [25] studied the dynamics of the difference equation

$$x_{n+1} = \frac{a + b x_{n-1}}{A + B x_{n-2}}. \tag{1.20}$$

Çinar [26, 27] got the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + a x_n x_{n-1}}. \tag{1.21}$$

In [28], Elabbasy et al. investigated the global stability and periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}. \tag{1.22}$$

Elabbasy et al. [29] investigated the global stability, boundedness, and periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}. \tag{1.23}$$

In [30], Ibrahim got the form of the solution of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}. \tag{1.24}$$

Karatas et al. [31] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}. \tag{1.25}$$

Yalçinkaya and Çinar [32] considered the dynamics of the difference equation

$$x_{n+1} = \frac{a x_{n-k}}{b + c x_n^p}. \tag{1.26}$$

Yang [33] investigated the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{x_{n-1} x_{n-2} + x_{n-3} + a}{x_{n-1} + x_{n-2} x_{n-3} + a}. \tag{1.27}$$

See also [1, 2, 30, 31, 34–40]. Other related results on rational difference equations can be found in [32, 33, 41–48].

2. Local Stability of (1.1)

In this section we investigate the local stability character of the solutions of (1.1). Equation (1.1) has a unique equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x} + d\overline{x}}, \tag{2.1}$$

or

$$\overline{x}^2 (1 - a)(c + d) = b\overline{x}^2, \tag{2.2}$$

if $(c + d)(1 - a) \neq b$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0, \infty)^3 \to (0, \infty)$ be a function defined by

$$f(u, v, w) = av + \frac{buv}{cu + dw}. \tag{2.3}$$

Therefore it follows that

$$f_u(u, v, w) = \frac{bdvw}{(cu + dw)^2}, \qquad f_v(u, v, w) = a + \frac{bu}{cu + dw}, \qquad f_w(u, v, w) = \frac{-bduv}{(cu + dw)^2}, \tag{2.4}$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x}) = \frac{bd}{(c+d)^2}, \quad f_v(\overline{x},\overline{x},\overline{x}) = a + \frac{b}{c+d}, \quad f_w(\overline{x},\overline{x},\overline{x}) = \frac{-bd}{(c+d)^2}. \tag{2.5}$$

The linearized equation of (1.1) about $\overline{x}$ is

$$y_{n+1} - \frac{bd}{(c+d)^2}y_n - \left(a + \frac{b}{c+d}\right)y_{n-1} + \frac{bd}{(c+d)^2}y_{n-1} = 0. \tag{2.6}$$

Theorem 2.1. *Assume that*

$$b(c+3d) < (1-a)(c+d)^2. \tag{2.7}$$

Then the equilibrium point of (1.1) *is locally asymptotically stable.*

Proof. It follows from Theorem A that (2.6) is asymptotically stable if

$$\left|\frac{bd}{(c+d)^2}\right| + \left|a + \frac{b}{c+d}\right| + \left|\frac{bd}{(c+d)^2}\right| < 1, \tag{2.8}$$

or

$$a + \frac{b}{c+d} + \frac{2bd}{(c+d)^2} < 1, \tag{2.9}$$

and so,

$$\frac{b(c+3d)}{(c+d)^2} < (1-a). \tag{2.10}$$

The proof is complete. □

3. Global Attractor of the Equilibrium Point of (1.1)

In this section we investigate the global attractivity character of solutions of (1.1).

Theorem 3.1. *The equilibrium point* $\overline{x}$ *of* (1.1) *is global attractor if* $c(1-a) \neq b$.

Proof. Let p, q be real numbers and assume that $g : [p,q]^3 \to [p,q]$ is a function defined by $g(u,v,w) = av + buv/(cu + dw)$, then we can easily see that the function $g(u,v,w)$ is increasing in u, v and decreasing in w.Suppose that (m, M) is a solution of the system

$$M = g(M,M,m), \qquad m = g(m,m,M). \tag{3.1}$$

Then from (1.1), we see that

$$M = aM + \frac{bM^2}{cM + dm}, \qquad m = am + \frac{bm^2}{cm + dM}, \tag{3.2}$$

or

$$M(1-a) = \frac{bM^2}{cM + dm}, \qquad m(1-a) = \frac{bm^2}{cm + dM}, \tag{3.3}$$

then

$$d(1-a)Mm + c(1-a)M^2 = bM^2, \qquad d(1-a)Mm + c(1-a)m^2 = bm^2, \tag{3.4}$$

subtracting, we obtain

$$c(1-a)\left(M^2 - m^2\right) = b\left(M^2 - m^2\right), \quad c(1-a) \neq b. \tag{3.5}$$

Thus

$$M = m. \tag{3.6}$$

It follows from Theorem B that $\overline{x}$ is a global attractor of (1.1), and then the proof is complete. □

4. Boundedness of Solutions of (1.1)

In this section we study the boundedness of solutions of (1.1).

Theorem 4.1. *Every solution of* (1.1) *is bounded if* $(a + b/c) < 1$.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (1.1). It follows from (1.1) that

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}} \leq ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n} = \left(a + \frac{b}{c}\right)x_{n-1}. \tag{4.1}$$

Then

$$x_{n+1} \leq x_{n-1} \quad \forall n \geq 0. \tag{4.2}$$

Then the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-2}, x_{-1}, x_0\}$. □

5. Special Cases of (1.1)

Our goal in this section is to find a specific form of the solutions of some special cases of (1.1) when a, b, c, and d are integers and give numerical examples of each case and draw it by using MATLAB 6.5.

5.1. On the Difference Equation $x_{n+1} = x_{n-1} + x_n x_{n-1}/(x_n + x_{n-2})$

In this subsection we study the following special case of (1.1):

$$x_{n+1} = x_{n-1} + \frac{x_n x_{n-1}}{x_n + x_{n-2}}, \tag{5.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.

Theorem 5.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (5.1). *Then for $n = 0, 1, 2, \ldots$*

$$x_{2n-1} = k\prod_{i=0}^{n-1}\left(\frac{F_{4i+3}h + F_{4i+2}r}{F_{4i+2}h + F_{4i+1}r}\right), \qquad x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{F_{4i+5}h + F_{4i+4}r}{F_{4i+4}h + F_{4i+3}r}\right), \tag{5.2}$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{F_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is,

$$\begin{gathered} x_{2n-3} = k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h + F_{4i+2}r}{F_{4i+2}h + F_{4i+1}r}\right), \qquad x_{2n-2} = h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h + F_{4i+4}r}{F_{4i+4}h + F_{4i+3}r}\right), \\ x_{2n-4} = h\prod_{i=0}^{n-3}\left(\frac{F_{4i+5}h + F_{4i+4}r}{F_{4i+4}h + F_{4i+3}r}\right). \end{gathered} \tag{5.3}$$

Now, it follows from (5.1) that

$$\begin{aligned} x_{2n-1} &= x_{2n-3} + \frac{x_{2n-2}x_{2n-3}}{x_{2n-2} + x_{2n-4}} \\ &= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h + F_{4i+2}r}{F_{4i+2}h + F_{4i+1}r}\right) \\ &\quad + \frac{h\prod_{i=0}^{n-2}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h+F_{4i+3}r))k\prod_{i=0}^{n-2}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h + F_{4i+1}r))}{h\prod_{i=0}^{n-2}((F_{4i+5}h + F_{4i+4}r)/(F_{4i+4}h + F_{4i+3}r)) + h\prod_{i=0}^{n-3}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h + F_{4i+3}r))} \\ &= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h + F_{4i+2}r}{F_{4i+2}h + F_{4i+1}r}\right) \\ &\quad + \frac{((F_{4n-3}h + F_{4n-4}r)/(F_{4n-4}h + F_{4n-5}r))k\prod_{i=0}^{n-2}((F_{4i+3}h + F_{4i+2}r)/(F_{4i+2}h + F_{4i+1}r))}{((F_{4n-3}h + F_{4n-4}r)/(F_{4n-4}h + F_{4n-5}r)) + 1} \end{aligned}$$

$$
\begin{aligned}
&= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h+F_{4i+2}r}{F_{4i+2}h+F_{4i+1}r}\right)+\frac{(F_{4n-3}h+F_{4n-4}r)k\prod_{i=0}^{n-2}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h+F_{4i+1}r))}{F_{4n-3}h+F_{4n-4}r+F_{4n-4}h+F_{4n-5}r}\\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h+F_{4i+2}r}{F_{4i+2}h+F_{4i+1}r}\right)+\frac{(F_{4n-3}h+F_{4n-4}r)k\prod_{i=0}^{n-2}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h+F_{4i+1}r))}{F_{4n-2}h+F_{4n-3}r}\\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h+F_{4i+2}r}{F_{4i+2}h+F_{4i+1}r}\right)\left(1+\frac{F_{4n-3}h+F_{4n-4}r}{F_{4n-2}h+F_{4n-3}r}\right)\\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{4i+3}h+F_{4i+2}r}{F_{4i+2}h+F_{4i+1}r}\right)\left(\frac{F_{4n-1}h+F_{4n-2}r}{F_{4n-2}h+F_{4n-3}r}\right).
\end{aligned}
\tag{5.4}
$$

Therefore

$$
x_{2n-1}=k\prod_{i=0}^{n-1}\left(\frac{F_{4i+3}h+F_{4i+2}r}{F_{4i+2}h+F_{4i+1}r}\right). \tag{5.5}
$$

Also, we see from (5.1) that

$$
\begin{aligned}
x_{2n} &= x_{2n-2}+\frac{x_{2n-1}x_{2n-2}}{x_{2n-1}+x_{2n-3}}\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)\\
&\quad+\frac{k\prod_{i=0}^{n-1}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h+F_{4i+1}r))h\prod_{i=0}^{n-2}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h+F_{4i+3}r))}{k\prod_{i=0}^{n-1}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h+F_{4i+1}r))+k\prod_{i=0}^{n-2}((F_{4i+3}h+F_{4i+2}r)/(F_{4i+2}h+F_{4i+1}r))}\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)\\
&\quad+\frac{((F_{4n-1}h+F_{4n-2}r)/(F_{4n-2}h+F_{4n-3}r))h\prod_{i=0}^{n-2}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h+F_{4i+3}r))}{(F_{4n-1}h+F_{4n-2}r/F_{4n-2}h+F_{4n-3}r)+1}\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)+\frac{(F_{4n-1}h+F_{4n-2}\mathrm{r})h\prod_{i=0}^{n-2}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h+F_{4i+3}r))}{F_{4n-1}h+F_{4n-2}r+F_{4n-2}h+F_{4n-3}r}\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)+\frac{(F_{4n-1}h+F_{4n-2}r)h\prod_{i=0}^{n-2}((F_{4i+5}h+F_{4i+4}r)/(F_{4i+4}h+F_{4i+3}r))}{F_{4n}h+F_{4n-1}r}\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)\left(1+\frac{F_{4n-1}h+F_{4n-2}r}{F_{4n}h+F_{4n-1}r}\right)\\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{4i+5}h+F_{4i+4}r}{F_{4i+4}h+F_{4i+3}r}\right)\left(\frac{F_{4n+1}h+F_{4n}r}{F_{4n}h+F_{4n-1}r}\right).
\end{aligned}
\tag{5.6}
$$

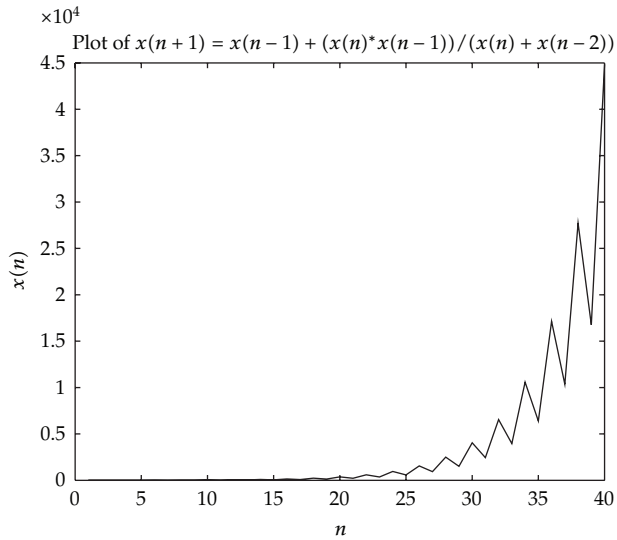

Figure 1

Thus

$$x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{F_{4i+5}h + F_{4i+4}r}{F_{4i+4}h + F_{4i+3}r}\right). \tag{5.7}$$

Hence, the proof is completed. □

For confirming the results of this section, we consider numerical example for $x_{-2} = 7, x_{-1} = 6,\ x_0 = 3$. (See Figure 1).

5.2. On the Difference Equation $x_{n+1} = x_{n-1} + x_n x_{n-1}/(x_n - x_{n-2})$

In this subsection we give a specific form of the solutions of the difference equation

$$x_{n+1} = x_{n-1} + \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \tag{5.8}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers with $x_{-2} \neq x_0$.

Theorem 5.2. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of* (5.8). *Then for* $n = 0, 1, 2, \ldots$

$$x_{2n-1} = k\prod_{i=0}^{n-1}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right), \qquad x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right), \tag{5.9}$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{F_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1, \quad n-2$. That is,

$$x_{2n-3} = k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right), \qquad x_{2n-2} = h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right),$$
$$x_{2n-4} = h\prod_{i=0}^{n-3}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right). \tag{5.10}$$

Now, it follows from (5.8) that

$$\begin{aligned}
x_{2n-1} &= x_{2n-3} + \frac{x_{2n-2}x_{2n-3}}{x_{2n-2} - x_{2n-4}} \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right) \\
&\quad + \frac{h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - \mathrm{F}_{2i}r))k\prod_{i=0}^{n-2}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r))}{h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r)) - h\prod_{i=0}^{n-3}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r))} \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right) \\
&\quad + \frac{((F_{2n}h - F_{2n-2}r)/(F_{2n-2}h - F_{2n-4}r))k\prod_{i=0}^{n-2}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r))}{(F_{2n}h - F_{2n-2}r/F_{2n-2}h - F_{2n-4}r) - 1} \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right) + \frac{(F_{2n}h - F_{2n-2}r)k\prod_{i=0}^{n-2}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r))}{F_{2n-1}h - F_{2n-3}r} \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right)\left(1 + \frac{F_{2n}h - F_{2n-2}r}{F_{2n-1}h - F_{2n-3}r}\right) \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right)\left(\frac{F_{2n-1}h - F_{2n-3}r + F_{2n}h - F_{2n-2}r}{F_{2n-1}h - F_{2n-3}r}\right) \\
&= k\prod_{i=0}^{n-2}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right)\left(\frac{F_{2n+1}h - F_{2n-1}r}{F_{2n-1}h - F_{2n-3}r}\right).
\end{aligned} \tag{5.11}$$

Therefore

$$x_{2n-1} = k\prod_{i=0}^{n-1}\left(\frac{F_{2i+3}h - F_{2i+1}r}{F_{2i+1}h - F_{2i-1}r}\right). \tag{5.12}$$

Also, we see from (5.8) that

$$\begin{aligned}
x_{2n} &= x_{2n-2} + \frac{x_{2n-1}x_{2n-2}}{x_{2n-1} - x_{2n-3}} \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right) \\
&\quad + \frac{k\prod_{i=0}^{n-1}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r))h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r))}{k\prod_{i=0}^{n-1}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r)) - k\prod_{i=0}^{n-2}((F_{2i+3}h - F_{2i+1}r)/(F_{2i+1}h - F_{2i-1}r))} \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right) \\
&\quad + \frac{((F_{2n+1}h - F_{2n-1}r)/(F_{2n-1}h - F_{2n-3}r))h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r))}{((F_{2n+1}h - F_{2n-1}r)/(F_{2n-1}h - F_{2n-3}r)) - 1} \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right) + \frac{(F_{2n+1}h - F_{2n-1}r)h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r))}{F_{2n+1}h - F_{2n-1}r - F_{2n-1}h + F_{2n-3}r} \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right) + \frac{(F_{2n+1}h - F_{2n-1}r)h\prod_{i=0}^{n-2}((F_{2i+4}h - F_{2i+2}r)/(F_{2i+2}h - F_{2i}r))}{F_{2n}h - F_{2n-2}\mathrm{r}} \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right)\left(1 + \frac{F_{2n+1}h - F_{2n-1}r}{F_{2n}h - F_{2n-2}r}\right) \\
&= h\prod_{i=0}^{n-2}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right)\left(\frac{F_{2n+2}h - F_{2n}r}{F_{2n}h - F_{2n-2}r}\right).
\end{aligned} \tag{5.13}$$

Thus

$$x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{F_{2i+4}h - F_{2i+2}r}{F_{2i+2}h - F_{2i}r}\right). \tag{5.14}$$

Hence, the proof is completed. □

Assume that $x_{-2} = 0.7, x_{-1} = 1.6,\ x_0 = 13$. (See Figure 2), and for $x_{-2} = 9, x_{-1} = 5,\ x_0 = 2$. (See Figure 3).

The following cases can be treated similarly.

5.3. On the Difference Equation $x_{n+1} = x_{n-1} - x_n x_{n-1}/(x_n + x_{n-2})$

In this subsection we obtain the solution of the following difference equation

$$x_{n+1} = x_{n-1} - \frac{x_n x_{n-1}}{x_n + x_{n-2}}, \tag{5.15}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

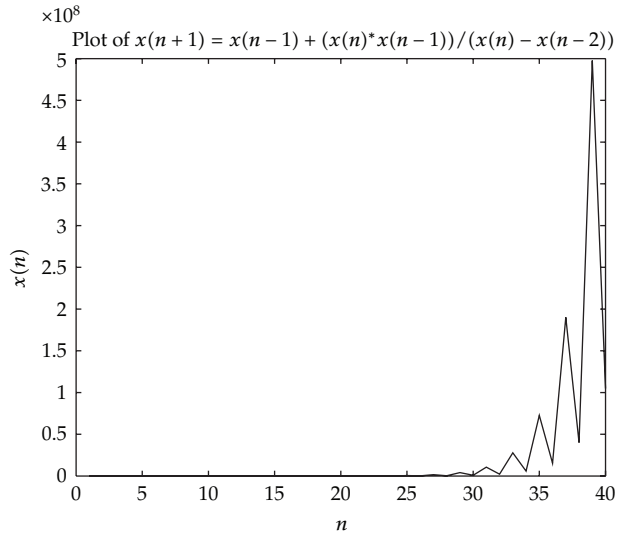

Figure 2

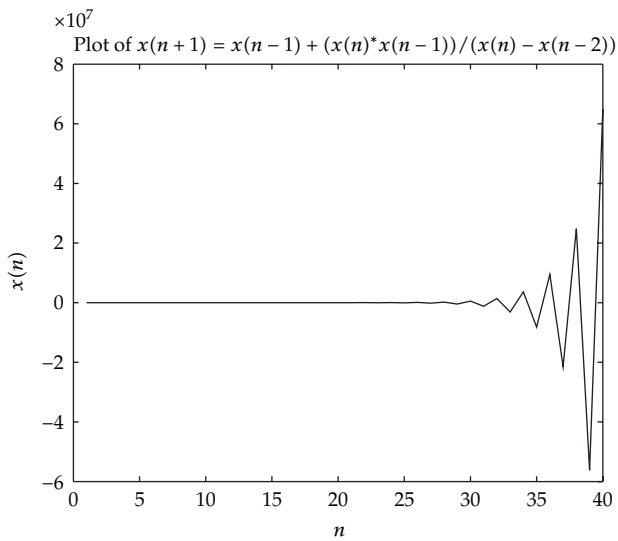

Figure 3

Theorem 5.3. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of* (5.15). *Then for* $n = 0, 1, 2, \ldots$

$$x_{2n-1} = k\prod_{i=0}^{n-1}\left(\frac{F_{2i}h + F_{2i+1}r}{F_{2i+1}h + F_{2i+2}r}\right), \qquad x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{F_{2i+1}h + F_{2i+2}r}{F_{2i+2}h + F_{2i+3}r}\right), \tag{5.16}$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{F_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Figure 4 shows the solution when $x_{-2} = 3,\ x_{-1} = 7,\ x_0 = 12$.

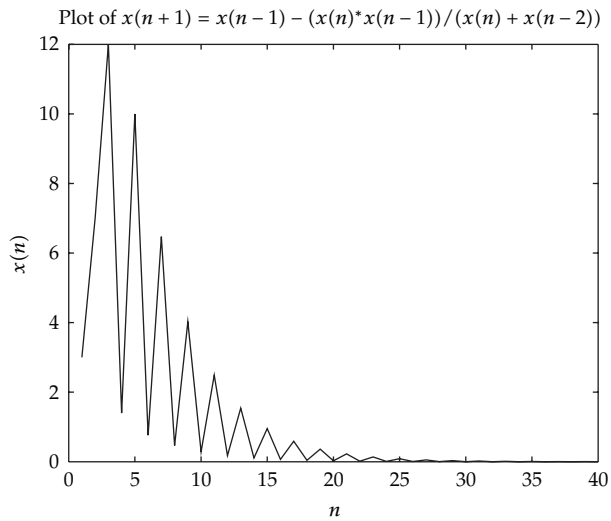

Figure 4

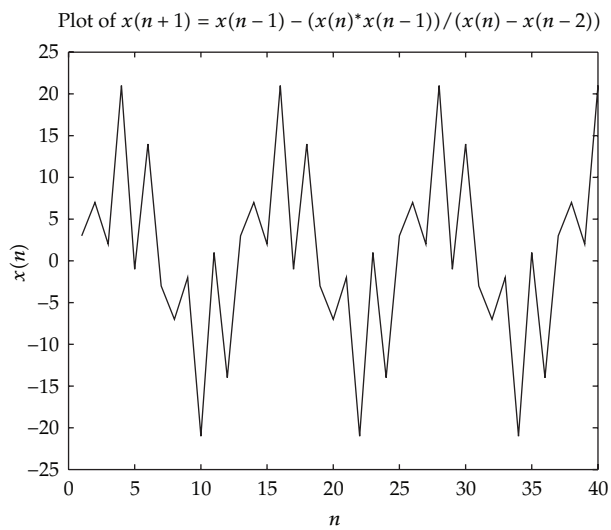

Figure 5

5.4. On the Difference Equation $x_{n+1} = x_{n-1} - x_n x_{n-1}/(x_n - x_{n-2})$

In this subsection we give the solution of the following special case of (1.1)

$$x_{n+1} = x_{n-1} - \frac{x_n x_{n-1}}{x_n - x_{n-2}} \tag{5.17}$$

where the initial conditions $x_{-2},\ x_{-1}, x_0$ are arbitrary real numbers. with $x_{-2} \neq x_0,\ x_{-2},\ x_{-1},\ x_0 \neq 0$.

Theorem 5.4. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (5.17). *Then every solution of* (5.17) *is periodic with period 12. Moreover $\{x_n\}_{n=-2}^{\infty}$ takes the form*

$$\left\{r, k, h, \frac{kr}{r-h}, h-r, \frac{hk}{r-h}, -r, -k, -h, \frac{kr}{h-r}, -(h-r), \frac{hk}{h-r}, r, k, h, \ldots\right\} \tag{5.18}$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h$.

Figure 5 shows the solution when $x_{-2} = 3,\ x_{-1} = 7,\ x_0 = 2$.

6. Conclusion

This paper discussed global stability, boundedness, and the solutions of some special cases of (1.1). In Section 2 we proved when $b(c+3d) < (1-a)(c+d)^2$, (1.1) local stability. In Section 3 we showed that the unique equilibrium of (1.1) is globally asymptotically stable if $c(1-a) \neq b$. In Section 4 we proved that the solution of (1.1) is bounded if $(a + b/c) < 1$. In Section 5 we gave the form of the solution of four special cases of (1.1) and gave numerical examples of each case and drew them by using Matlab 6.5.

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