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Research Article

Application of Multistep Generalized Differential Transform Method for the Solutions of the Fractional-Order Chua's System

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We numerically investigate the dynamical behavior of the fractional-order Chua's system. By utilizing the multistep generalized differential transform method (MSGDTM), we find that the fractional-order Chua's system with "effective dimension" less than three can exhibit chaos as well as other nonlinear behavior. Numerical results are presented graphically and reveal that the multistep generalized differential transform method is an effective and convenient method to solve similar nonlinear problems in fractional calculus.

1. Introduction

The characterization of real dynamical systems using fractional-order dynamical models has proved to be superior to the traditional calculus [1]. Many generalized fundamentals were extracted and were reduced to their known responses when the fractional orders converge to integer values. Fractional derivatives provide an excellent instrument to describe memory and hereditary properties of various materials and processes. Fractional differentiation and integration operators are used to model problems in astrophysics [2–5], chemical physics, signal processing, systems identification, control and robotics [6, 7], and many other areas [8–15].

It is well known that chaos cannot occur in continuous systems of total order less than three. This assertion is based on the usual concepts of order, such as the number of states in a system or the total number of separate differentiations or integrations in the system. The model of system can be rearranged to three single differential equations, where the equations contain the noninteger (fractional) order derivative. The total order of system is changed from 3 to the sum of each particular order. Petrás [16] has studied the dynamics of fractional-order

Chua's system. The key finding of his study is that chaotic behavior exists in the fractional-order Chua's system with total order less than three. Hartley et al. [17] introduced a particular form of the fractional-order Chua's system which is known as the Chua-Hartley's system. This system is different from the usual Chua's system in that the piecewise-linear nonlinearity is replaced by an appropriate cubic nonlinearity which yields very similar behavior. They found that the lowest value of the total order to have chaos in this system is 2.7.

In this paper, we intend to obtain the approximate solution of the fractional-order Chua's system and Chua-Hartley's system via the multistep generalized differential transform method (MSGDTM). This method is only a simple modification of the generalized differential transform method (GDTM) [18–21], in which it is treated as an algorithm in a sequence of small intervals (i.e., time step) for finding accurate approximate solutions to the corresponding systems. The approximate solutions obtained by using GDTM are valid only for a short time. The ones obtained by using the MSGDTM [22–24] are more valid and accurate during a long time and are in good agreement with the classical Runge-Kutta method numerical solution when the order of the derivative is one.

This paper is organized as follows. In Section 2, we present some necessary definitions and notations related to fractional calculus. In Section 3, the proposed method is described. In Section 4, the method is applied to the fractional-order Chua's system and the Chua-Hartley's system. In Section 5, the numerical simulations are presented graphically. Finally, the conclusions are given in Section 6.

2. Fractional Calculus

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [1–5].

Definition 2.1. A function $f(x)$ $(x > 0)$ is said to be in the space C_α $(\alpha \in R)$ if it can be written as $f(x) = x^p f_1(x)$ for some $p > \alpha$ where $f_1(x)$ is continuous in $[0, \infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in \mathbf{N}$.

Definition 2.2. The Riemann-Liouville integral operator of order $\alpha > 0$ with $a \geq 0$ is defined as

$$\begin{gathered}
(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \\
\left(J_a^0 f\right)(x) = f(x).
\end{gathered} \tag{2.1}$$

Properties of the operator can be found in [1–5]. We only need here the following: for $f \in C_\alpha$, $\alpha, \beta > 0$, $a \geq 0$, $c \in R$ and $\gamma > -1$, we have

$$\begin{gathered}
\left(J_a^\alpha J_a^\beta f\right)(x) = \left(J_a^\beta J_a^\alpha f\right)(x) = \left(J_a^{\alpha+\beta} f\right)(x), \\
J_a^\alpha x^\gamma = \frac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_{(x-a)/x}(\alpha, \gamma+1),
\end{gathered} \tag{2.2}$$

where $B_\tau(\alpha, \gamma+1)$ is the incomplete beta function which is defined as

$$B_\tau(\alpha,\gamma+1) = \int_0^\tau t^{\alpha-1}(1-t)^\gamma dt,$$
$$J_a^\alpha e^{cx} = e^{ac}(x-a)^\alpha \sum_{k=0}^{\infty} \frac{[c(x-a)]^k}{\Gamma(\alpha+k+1)}. \tag{2.3}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we will introduce a modified fractional differential operator D_a^α proposed by Caputo in his work on the theory of viscoelasticity.

Definition 2.3. The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ with $a \geq 0$ is defined as

$$(D_a^\alpha f)(x) = \left(J_a^{m-\alpha} f^{(m)}\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \tag{2.4}$$

for $m-1 < \alpha \leq m$, $m \in \mathbf{N}$, $x \geq a$, $f(x) \in C_{-1}^m$.

The Caputo fractional derivative was investigated by many authors, for $m-1 < \alpha \leq m$, $f(x) \in C_\alpha^m$ and $\alpha \geq -1$, we have

$$(J_a^\alpha D_a^\alpha f)(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(x-a)^k}{k!}. \tag{2.5}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. Multistep Generalized Differential Transform Method (MSGDTM)

To describe the multistep generalized differential transform method (MSGDTM) [22–24], we consider the following initial value problem for systems of fractional differential equations

$$\begin{aligned} D_*^{\alpha_1} y_1(t) &= f_1(t, y_1, y_2, \ldots, y_n), \\ D_*^{\alpha_2} y_2(t) &= f_1(t, y_1, y_2, \ldots, y_n), \\ &\vdots \\ D_*^{\alpha_n} y_n(t) &= f_1(t, y_1, y_2, \ldots, y_n), \end{aligned} \tag{3.1}$$

subject to the initial conditions

$$y_i(t_0) = c_i, \quad i = 1, 2, \ldots, n, \tag{3.2}$$

where $D_*^{\alpha_i}$ is the Caputo fractional derivative of order α_i, where $0 < \alpha_i \leq 1$, for $i = 1, 2, \ldots, n$. Let $[t_0, T]$ be the interval over which we want to find the solution of the initial value problem ((3.1)-(3.2)). In actual applications of the generalized differential transform method (GDTM), the Kth-order approximate solution of the initial value problem ((3.1)-(3.2)) can be expressed by the finite series

$$y_i(t) = \sum_{i=0}^{K} Y_i(k)(t - t_0)^{k\alpha_i}, \quad t \in [t_0, T], \tag{3.3}$$

where $Y_i(k)$ satisfied the recurrence relation

$$\frac{\Gamma((k+1)\alpha_i + 1)}{\Gamma(k\alpha_i + 1)} Y_i(k+1) = F_i(k, Y_1, Y_2, \ldots, Y_n), \tag{3.4}$$

$Y_i(0) = c_i$, and $F_i(k, Y_1, Y_2, \ldots, Y_n)$ is the differential transform of function $f_i(t, y_1, y_2, \ldots, y_n)$ for $i = 1, 2, \ldots, n$. The basics steps of the GDTM can be found in [18–21].

Assume that the interval $[t_0, T]$ is divided into M subintervals $[t_{m-1}, t_m], m = 1, 2, \ldots, M$ of equal step size $h = (T - t_0)/M$ by using the nodes $t_m = t_0 + mh$. The main ideas of the MSGDTM are as follows.

First, we apply the GDTM to the initial value problem ((3.1)-(3.2)) over the interval $[t_0, t_1]$, we will obtain the approximate solution $y_{i,1}(t),\ t \in [t_0, t_1]$, using the initial condition $y_i(t_0) = c_i$, for $i = 1, 2, \ldots, n$. For $m \geq 2$ and at each subinterval $[t_{m-1}, t_m]$, we will use the initial condition $y_{i,m}(t_{m-1}) = y_{i,m-1}(t_{m-1})$ and apply the GDTM to the initial value problem ((3.1)-(3.2)) over the interval $[t_{m-1}, t_m]$. The process is repeated and generates a sequence of approximate solutions $y_{i,m}(t)$, $m = 1, 2, \ldots, M$, for $i = 1, 2, \ldots, n$. Finally, the MSGDTM assumes the following solution:

$$y_i(t) = \begin{cases} y_{i,1}(t), & t \in [t_0, t_1] \\ y_{i,2}(t), & t \in [t_1, t_2] \\ \vdots & \\ y_{i,M}(t), & t \in [t_{M-1}, t_M]. \end{cases} \tag{3.5}$$

The new algorithm, MSGDTM, is simple for computational performance for all values of h. As we will see in the next section, the main advantage of the new algorithm is that the obtained solution converges for wide time regions.

4. Solving the Fractional-Order Chua's System and Chua-Hartley's System Using the MSGDTM Algorithm

In order to demonstrate the performance and efficiency of the multistep generalized differential transform method for solving linear and nonlinear fractional-order equations, we have applied the method to two examples. In the first example, we consider the fractional-order Chua's chaotic system, while in the second example, we consider the fractional-order Chua-Hartley's system.

4.1. The Fractional-Order Chua's System

The classical Chua's oscillator is a simple electronic circuit that exhibits nonlinear dynamical phenomena such as bifurcation and chaos [16]. Now, we consider a fractional-order Chua's system, where integer-order derivatives are replaced by fractional-order ones. This system is expressed as

$$D^{q_1}x(t) = \alpha\big(y - z - f(x)\big), \tag{4.1}$$

$$D^{q_2}y(t) = x - y + z, \tag{4.2}$$

$$D^{q_3}z(t) = -\beta y - \gamma z, \tag{4.3}$$

$$f(x) = m_1 x(t) + \frac{1}{2}(m_0 - m_1)(|x(t)+1| - |x(t)-1|), \tag{4.4}$$

where (x, y, z) are the state variables, $m_0,\ m_1,\ \alpha,\ \beta,\ \gamma$ are constants, and $q_i,\ i = 1,\ 2,\ 3$ are parameters describing the order of the fractional time derivatives in the Caputo sense.

Applying the MSGDTM Algorithm to (4.1)–(4.3) gives

$$\begin{aligned}
X(k+1) &= \frac{\Gamma(q_1 k + 1)}{\Gamma(q_1(k+1)+1)} \\
&\quad \times \alpha\left(Y(k) - X(k) - \left(m_1 X(k) + \frac{1}{2}(m_0 - m_1)(|X(k)+\delta(k)| - |X(k) - \delta(k)|)\right)\right), \\
Y(k+1) &= \frac{\Gamma(q_2 k + 1)}{\Gamma(q_2(k+1)+1)}(X(k) - Y(k) + Z(k)), \\
Z(k+1) &= \frac{\Gamma(q_3 k + 1)}{\Gamma(q_3(k+1)+1)}(-\beta Y(k) - \gamma Z(k)),
\end{aligned} \tag{4.5}$$

where $X(k)$, $Y(k)$, and $Z(k)$ are the differential transformation of $x(t)$, $y(t)$, and $z(t)$, respectively, and $\delta(k)$ equals 1 when $k = 0$ and equals 0 otherwise. The differential transform of the initial conditions is given by $X(0) = c_1$, $Y(0) = c_2$, $Z(0) = c_3$. In view of the differential inverse transform, the differential transform series solution for the system (4.1)–(4.3) can be obtained as

$$\begin{aligned}
x(t) &= \sum_{n=0}^{N} X(n) t^{q_1 n}, \\
y(t) &= \sum_{n=0}^{N} Y(n) t^{q_2 n}, \\
z(t) &= \sum_{n=0}^{N} Z(n) t^{q_3 n}.
\end{aligned} \tag{4.6}$$

According to the multistep generalized differential transform method, the series solution for the system (4.1)–(4.3) is suggested by

$$x(t)=\begin{cases}\sum_{n=0}^{K}X_1(n)t^{q_1 n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}X_2(n)(t-t_1)^{q_1 n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}X_M(n)(t-t_{M-1})^{q_1 n}, & t\in[t_{M-1},t_M],\end{cases}$$

$$y(t)=\begin{cases}\sum_{n=0}^{K}Y_1(n)t^{q_2 n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}Y_2(n)(t-t_1)^{q_2 n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}Y_M(n)(t-t_{M-1})^{q_2 n}, & t\in[t_{M-1},t_M],\end{cases} \tag{4.7}$$

$$z(t)=\begin{cases}\sum_{n=0}^{K}Z_1(n)t^{q_3 n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}Z_2(n)(t-t_1)^{q_3 n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}Z_M(n)(t-t_{M-1})^{q_3 n}, & t\in[t_{M-1},t_M],\end{cases}$$

where $X_i(n)$, $Y_i(n)$, and $Z_i(n)$ for $i=1,2,\ldots,M$ satisfy the following recurrence relations:

$$\begin{aligned}X_i(k+1)&=\frac{\Gamma(q_1k+1)}{\Gamma(q_1(k+1)+1)}\\&\quad\times\alpha\Bigg(Y_i(k)-X_i(k)\\&\qquad-\left(m_1X(k)+\frac{1}{2}(m_0-m_1)(|X_i(k)+\delta(k)|-|X_i(k)-\delta(k)|)\right)\Bigg),\\Y_i(k+1)&=\frac{\Gamma(q_2k+1)}{\Gamma(q_2(k+1)+1)}(X_i(k)-Y_i(k)+Z_i(k)),\\Z_i(k+1)&=\frac{\Gamma(q_3k+1)}{\Gamma(q_3(k+1)+1)}(-\beta Y_i(k)-\gamma Z_i(k)),\end{aligned} \tag{4.8}$$

such that $X_i(0)=x_i(t_{i-1})=x_{i-1}(t_{i-1})$, $Y_i(0)=y_i(t_{i-1})=y_{i-1}(t_{i-1})$, and $Z_i(0)=z_i(t_{i-1})=z_{i-1}(t_{i-1})$.

Finally, we start with $X_0(0) = c_1$, $Y_0(0) = c_2$, and $Z_0(0) = c_3$, using the recurrence relation given in (4.8), then we can obtain the multistep solution given in (4.7).

4.2. The Fractional-Order Chua-Hartley's System

The Chua-Hartley's system is different from the usual Chua's system in that the piecewise-linear nonlinearity is replaced by an appropriate cubic nonlinearity which yields a very similar behavior. Derivatives on the left side of the differential equations are also replaced by the fractional derivatives as follows [16, 17]:

$$D^{q_1}x(t) = \alpha\left(y + \frac{1}{7}\left(x - 2x^3\right)\right), \tag{4.9}$$

$$D^{q_2}y(t) = x - y + z, \tag{4.10}$$

$$D^{q_3}z(t) = -\frac{100}{7}y, \tag{4.11}$$

where (x, y, z) are the state variables, α is a positive constant and q_i, $i = 1, 2, 3$ are parameters describing the order of the fractional time-derivatives in the Caputo sense.

Following the same procedure as the previous system and applying MSGDTM algorithm to (4.9)–(4.11) yields

$$\begin{aligned} X(k+1) &= \frac{\Gamma(q_1k+1)}{\Gamma(q_1(k+1)+1)} \\ &\quad\times \alpha\left(Y(k) + \frac{1}{7}\left(X(k) - 2\sum_{j=0}^{k}\sum_{i=0}^{j}X(i)X(j-i)X(k-j)\right)\right), \\ Y(k+1) &= \frac{\Gamma(q_2k+1)}{\Gamma(q_2(k+1)+1)}(X(k) - Y(k) + Z(k)), \\ Z(k+1) &= \frac{\Gamma(q_3k+1)}{\Gamma(q_3(k+1)+1)}\left(-\frac{100}{7}Y(k)\right), \end{aligned} \tag{4.12}$$

where $X(k)$, $Y(k)$, and $Z(k)$ are the differential transformation of $x(t)$, $y(t)$, and $z(t)$ respectively. The differential transform of the initial conditions is given by $X(0) = c_1$, $Y(0) = c_2$, and $Z(0) = c_3$. In view of the differential inverse transform, the differential transform series solution for the system (4.9)–(4.11) can be obtained as

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} X(n)t^{q_1n}, \\ y(t) &= \sum_{n=0}^{N} Y(n)t^{q_2n}, \\ z(t) &= \sum_{n=0}^{N} Z(n)t^{q_3n}. \end{aligned} \tag{4.13}$$

Now, according to the MSGDTM algorithm, the series solution for the system (4.8)–(4.11) is suggested by

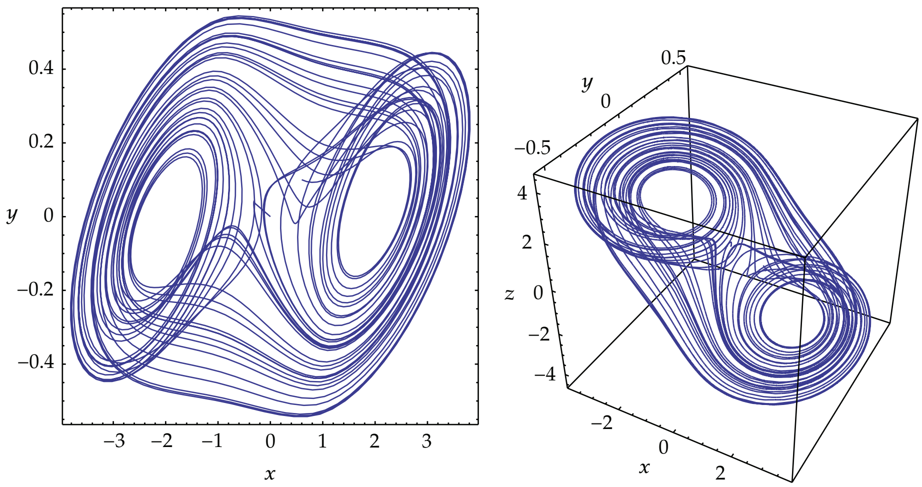
$$x(t)=\begin{cases}\sum_{n=0}^{K}X_1(n)t^{q_1n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}X_2(n)(t-t_1)^{q_1n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}X_M(n)(t-t_{M-1})^{q_1n}, & t\in[t_{M-1},t_M],\end{cases}$$
$$y(t)=\begin{cases}\sum_{n=0}^{K}Y_1(n)t^{q_2n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}Y_2(n)(t-t_1)^{q_2n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}Y_M(n)(t-t_{M-1})^{q_2n}, & t\in[t_{M-1},t_M],\end{cases}\tag{4.14}$$
$$z(t)=\begin{cases}\sum_{n=0}^{K}Z_1(n)t^{q_3n}, & t\in[0,t_1]\\ \sum_{n=0}^{K}Z_2(n)(t-t_1)^{q_3n}, & t\in[t_1,t_2]\\ \vdots & \\ \sum_{n=0}^{K}Z_M(n)(t-t_{M-1})^{q_3n}, & t\in[t_{M-1},t_M],\end{cases}$$

where $X_i(n)$, $Y_i(n)$, and $Z_i(n)$ for $i=1,2,\ldots,M$ satisfy the following recurrence relations:

$$\begin{aligned}X_i(k+1)&=\frac{\Gamma(q_1k+1)}{\Gamma(q_1(k+1)+1)}\\&\quad\times\alpha\left(Y_i(k)+\frac{1}{7}\left(X_i(k)-2\sum_{J=0}^{k}\sum_{r=0}^{J}X_i(I)X_i(J-I)X_i(k-J)\right)\right),\\ Y_i(k+1)&=\frac{\Gamma(q_2k+1)}{\Gamma(q_2(k+1)+1)}(X_i(k)-Y_i(k)+Z_i(k)),\\ Z_i(k+1)&=\frac{\Gamma(q_3k+1)}{\Gamma(q_3(k+1)+1)}\left(-\frac{100}{7}Y_i(k)\right)\end{aligned}\tag{4.15}$$

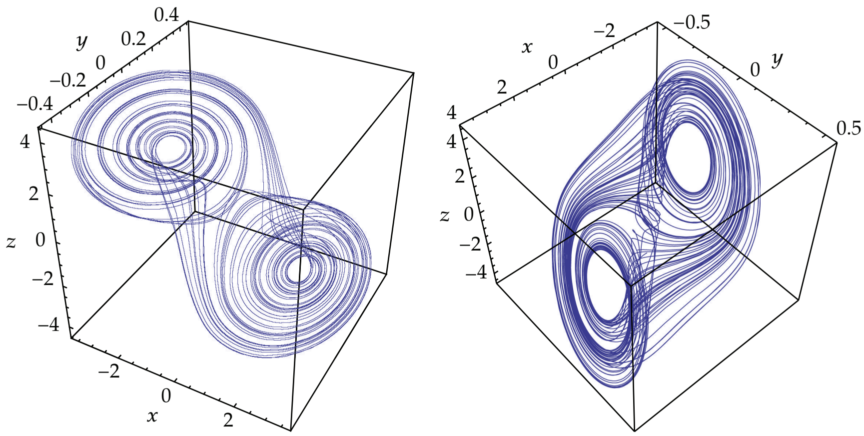
such that $X_i(0)=x_i(t_{i-1})=x_{i-1}(t_{i-1})$, $Y_i(0)=y_i(t_{i-1})=y_{i-1}(t_{i-1})$, and $Z_i(0)=z_i(t_{i-1})=z_{i-1}(t_{i-1})$.

Starting with $X_0(0)=c_1$, $Y_0(0)=c_2$, $Z_0(0)=c_3$ and using the recurrence relation given in (4.15), then we can obtain the multistep solution given in (4.14).



(a) (b)

Figure 1: Phase plot of Chua's chaotic attractor with $q_1 = q_2 = q_3 = 1$, $\alpha = 9.085$: (a) in the x-y space, (b) in the x-y-z space.



(a) (b)

Figure 2: Phase plot of Chua's chaotic attractor in the x-y-z space: (a) $q_1 = q_2 = 0.98$, $q_3 = 0.94$, $\alpha = 10.1911$, (b) $q_1 = q_3 = 0.87$, $q_2 = 0.88$, $\alpha = 9.085$.

5. Numerical Results

The MSGDTM is coded in the computer algebra package Mathemtica. The Mathemtica environment variable digits controlling the number of significant digits is set to 20 in all the calculations done in this paper. The time range studied in this work is $[0, 200]$ and the step size $\Delta t = 0.02$. We take the initial conditions for Chua's system and Chua-Hartley's system: $x(0) = 0.6$, $y(0) = 0.1$ and $z(0) = -0.6$.

We consider the case $q_1 = q_2 = q_3 = 1$ which corresponds to the classical Chua's system. Figure 1 represents the phase portrait for chaotic solutions. The effective dimension Σ of (4.1)–(4.3) is defined as the sum of orders $q_1 + q_2 + q_3 = \Sigma$. We can see that the

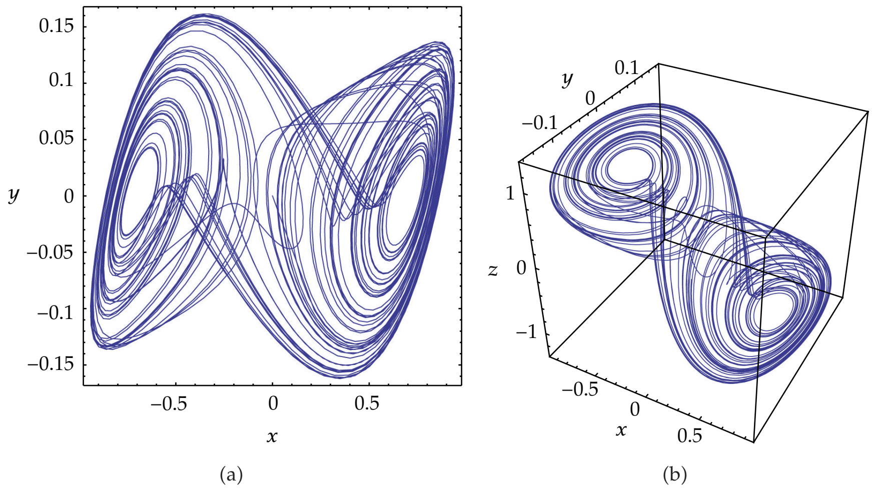


Figure 3: Phase plot of Chua-Hartley's chaotic attractor with $\alpha = 9.5$ and $q_1 = q_2 = q_3 = 1$: (a) in the x-y space, (b) in the x-y-z space.

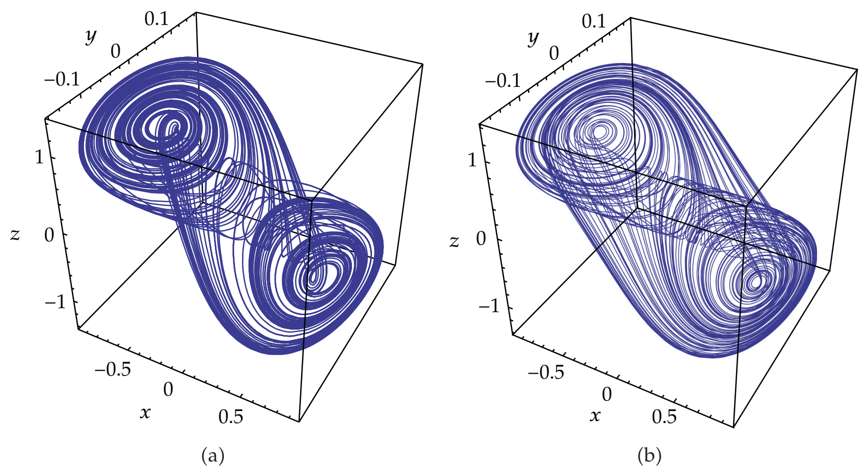


Figure 4: Phase plot of Chua-Hartley's chaotic attractor in the x-y-z space: (a) $q_1 = q_2 = 0.92$, $q_3 = 0.86$, $\alpha = 12.75$, (b) $q_1 = 0.90$, $q_2 = 0.89$, $q_3 = 0.86$, $\alpha = 12.75$.

chaotic attractors of the fractional-order system are similar to that of the integer-order Chua's attractor as shown in Figure 2. Figure 2(b) shows the lowest order we found to yield chaos in this system is 2.62. From the numerical results in Figure 2, it is clear that the approximate solutions depend continuously on the time-fractional derivative q_i, $i = 1, 2, 3$.

Simulations were performed for the classical integer-order Chua-Hartley's system in Figure 3. Figure 4(b) shows the lowest order we found to yield chaos in the fractional-order Chua-Hartley's system is 2.65. From the graphical results in Figures 1–4, it is to conclude that the approximate solutions obtained using the multistep generalized differential transform method are in good agreement with the approximate solutions obtained in [16, 17].

6. Conclusions

In this present work, the multistep generalized differential transform method was introduced to obtain the solutions of the fractional-order Chua's system and Chua-Hartley's system by time discretization. This method has the advantage of giving an analytical form of the solution within each time interval which is not possible using purely numerical techniques like the fourth-order Runge-Kutta method (RK4). We conclude that MSGDTM is a very reliable method in solving a broad array of dynamical problems in fractional calculus due to its consistency used in a longer time frame.

References

[1] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, *Advances in Fractional Calculus; Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.
[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.
[3] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
[4] I. Podlubny, *Fractional Differential Equations*, vol. 198, Academic Press, San Diego, Calif, USA, 1999.
[5] C. Li and W. Deng, "Remarks on fractional derivatives," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 777–784, 2007.
[6] G. M. Zaslavsky, "Chaos, fractional kinetics, and anomalous transport," *Physics Reports*, vol. 371, no. 6, pp. 461–580, 2002.
[7] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, Singapore, 2009.
[8] R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus," *Journal of Rheology*, vol. 27, pp. 201–210, 1983.
[9] R. L. Bagley and R. A. Calico, "Fractional order state equations for the control of viscoelastically damped structures," *Journal of Guidance, Control, and Dynamics*, vol. 14, no. 2, pp. 304–311, 1991.
[10] E. J. S. Pires, J. A. T. Machado, and P. B. de Moura, "Fractional order dynamics in a GA planner," *Signal Processing*, vol. 83, pp. 2377–2386, 2003.
[11] K. S. Hedrih and V. N. Stanojević, "A model of gear transmission: fractional order system dynamics," *Mathematical Problems in Engineering*, vol. 2010, Article ID 972873, 2010.
[12] J. Cao, C. Ma, H. Xie, and Z. Jiang, "Nonlinear dynamics of duffing system with fractional order damping," *Journal of Computational and Nonlinear Dynamics*, vol. 5, no. 4, pp. 041012–041018, 2010.
[13] C. Li and G. Chen, "Chaos and hyperchaos in the fractional-order Rössler equations," *Physica A*, vol. 341, no. 1–4, pp. 55–61, 2004.
[14] K. Moaddy, S. Momani, and I. Hashim, "The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1209–1216, 2011.
[15] A. G. Radwan, K. Moaddy, and S. Momani, "Stability and non-standard finite difference method of the generalized Chua's circuit," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 961–970, 2011.
[16] I. Petrás, "A note on the fractional-order Chua's system," *Chaos Solution & Fractals*, vol. 38, pp. 140–147, 2008.
[17] T. T. Hartley, C. F. Lorenzo, and H. K. Qammer, "Chaos on a fractional Chua's system," *IEEE Transactions on Circuits and Systems*, vol. 42, no. 8, pp. 485–490, 1995.
[18] Z. Odibat, S. Momani, and V. S. Erturk, "Generalized differential transform method: application to differential equations of fractional order," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 467–477, 2008.
[19] S. Momani and Z. Odibat, "A novel method for nonlinear fractional partial differential equations: combination of DTM and generalized Taylor's formula," *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 85–95, 2008.
[20] Z. Odibat and S. Momani, "A generalized differential transform method for linear partial differential equations of fractional order," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 194–199, 2008.

[21] V. S. Erturk, S. Momani, and Z. Odibat, “Application of generalized differential transform method to multi-order fractional differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.

[22] Z. M. Odibat, C. Bertelle, M. A. Aziz-Alaoui, and G. H. E. Duchamp, “A multi-step differential transform method and application to non-chaotic or chaotic systems,” *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1462–1472, 2010.

[23] V. S. Ertürk, Z. M. Odibat, and S. Momani, “An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^+$ T-cells,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 996–1002, 2011.

[24] V. Erturk, G. Zaman, and S. Momani, “A numericanalytic method for approximating a giving up smoking model containing fractional derivatives,” *Computers & Mathematics With Applications*, vol. 64, no. 10, pp. 3065–3074, 2012.