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Research Article

Integral Formulae of Bernoulli Polynomials

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Recently, some interesting and new identities are introduced in (Hwang et al., Communicated). From these identities, we derive some new and interesting integral formulae for the Bernoulli polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by generating functions as follows:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

(see [1–11]). In the special case, $x=0$, $B_n(0)=B_n$ are called the nth Bernoulli numbers. The Euler polynomials are also defined by

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \tag{1.2}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–11]). From (1.1) and (1.2),

we can easily derive the following equation:

$$\begin{aligned}\frac{t}{e^t-1}e^{xt} &= \frac{t}{2}\left(\frac{2e^{xt}}{e^t+1}\right)+\left(\frac{t}{e^t-1}\right)\left(\frac{2e^{xt}}{e^t+1}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_kE_{n-k}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{1.3}$$

By (1.1) and (1.3), we get

$$B_n(x) = \sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_kE_{n-k}(x), \quad (n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}). \tag{1.4}$$

From (1.1), we have

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}B_lx^{n-l}. \tag{1.5}$$

Thus, by (1.5), we get

$$\frac{d}{dx}B_n(x) = n\sum_{l=0}^{n-1}\binom{n-1}{l}B_lx^{n-1-l} = nB_{n-1}(x). \tag{1.6}$$

It is known that $E_n(0)=E_n$ are called the nth Euler numbers (see [7]). The Euler polynomials are also given by

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n}\binom{n}{l}E_lx^{n-l}, \tag{1.7}$$

(see [6]). From (1.7), we can derive the following equation:

$$\frac{d}{dx}E_n(x) = n\sum_{l=0}^{n-1}\binom{n}{l}E_lx^{n-1-l} = nE_{n-1}(x). \tag{1.8}$$

By the definition of Bernoulli and Euler numbers, we get the following recurrence formulae:

$$E_0=1, \qquad E_n(1)+E_n=2\delta_{0,n}, \qquad B_0=1, \qquad B_n(1)-B_n=\delta_{1,n}, \tag{1.9}$$

where $\delta_{n,k}$ is the kronecker symbol (see [5]). From (1.6), (1.8), and (1.9), we note that

$$\int_0^1 B_n(x)dx = \frac{\delta_{0,n}}{n+1}, \qquad \int_0^1 E_n(x)dx = -\frac{2E_{n+1}}{n+1}, \tag{1.10}$$

where $n \in \mathbb{Z}_+$. The following identity is known in [5]:

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}\frac{(-1)^{j+l}}{j+l+1} \\
&\quad+\sum_{j=0}^{m}\sum_{l=0}^{k}\left((a+b+1)^{m-j}(a+1)^{k-l}-(a+b)^{m-j}a^{k-l}\right)\binom{m}{j}\binom{k}{l}\frac{B_{j+l+1}(x)}{j+l+1} \\
&\quad=(x+a)^{k}(x+a+b)^{m}, \quad \text{where } a,b\in\mathbb{Z}.
\end{aligned} \tag{1.11}$$

From the identities of Bernoulli polynomials, we derive some new and interesting integral formulae of an arithmetical nature on the Bernoulli polynomials.

2. Integral Formulae of Bernoulli Polynomials

From (1.1) and (1.2), we note that

$$\begin{aligned}
\frac{2}{e^t+1}e^{xt} &= \frac{1}{t}\left(\frac{2(e^t-1)}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right) \\
&= \frac{1}{t}\left(2-2\frac{2}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right) \\
&= \frac{1}{t}\left(-\sum_{l=1}^{\infty}\frac{2E_l}{l!}t^l\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right) \\
&= -2\left(\sum_{l=0}^{\infty}\frac{E_{l+1}}{l+1}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right) \\
&= -2\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{E_{l+1}}{l+1}\binom{n}{l}B_{n-l}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

Therefore, by (1.2) and (2.1), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$E_n(x) = -2\sum_{l=0}^{n}\binom{n}{l}\frac{E_{l+1}}{l+1}B_{n-l}(x). \tag{2.2}$$

Let us take the definite integral from 0 to 1 on both sides of (1.4): for $n \geq 2$,

$$0 = -2\sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_k\frac{E_{n-k+1}}{n-k+1} = -2B_nE_1 - 2\sum_{\substack{k=0\\k\neq 1}}^{n-1}\binom{n}{k}B_k\frac{E_{n-k+1}}{n-k+1}. \tag{2.3}$$

By (2.3), we get

$$B_n = 2\sum_{\substack{k=0\\k\neq 1}}^{n-1}\binom{n}{k}\frac{B_k E_{n-k+1}}{n-k+1}. \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, with $n \geq 2$, one has*

$$B_n = 2\sum_{\substack{k=0\\k\neq 1}}^{n-1}\binom{n}{k}\frac{B_k E_{n-k+1}}{n-k+1}. \tag{2.5}$$

Let us take $k = m$, $a = 0$, and $b = -2$ in (1.11). Then we have

$$\begin{aligned}\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{(-1)^{j+l}}{j+l+1} &+ \sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{B_{j+l+1}(x)}{j+l+1}\\ &- \sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{B_{j+m+1}(x)}{j+m+1} = x^m(x-2)^m.\end{aligned} \tag{2.6}$$

It is easy to show that

$$\begin{aligned}\int_0^1 x^m(x-2)^m dx &= 2\int_0^{1/2}(2t-2)^m(2t)^m dt\\ &= (-1)^m 2^{2m}\left(2\int_0^{1/2} t^m(1-t)^m dt\right) = (-1)^m 2^{2m}\int_0^1 t^m(1-t)^m dt\\ &= (-1)^m 2^{2m}\frac{m!m!}{(2m+1)!} = \frac{(-1)^m 2^{2m}}{2m+1}\frac{1}{\binom{2m}{m}}.\end{aligned} \tag{2.7}$$

Let us consider the integral from 0 to 1 in (2.6):

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{j}\binom{m}{l}\frac{1}{j+l+1} = \frac{(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}} \quad (m \in \mathbb{N}). \tag{2.8}$$

By (2.6) and (2.8), we get

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{B_{j+l+1}}{j+l+1} \\
&\quad = 2\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{1}{j+m+1}\sum_{\substack{k=0\\k\neq 1}}^{j+m}\binom{j+m+1}{k}\frac{B_k E_{j+m+2-k}}{j+m+2-k} \\
&\qquad + \frac{(-1)^{m+1}2^{2m}}{2m+1}\frac{1}{\binom{2m}{m}}, \quad \text{for } m\in\mathbb{N}.
\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. *For $m\in\mathbb{N}$, one has*

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{B_{j+l+1}}{j+l+1} \\
&\quad = 2\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{1}{j+m+1}\sum_{\substack{k=0\\k\neq 1}}^{j+m}\binom{j+m+1}{k}\frac{B_k E_{j+m+2-k}}{j+m+2-k} \\
&\qquad + \frac{(-1)^{m+1}2^{2m}}{2m+1}\frac{1}{\binom{2m}{m}}.
\end{aligned} \tag{2.10}$$

Lemma 2.4. *Let $a,b\in\mathbb{Z}$. For $m,k\in\mathbb{Z}_+$, one has*

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x) \\
&\qquad + \sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x) = 2(x+a+b)^m(x+a)^k,
\end{aligned} \tag{2.11}$$

(see [5]).

Let us take $k=m$, $a=1$, $b=-2$ in Lemma 2.4. Then we have

$$\sum_{l=0}^{m}2^{m-l}\binom{m}{l}E_{m+l}(x) + \sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}E_{j+l}(x) = 2\left(x^2-1\right)^m. \tag{2.12}$$

Taking integral from 0 to 1 in (2.12), we get

$$-2\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{E_{m+l+1}}{m+l+1} - 2\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1} = 2\int_0^1\left(x^2-1\right)^m dx. \tag{2.13}$$

It is easy to show that

$$\int_0^1 \left(x^2-1\right)^m dx = (-1)^m \prod_{k=1}^{m}\left(\frac{2k}{2k+1}\right) = \frac{(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}}. \tag{2.14}$$

Thus, by (2.13) and (2.14), we get

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1} = -\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{E_{m+l+1}}{m+l+1} + \frac{(-1)^{m+1}2^{2m}}{(2m+1)\binom{2m}{m}}. \tag{2.15}$$

Therefore, by (2.2) and (2.15), we obtain the following theorem.

Theorem 2.5. *For $m \in \mathbb{Z}_+$, one has*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1} + \frac{(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}}\\ &\quad = 2\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{1}{m+l+1}\sum_{k=0}^{m+l+1}\binom{m+l+1}{k}\frac{E_{k+1}}{k+1}B_{m+l+1-k}.\end{aligned} \tag{2.16}$$

3. p-Adic Integral on $\mathbb{Z}_p$ Associated with Bernoulli and Euler Numbers

Let p be a fixed odd prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x), \tag{3.1}$$

(see [8]). Thus, by (3.1), we get

$$\int_{\mathbb{Z}_p} f_1(x)d\mu(x) = \int_{\mathbb{Z}_p} f(x)d\mu(x) + f'(0), \tag{3.2}$$

where $f_1(x) = f(x+1)$, and $f'(0) = df(x)/dx|_{x=0}$. Let us take $f(y) = e^{t(x+y)}$. Then we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)}d\mu(y) = \frac{t}{e^t-1}e^{tx} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{3.3}$$

From (3.3), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \qquad \int_{\mathbb{Z}_p} y^n d\mu(y) = B_n. \tag{3.4}$$

From (1.2), we can derive the following integral equation:

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \quad (n \in \mathbb{N}). \tag{3.5}$$

Thus, from (3.4) and (3.5), we get

$$\int_{\mathbb{Z}_p} (x+n)^m d\mu(x) = \int_{\mathbb{Z}_p} x^m d\mu(x) + m\sum_{i=0}^{n-1} i^{m-1}. \tag{3.6}$$

From (3.6), we have

$$B_m(n) - B_m = m\sum_{i=0}^{n-1} i^{m-1} \quad (m \in \mathbb{Z}_+). \tag{3.7}$$

The fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows [6, 7]:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)(-1)^x. \tag{3.8}$$

Let $f_1(x) = f(x+1)$. Then we have

$$\begin{aligned} I_{-1}(f_1) &= -I_{-1}(f) + 2f(0), \\ I_{-1}(f_2) &= -I_{-1}(f_1) + 2f_1(0) = -I_{-1}(f_1) + 2f(1) \\ &= (-1)^2 I_{-1}(f) + 2(-1)^{2-1} f(0) + 2f(1). \end{aligned} \tag{3.9}$$

Continuing this process, we obtain the following equation:

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l), \quad \text{where } f_n(x) = f(x+n). \tag{3.10}$$

Thus, by (3.10), we have

$$\int_{\mathbb{Z}_p} (x+n)^m d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} l^m. \tag{3.11}$$

Let us take $f(y) = e^{t(x+y)}$. By (3.9), we get

$$\int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{3.12}$$

From (3.2), we have the Witt's formula for the nth Euler polynomials and numbers as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = E_n, \quad \text{where } n \in \mathbb{Z}_+. \tag{3.13}$$

By (3.11) and (3.13), we get

$$E_m(n) = (-1)^n \left(E_m + 2\sum_{l=0}^{n-1} (-1)^{l-1} l^m \right), \quad (m \in \mathbb{Z}_+, n \in \mathbb{N}). \tag{3.14}$$

Let us consider the following p-adic integral on $\mathbb{Z}_p$:

$$\begin{aligned} K_1 &= \int_{\mathbb{Z}_p} B_n(x) d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l. \end{aligned} \tag{3.15}$$

From (1.4) and (3.15), we have

$$\begin{aligned} K_1 &= \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k \sum_{l=0}^{n-k} E_{n-k-l} \binom{n-k}{l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{\substack{k=0 \\ k\neq 1}}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} B_k B_l E_{n-k-l}. \end{aligned} \tag{3.16}$$

Therefore, by (3.15) and (3.16), we obtain the following theorem.

Theorem 3.1. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} B_k B_l E_{n-k-l}. \tag{3.17}$$

Now, we set

$$K_2 = \int_{\mathbb{Z}_p} B_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} E_l. \tag{3.18}$$

By (1.4), we get

$$\begin{aligned} K_2 &= \sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_k\sum_{l=0}^{n-k}E_{n-k-l}\binom{n-k}{l}\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x) \\ &= \sum_{\substack{k=0\\k\neq 1}}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}B_kE_{n-k-l}E_l. \end{aligned} \tag{3.19}$$

Therefore, by (3.18) and (3.19), we obtain the following theorem.

Theorem 3.2. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l = \sum_{\substack{k=0\\k\neq 1}}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}B_kE_{n-k-l}E_l. \tag{3.20}$$

Let us consider the following integral on $\mathbb{Z}_p$:

$$K_3 = \int_{\mathbb{Z}_p}E_n(x)d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}E_l. \tag{3.21}$$

From (2.2), we have

$$\begin{aligned} K_3 &= -2\sum_{l=0}^{n}\frac{E_{l+1}}{l+1}\binom{n}{l}\sum_{k=0}^{n-l}\binom{n-l}{k}B_{n-l-k}\int_{\mathbb{Z}_p}x^k d\mu_{-1}(x) \\ &= -2\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{E_{l+1}}{l+1}B_{n-l-k}E_k. \end{aligned} \tag{3.22}$$

Therefore, by (3.21) and (3.22), we obtain the following theorem.

Theorem 3.3. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}E_{n-l}E_l = -2\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{E_{l+1}}{l+1}E_kB_{n-l-k}. \tag{3.23}$$

Now, we set

$$K_4 = \int_{\mathbb{Z}_p}E_n(x)d\mu(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_l. \tag{3.24}$$

By (2.2), we get

$$K_4 = -2\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{E_{l+1}}{l+1}B_{n-l-k}B_k. \tag{3.25}$$

Therefore, by (3.24) and (3.25), we obtain the following corollary.

Corollary 3.4. *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}E_lB_l = -2\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{E_{l+1}}{l+1}B_{n-l-k}B_k. \tag{3.26}$$

Let us assume that $a, b, c, d \in \mathbb{Z}$. From Lemma 2.4 and (3.13), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}((a+b+1)+(x+y))^m((a+1)+(x+y))^k d\mu_{-1}(y) \\
&\quad + \int_{\mathbb{Z}_p}((a+b)+(x+y))^m((a+(x+y))^k d\mu_{-1}(y) \\
&\quad = 2(x+a+b)^m(x+a)^k.
\end{aligned} \tag{3.27}$$

By (3.27), we get

$$\begin{aligned}
2(x+a+b)^m(x+a)^k &= \int_{\mathbb{Z}_p}((a+b-c+1)+(x+c+y))^m((a-c+1)+(x+c+y))^k d\mu_{-1}(y) \\
&\quad + \int_{\mathbb{Z}_p}((a+b-d)+(x+y+d))^m((a-d)+(x+y+d))^k d\mu_{-1}(y).
\end{aligned} \tag{3.28}$$

Thus, by (3.28) and (3.13), we obtain the following lemma (see [5]).

Lemma 3.5. *Let $a, b, c, d \in \mathbb{Z}$. For $m, k \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-c+1)^{m-j}(a-c+1)^{k-l}E_{j+l}(x+c) \\
&\quad + \sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-d)^{m-j}(a-d)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x+d) = 2(x+a+b)^m(x+a)^k.
\end{aligned} \tag{3.29}$$

Let us consider the formula in Lemma 3.5 with $d = c - 1$. Then we have

$$\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l}\left(E_{j+l}(x+c)+E_{j+l}(x+c-1)\right) = 2(x+a+b)^{m}(x+a)^{k}. \tag{3.30}$$

Taking $\int_{\mathbb{Z}_p} d\mu(x)$ on both sides of (3.30),

$$\begin{aligned}
\text{LHS} &= \sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s} \\
&\quad\times E_{j+l-s}\int_{\mathbb{Z}_p}\left((x+c)^{s}+(x+c-1)^{s}\right)d\mu(x) \\
&= 2\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s} \\
&\quad\times E_{j+l-s}B_{s}(c-1)+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l} \\
&\quad\times (j+l)E_{j+l-1}(c-1).
\end{aligned} \tag{3.31}$$

By the same method, we get

$$\begin{aligned}
\text{RHS} &= 2\sum_{s=0}^{m}\binom{m}{s}b^{m-s}\int_{\mathbb{Z}_p}(x+a)^{s+k}d\mu(x) \\
&= 2\sum_{s=0}^{m}\binom{m}{s}b^{m-s}B_{s+k}(a).
\end{aligned} \tag{3.32}$$

Therefore, by (3.31) and (3.32), we obtain the following proposition.

Proposition 3.6. *Let $a, b, c \in \mathbb{Z}$. Then one has*

$$\begin{aligned}
&2\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s}E_{j+l-s} \\
&\quad\times B_{s}(c-1)+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\binom{m}{j}\binom{k}{l}(j+l)E_{j+l-1}(c-1) \\
&= 2\sum_{s=0}^{m}\binom{m}{s}b^{m-s}B_{s+k}(a).
\end{aligned} \tag{3.33}$$

Replacing c by $c+1$, we have

$$\begin{aligned}
&2\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s}E_{j+l-s}B_s(c)\\
&\quad+\sum_{j=0}^{m}\sum_{l=0}^{k}(j+l)(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l-s}(c)\\
&\quad=2\sum_{s=0}^{m}\binom{m}{s}b^{m-s}B_{s+k}(a).
\end{aligned}\tag{3.34}$$

From (3.4) and (3.7), we derive some identity for the first term of the LHS of (3.34). The first term of the LHS of (3.34)

$$\begin{aligned}
&=2\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s}E_{j+l-s}\\
&\quad\times\left(B_s+s\sum_{i=0}^{c-1}i^{s-1}\right)\\
&=2\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}\sum_{s=0}^{j+l}\binom{j+l}{s}E_{j+l-s}B_s\\
&\quad+2\sum_{i=0}^{c-1}\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}(j+l)(-1)^0\\
&\quad\times\left(E_{j+l-1}+2\sum_{e=0}^{i-1}(-1)^{e-1}e^{j+l-1}\right)\\
&=2\sum_{j=0}^{m}\sum_{l=0}^{k}\sum_{s=0}^{j+l}\binom{m}{j}\binom{k}{l}\binom{j+l}{s}(a+b-c)^{m-j}(a-c)^{k-l}\\
&\quad\times E_{j+l-s}B_s+2m\sum_{j=0}^{m-1}\sum_{l=0}^{k}\binom{m-1}{j}\binom{k}{l}(a+b-c)^{m-1-j}(a-c)^{k-l}E_{j+l}\delta_{c\equiv1}\\
&\quad+2k\sum_{j=0}^{m}\sum_{l=0}^{k-1}\binom{m}{j}\binom{k-1}{l}(a+b-c)^{m-j}(a-c)^{k-1-l}E_{j+l}\delta_{c\equiv1}\\
&\quad+4m\sum_{e=0}^{c-2}(a+b-c+e)^{m-1}(a-c+e)^{k}\delta_{c\equiv e}\\
&\quad+4k\sum_{e=0}^{c-2}(a+b-c+e)^{m}(a-c+e)^{k-1}\delta_{c\equiv e},
\end{aligned}\tag{3.35}$$

where

$$\delta_{c\equiv k} = \begin{cases} 1 & \text{if } c \equiv k (\text{mod } 2), \\ 0 & \text{if } c \neq k (\text{mod } 2). \end{cases} \tag{3.36}$$

The second term of the LHS of (3.34)

$$\begin{aligned}
&= \sum_{j=0}^{m}\sum_{l=0}^{k}(j+l)(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}(-1)^{c}\left(E_{j+l-1}+2\sum_{i=0}^{c-1}(-1)^{i-1}i^{j+l-1}\right) \\
&= (-1)^{c}\sum_{j=0}^{m}\sum_{l=0}^{k}(j+l)(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l-1} \\
&\quad + 2(-1)^{c}\sum_{i=0}^{c-1}(-1)^{i-1}m(a+b-c+i)^{m-1}(a-c+i)^{k} \\
&\quad + 2(-1)^{c}\sum_{i=0}^{c-1}(-1)^{i-1}(a+b-c+i)^{m}k(a-c+i)^{k-1} \\
&= (-1)^{c}\sum_{j=0}^{m}\sum_{l=0}^{k}(j+l)(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l-1} \\
&\quad + 2m(-1)^{c}\sum_{i=0}^{c-1}(-1)^{i-1}(a+b-c+i)^{m-1}(a-c+i)^{k} \\
&\quad + 2k(-1)^{c}\sum_{i=0}^{c-1}(-1)^{i-1}(a+b-c+i)^{m}(\mathrm{a}-c+i)^{k-1}.
\end{aligned} \tag{3.37}$$

Therefore, by (3.34), (3.35), and (3.37), we obtain the following theorem.

Theorem 3.7. *Let $a, b, c \in \mathbb{Z}$ with $c \geq 1$. Then one has*

$$\begin{aligned}
&2\sum_{j=0}^{m}\sum_{l=0}^{k}\sum_{s=0}^{j+l}\binom{m}{j}\binom{k}{l}\binom{j+l}{s}(a+b-c)^{m-j}(a-c)^{k-l}E_{j+l-s}B_{s} \\
&\quad + 2m\delta_{c\equiv 1}\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m-1}{j}\binom{k}{l}(a+b-c)^{m-1-j}(a-c)^{k-l}E_{j+l} \\
&\quad + 2k\delta_{c\equiv 1}\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k-1}{l}(a+b-c)^{m-j}(a-c)^{k-1-l}E_{j+l} \\
&\quad + (-1)^{c}\sum_{j=0}^{m}\sum_{l=0}^{k}(j+l)(a+b-c)^{m-j}(a-c)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l-1}
\end{aligned}$$

$$
\begin{aligned}
&+ 2m\sum_{e=0}^{c-1}(a+b-c+e)^{m-1}(a-c+e)^{k} \\
&+ 2k\sum_{e=0}^{c-1}(a+b-c+e)^{m}(a-c+e)^{k-1} \\
&= 2\sum_{s=0}^{m}\binom{m}{s}b^{m-s}B_{s+k}(a),
\end{aligned}
\tag{3.38}
$$

where

$$
\delta_{c\equiv k} = \begin{cases} 1 & \text{if } c \equiv k (\text{mod } 2), \\ 0 & \text{if } c \not\equiv k (\text{mod } 2). \end{cases}
\tag{3.39}
$$

Remark 3.8. Here, we note that

$$
\begin{aligned}
&4m\sum_{e=0}^{c-2}(a+b-c+e)^{m-1}(a-c+e)^{k}\delta_{c\equiv e} + 2m(-1)^{c}\sum_{i=0}^{c-1}(-1)^{i-1}(a+b-c+i)^{m-1}(a-c+i)^{k} \\
&\quad = 2m\sum_{e=0}^{c-1}(a+b-c+e)^{m-1}(a-c+e)^{k}.
\end{aligned}
\tag{3.40}
$$

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References

[1] A. Bayad and T. Kim, "Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 2, pp. 247–253, 2010.

[2] D. Ding and J. Yang, "Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 1, pp. 7–21, 2010.

[3] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, "On the higher-order w-q-Genocchi numbers," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 39–57, 2009.

[4] K.-W. Hwang, D. V. Dolgy, T. Kim, and S. H. Lee, "On the higher-order q-Euler numbers and polynomials with weight α," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 354329, 12 pages, 2011.

[5] K. W. Hwang, D. V. Dolgy, D. S. Kim, T. Kim, and S. H. Lee, "Some theorem on Bernoulli and Euler numbers," (Communicated).

[6] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.

[7] T. Kim, "New approach to q-Euler polynomials of higher order," *Russian Journal of Mathematical Physics*, vol. 17, no. 2, pp. 218–225, 2010.

[8] T. Kim, "Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.

[9] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on q-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.

[10] C. S. Ryoo, "Some identities of the twisted q-Euler numbers and polynomials associated with q-Bernstein polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 14, no. 2, pp. 239–248, 2011.

[11] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.