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Research Article

On Properties of Differences Polynomials about Meromorphic Functions

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We study the value distribution of a special class difference polynomial about finite order meromorphic function. Our methods of the proof are also different from ones in the previous results by Chen (2011), Liu and Laine (2010), and Liu and Yang (2009).

1. Introduction and Results

A function $f(z)$ is called meromorphic function, if it is analytic in the complex plane $\mathbb{C}$ except at isolated poles. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$, and the first and second main theorem (see [1–3]). The notation $S(r, f)$ denotes any quantity that satisfies the condition: $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside an exceptional set of r of finite linear measure. We use the notation $\tau(f)$ to denote the exponent of convergence of zeros of $f(z)$, and use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$. Also, we give an estimate of numbers of b-points, namely, $\tau(f - b)$ for every $b \in \mathbb{C}$.

Next, we will introduce the notation of Borel exceptional value (see [1]).

Definition 1.1. Let f be a transcendental meromorphic function in $\mathbb{C}$ with the order $\sigma(f)$. A complex number a is said to be a Borel exceptional value if

$$\overline{\lim} \frac{\log^{+} n(r, 1/(f-a))}{\log r} < \sigma(f). \tag{1.1}$$

Here $\log^{+} n(r, 1/(f-a))$ can be replaced by $\log^{+} N(r, 1/(f-a))$.

In 1959, Hayman [4] proved the following Theorem.

Theorem A. *Let f be a meromorphic function in $\mathbb{C}$, if $f' - af^{n} \neq b$, where n is a positive integer and a, b are two finite complex numbers such that $n \geq 5$ and $a \neq 0$, then f is a constant.*

On the other hand, Mues [5] showed that for $n = 3, 4$ the conclusion is not valid.

Recently, as the significant results on Nevanlinna theory with respect to difference operators, see the papers [6, 7] by Halburd and Korhonen and [8] by Chiang and Feng. Many papers (see [2–4, 9–17]) have focused on complex differences and given many difference analogues in value distribution theory of entire functions.

In 2010, replacing f' by $f(z+c) - f(z)$ in Theorem A, Liu and Laine [17] obtained the following result.

Theorem B (see [17]). *Let f be a transcendental entire function of finite order, not of period c, where c is a nonzero constant, and let $s(z)$ be a nonzero small function of f. Then the difference polynomial $f^{n}(z) + f(z+c) - f(z) - s(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 3$.*

In 2011, Chen [18] considered the difference counterpart of Theorem A and proved an almost direct difference analogue of Hayman's Theorem.

Theorem C (see [18, Theorem 1.1]). *Let f be a transcendental entire function of finite order, not of period c, and let $a(\neq 0)$, b, $c(\neq 0)$ be three complex numbers. Then $\Psi_{n}(z) = f(z+c) - f(z) - af^{n}(z)$ assumes all finite values infinitely often, provided that $n \geq 3$ and $\tau(\Psi_{n}(z) - b) = \sigma(f)$ for every b.*

In 1994, Ye [19] considered a similar problem and obtained that if f is a transcendental meromorphic function and a is a nonzero finite complex number, then $f + a(f')^{n}$ assumes every finite complex value infinitely often for $n \geq 3$. Ye [19] also asked whether the conclusion remains valid for $n = 2$.

In 2008, Fang and Zalcman [20] solved this problem and obtained the following result.

Theorem D. *Let f be a transcendental meromorphic function and a be a nonzero complex number. Then $f + a(f')^{n}$ assumes every complex value infinitely often for each positive integer $n \geq 2$.*

Just like Theorem B, it is natural to ask whether Theorem D can be improved by the ideas of difference operator. The purpose of this paper is to study value distribution of meromorphic function with respect to difference. Our methods of proof are also different from ones in previous Theorems (see [17, 18, 21]). We obtain the following results.

Theorem 1.2. *Let f be a transcendental meromorphic function of finite order, not of period c, where c is a nonzero constant, and let $s(z)$ be a small function of f, let a be a nonzero constant. Then* *the difference polynomial* $af^n(z+c)+f(z)-s(z)$ *has infinitely many zeros in the complex plane, provided that* $n \geq 5$.

Corollary 1.3. *Let* f *be a transcendental entire function of finite order, not of period* c, *where* c *is a nonzero constant, and let* $s(z)$ *be a small function of* f, *let* a *be a nonzero constant. Then the difference polynomial* $af^n(z+c)+f(z)-s(z)$ *has infinitely many zeros in the complex plane, provided that* $n \geq 3$.

Recently, Qi and Liu [22] obtained the following result.

Theorem E (see [22, Theorem 2]). *Let* f *be a transcendental entire function of finite order,* c *be a nonzero constant,* m *and* n *be integers satisfying* $n \geq m > 0$, *and let* λ, μ *be two complex numbers such that* $|\lambda|+|\mu| \neq 0$. *If* $n \geq 2$, *then either* $f^n(z)(\lambda f^m(z+c)+\mu f^m(z))$ *assumes every nonzero finite value infinitely often or* $f(z) = \exp\{(\log t/c)z\}g(z)$, *where* $t = (-\mu/\lambda)^{1/m}$, *and* $g(z)$ *is periodic function with period* c.

Thus, it is natural to ask, what happens if $n = 1$ in Theorem E?

By the same method of [18, 23], we investigate this problem and obtain the following results.

Theorem 1.4. *Let* f *be a transcendental entire function with finite order with a Borel exceptional value* 0, c *be a nonzero complex constant, and let* λ, μ *be two complex numbers such that* $|\lambda|+|\mu| \neq 0$ *and* $\lambda f(z+c)+\mu f(z) \not\equiv 0$, *then* $H(z) := f(z)(\lambda f(z+c)+\mu f(z))$ *assumes every nonzero value* $a \in \mathbb{C}$ *infinitely often and* $\tau(H-a) = \sigma(f)$.

Theorem 1.5. *Let* $f(z)$ *be a transcendental entire function of finite order,* $c \in \mathbb{C} \setminus \{0\}$ *be a complex constant, and let* λ, μ *be two complex numbers such that* $|\lambda|+|\mu| \neq 0$ *and* $\lambda f(z+c)+\mu f(z) \not\equiv 0$. *If* $f(z)$ *has infinitely many multiple zeros, then* $H(z) := f(z)(\lambda f(z+c)+\mu f(z))$ *takes every value* $a \in \mathbb{C}$ *infinitely often.*

Example 1.6. $f(z) = e^z$ satisfies $f(z+1)-ef(z) \equiv 0$. However, $f(z)(\lambda f(z+c)+\mu f(z))$ cannot assume any nonzero value $a \in \mathbb{C}$.

Remark 1.7. From the Example 1.6, the condition $(\lambda f(z+c)+\mu f(z)) \not\equiv 0$ is necessary in Theorems 1.4 and 1.5.

Remark 1.8. Some ideas in this paper are based on [18, 23–25].

2. Some Lemmas

In order to prove our theorems, we need the following Lemmas.

The Lemma 2.1 is a difference analogue of the logarithmic derivative lemma, given by Halburd and Korhonen [7] and Chiang and Feng [8], independently.

Lemma 2.1 (see [7, Theorem 2.1]). *Let* $f(z)$ *be a meromorphic function of finite order, and let* $c \in \mathbb{C}$ *and* $\delta \in (0,1)$. *Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f). \tag{2.1}$$

Lemma 2.2 (see [1, Theorem 1.12]). *Let $f(z)$ be a nonconstant meromorphic function, and let $P(f) = a_0 f^n + a_1 f^{n-1} + \cdots a_n$, where $a_0(\neq 0)$, $a_1, \ldots, a_n$ are small function of f. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f). \tag{2.2}$$

By using the formulation (12) in [13], it is easy to get the following lemma.

Lemma 2.3. *Let $f(z)$ be a meromorphic function of finite order, $c \in \mathbb{C}$. Then*

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)). \tag{2.3}$$

Lemma 2.4. *Let $f(z)$ be a transcendental entire function of finite order ρ with a Borel exceptional value 0, $c \in \mathbb{C} \setminus \{0\}$ be complex constant, and let λ, μ be two complex numbers such that $|\lambda| + |\mu| \neq 0$ and $\lambda f(z+c) + \mu f(z) \not\equiv 0$, then $\sigma(H) = \sigma(f)$, where $H(z) := f(z)[\lambda f(z+c) + \mu f(z)]$.*

Proof. Rewrite $H(z)$ as the form

$$H(z) = f(z)^2 \frac{\lambda f(z+c) + \mu f(z)}{f(z)}. \tag{2.4}$$

For each $\varepsilon > 0$, by Lemma 2.1 and (2.4), we get that

$$m(r, H) \leq 2m(r, f) + m\left(r, \frac{\lambda f(z+c)}{f(z)}\right) + O\left(r^{\rho-1+\varepsilon}\right) = 2m(r, f) + O\left(r^{\rho-1+\varepsilon}\right), \tag{2.5}$$

$$\begin{aligned}
2T(r, f) = T\left(r, f^2\right) &\leq T(r, H) + T\left(r, \frac{f}{\lambda f(z+c) + \mu f(z)}\right) \\
&= T(r, H) + T\left(r, \frac{\lambda f(z+c) + \mu f(z)}{f(z)}\right) + O(1) \\
&= T(r, H) + N\left(r, \frac{f(z+c)}{f(z)}\right) + O\left(r^{\rho-1+\varepsilon}\right) + O(1) \\
&= T(r, H) + N\left(r, \frac{1}{f}\right) + N(r, f(z+c)) + O\left(r^{\rho-1+\varepsilon}\right) + O(1).
\end{aligned} \tag{2.6}$$

Because $f(z)$ is a transcendental entire function of finite order ρ with a Borel exceptional value 0. Then we obtain

$$T(r, f) \leq T(r, H) + O\left(r^{\rho-1+\varepsilon}\right). \tag{2.7}$$

Thus, (2.5) and (2.7) give that $\sigma(H) = \sigma(f)$. □

Lemma 2.5 (see [1]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions, and satisfy*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$,

(ii) *when* $1 \le j < k \le n$, $g_j(z) - g_k(z)$ *is not a constant,*

(iii) *when* $1 \le j \le n$, $1 \le h < k \le n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$,

where $E \subset (1, \infty)$ *is of finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 2.6 (see [1]). *Let* f *be a transcendental meromorphic function of order* $\sigma(f)$ *and* $\tau(f)$ *be the convergence exponent of its zeros. Then* $\tau(f) \le \sigma(f)$.

Lemma 2.7 (see [1], Hadamard's factorization theorem). *Let* f *be a transcendental entire function of finite order* $\sigma(f)$ *with zeros* $\{z_1, z_2, \ldots\} \subset \mathbb{C} \setminus \{0\}$ *and a k-fold zero at origin. Then*

$$f(z) = z^k \mathbf{E}(z) e^{Q(z)}, \tag{2.8}$$

where $\mathbf{E}(z)$ *is the canonical product of* f *formed with the nonnull zeros of* f, *and* $Q(z)$ *is a polynomial of degree* $\le \sigma(f)$.

Lemma 2.8 (see [1]). *Let* $\lambda(\mathbf{E})$ *be the order of the canonical product* $\mathbf{E}(z)$. *We use* $\tau(\mathbf{E})$ *to denote the exponent of convergence of zeros of* $\mathbf{E}(z)$. *Then* $\lambda(\mathbf{E}) = \tau(\mathbf{E})$.

3. Proofs of Theorems

Proof of Theorem 1.2. Set $\Phi(z) := af^n(z+c) + f(z) - s(z)$. Obviously, $\Phi(z) \not\equiv C$. If it is false, then $af^n(z+c) \equiv s(z) + C - f(z)$. Thus we have that

$$T(r, f^n(z+c)) = nT(r, f(z+c)) = T(r, f(z)) + S(r, f), \tag{3.1}$$

where $n \ge 5$. Using Lemmas 2.1 and 2.3, we deduce that

$$\begin{aligned} T(r, f(z+c)) &= m(r, f(z+c)) + N(r, f(z+c)) \\ &\le m(r, f(z)) + m\left(r, \frac{f(z+c)}{f(z)}\right) + N(r, f(z)) + S(r, f(z)) \\ &= T(r, f(z)) + S(r, f(z)), \\ T(r, f(z)) &= m(r, f(z)) + N(r, f(z)) \\ &\le m(r, f(z+c)) + m\left(r, \frac{f(z)}{f(z+c)}\right) + N(r, f(z)) + S(r, f(z)) \\ &= m(r, f(z+c)) + N(r, f(z+c)) + S(r, f(z)) \\ &= T(r, f(z+c)) + S(r, f(z)), \end{aligned} \tag{3.2}$$

Equations (3.1) and (3.2) imply $T(r, f(z+c)) = S(r, f(z))$, a contradiction, therefore $\Phi(z) \not\equiv C$.

Furthermore, we claim that

$$\frac{(f^n(z+c))'}{f^n(z+c)} - \frac{\Phi'}{\Phi} \not\equiv 0. \tag{3.3}$$

Otherwise, $(f^n(z+c))'/f^n(z+c) - \Phi'/\Phi \equiv 0$. By integration, we obtain $\Phi(z) = bf^n(z+c)$, where b is a constant, hence $(b-a)f^n(z+c) = f(z) - s(z)$.

If $b = a$, we can deduce $T(r, f(z)) = T(r, s(z))$. This contradicts the hypothesis.

If $b \neq a$, by the same arguments of the proof of Case $\Phi(z) \equiv C$, we get the same contradiction.

By a simple calculation we get that

$$af^n(z+c) = \frac{(\Phi'/\Phi)[f(z) - s(z)] - [f(z) - s(z)]'}{(f^n(z+c))'/(f^n(z+c)) - \Phi'/\Phi}. \tag{3.4}$$

From Lemmas 2.1 and 2.2 and some results of Nevanlinna Theory, we obtain that

$$\begin{aligned}
T(r, af^n(z+c)) &= nT(r, f(z+c)) + S(r, f(z+c)) \\
&= T\left(r, \frac{(\Phi'/\Phi)[f(z) - s(z)] - [f(z) - s(z)]'}{(f^n(z+c))'/(f^n(z+c)) - \Phi'/\Phi}\right) \\
&\le m(r, f(z)) + N\left(r, \frac{\Phi'}{\Phi}[f(z) - s(z)] - [f(z) - s(z)]'\right) \\
&\quad + m\left(r, \frac{\Phi'}{\Phi} - \frac{[f(z) - s(z)]'}{[f(z) - s(z)]}\right) + m\left(r, \frac{(f^n(z+c))'}{f^n(z+c)} - \frac{\Phi'}{\Phi}\right) \\
&\quad + N\left(r, \frac{(f^n(z+c))'}{f^n(z+c)} - \frac{\Phi'}{\Phi}\right) + S(r, f(z)).
\end{aligned} \tag{3.5}$$

Next, we will estimate $N(r, (\Phi'/\Phi)[f(z) - s(z)] - [f(z) - s(z)]')$ and $N(r, (f^n(z+c))'/f^n(z+c) - \Phi'/\Phi)$.

The poles of $\varphi_1(z) = (\Phi'/\Phi)[f(z) - s(z)] - [f(z) - s(z)]'$ come from the zeros of $\Phi(z)$, the poles of $f(z+c)$, the poles of $f(z)$, and the poles of $s(z)$. By the hypothesis, we ignore the poles of $s(z)$. If z_0 is a zero of $\Phi(z)$ or a pole of $f(z+c)$ but not a pole of $f(z)$, then z_0 is a simple pole of $\varphi_1(z)$. If z_0 is a common pole of $f(z+c)$ and $f(z)$, and the multiplicity is k and l, respectively, then z_0 is a pole of $\varphi_1(z)$ with the multiplicity of no more than $l+1$. If z_0 is a pole of $f(z)$ but not a pole of $f(z+c)$, we obtain that z_0 is at most a simple pole of $\varphi_1(z)$ because of (3.4). Using the Lemma 2.3, we can get that

$$\begin{aligned}
N\left(r, \frac{\Phi'}{\Phi}[f(z) - s(z)] - [f(z) - s(z)]'\right) &\le \overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}(r, f(z+c)) + N(r, f(z)) + S(r, f(z)) \\
&= \overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}(r, f(z)) + N(r, f(z)) + S(r, f(z)).
\end{aligned} \tag{3.6}$$

We deal with the poles of $s(z)$ as above. The zeros of $\Phi(z)$, the poles of $f(z+c)$, the poles of $f(z)$, and the zeros of $f(z+c)$ compose the poles of $\varphi_2(z) = (f^n(z+c))'/f^n(z+c) - \Phi'/\Phi$. If z_0 is a zero of $\Phi(z)$, zero of $f(z+c)$, or pole of $f(z)$, then z_0 is a simple pole of $\varphi_2(z)$. If z_0 is a pole of $f(z+c)$ but not a pole of $f(z)$, using the Laurent series, we can get that $\varphi_2(z)$ is analytic at z_0. Therefore, we conclude that

$$N\left(r, \frac{(f^n(z+c))'}{f(z+c)} - \frac{\Phi'}{\Phi}\right) \le \overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}(r, f(z)) + S(r, f(z)). \tag{3.7}$$

Combining (3.2), (3.4), (3.5), and (3.6), we have that

$$\begin{aligned} nT(r, f(z+c)) \le{}& 2m\left(r, \frac{\Phi'}{\Phi}\right) + m(r, f(z)) + m\left(r, \frac{(f^n(z+c))'}{f^n(z+c)}\right) \\ &+ m\left(r, \frac{[f(z)-s(z)]'}{f(z)-s(z)}\right) + 2\overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}(r, f(z+c)) + N(r, f(z)) \\ &+ \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}(r, f(z)) + S(r, f(z)). \end{aligned} \tag{3.8}$$

From (3.2) and Lemma 2.2, we deduce that $T(r, \Phi(z)) = O(T(r, f(z)))$. Therefore, we get that

$$m\left(r, \frac{f'(z+c)}{f(z+c)}\right) = S(r, f(z)), \qquad m\left(r, \frac{\Phi'}{\Phi}\right) = S(r, \Phi(z)) = S(r, f(z)). \tag{3.9}$$

By (3.2), (3.7), (3.9), and the First Fundamental Theorem, we can simplify (3.8) to be

$$(n-4)T(r, f(z)) \le 2\overline{N}\left(r, \frac{1}{\Phi(z)}\right) + S(r, f(z)). \tag{3.10}$$

Because $n \ge 5$, we deduce that

$$T(r, f(z)) \le C\overline{N}\left(r, \frac{1}{af^n(z+c) + f(z) - s(z)}\right) + S(r, f(z)). \tag{3.11}$$

If $af^n(z+c) + f(z) - s(z)$ has finite zeros, then $T(r, f(z)) = S(r, f(z))$, a contradiction. We complete the proof of the Theorem 1.2. □

Proof of Corollary 1.3. The proof of Corollary 1.3 is the same as the proof of Theorem 1.2; note that $f(z)$ is entire, some different places are stated below.

The poles of $\varphi_1(z) = (\Phi'/\Phi)[f(z) - s(z)] - [f(z) - s(z)]'$ come from the zeros of $\Phi(z)$. By the hypothesis, we ignore the poles of $s(z)$. If z_0 be a zero of $\Phi(z)$, then z_0 is a simple pole of $\varphi_1(z)$. Using the Lemma 2.3, we can get that

$$N\left(r, \frac{\Phi'}{\Phi}[f(z) - s(z)] - [f(z) - s(z)]'\right) \leq \overline{N}\left(r, \frac{1}{\Phi(z)}\right) + S(r, f(z)). \tag{3.12}$$

The zeros of $\Phi(z)$ and the zeros of $f(z+c)$ compose the poles of $\varphi_2(z) = (f^n(z+c))'/f^n(z+c) - \Phi'/\Phi$. If z_0 is a zero of $\Phi(z)$ or zero of $f(z+c)$, then z_0 is a simple pole of $\varphi_2(z)$. Therefore, we conclude that

$$N\left(r, \frac{(f^n(z+c))'}{f(z+c)} - \frac{\Phi'}{\Phi}\right) \leq \overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f(z)). \tag{3.13}$$

Combining (3.2), (3.4), (3.5), and (3.12), we have that

$$\begin{aligned} nT(r, f(z+c)) \leq{}& 2m\left(r, \frac{\Phi'}{\Phi}\right) + m(r, f(z)) + m\left(r, \frac{(f^n(z+c))'}{f^n(z+c)}\right) \\ &+ m\left(r, \frac{[f(z) - s(z)]'}{f(z) - s(z)}\right) + 2\overline{N}\left(r, \frac{1}{\Phi(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f(z)). \end{aligned} \tag{3.14}$$

By (3.2), (3.13), (3.9), and the First Fundamental Theorem, we can simplify (3.14), to be

$$(n-2)T(r, f(z)) \leq 2\overline{N}\left(r, \frac{1}{\Phi(z)}\right) + S(r, f(z)). \tag{3.15}$$

Because $n \geq 3$, we deduce that

$$T(r, f(z)) \leq C\overline{N}\left(r, \frac{1}{af^n(z+c) + f(z) - s(z)}\right) + S(r, f(z)). \tag{3.16}$$

If $af^n(z+c) + f(z) - s(z)$ has finite zeros, then $T(r, f(z)) = S(r, f(z))$, a contradiction. The proof of Corollary 1.3 is complete. □

Proof of Theorem 1.4. By Lemma 2.7, we write $f(z)$ as follows

$$f(z) = z^k \mathbf{E}(z) e^{Q(z)}, \tag{3.17}$$

where $\mathbf{E}(z)$ is the canonical product of f formed with the nonnull zeros of f, and $Q(z)$ is a polynomial of degree $\leq \sigma(f)$.

Since 0 is the Borel exceptional value of $f(z)$, by Definition 1.1 and Lemmas 2.6 and 2.8, we can rewrite $f(z)$ as the form

$$f(z) = P(z)e^{sz^k}, \tag{3.18}$$

where $P(z)$ is an entire function with $\sigma(P) < \sigma(f) = k$, $s(\neq 0)$ is a constant, k is a positive integer. Thus

$$f(z+c) = P(z+c)P_1(z)e^{sz^k}, \tag{3.19}$$

where

$$P_1(z) = e^{skz^{k-1}+\cdots+sc^k}, \qquad \sigma(P_1) = k-1. \tag{3.20}$$

Now we suppose that $\tau(H-b) < \sigma(f)$. By Lemma 2.1, we see that $\sigma(H) = \sigma(f) = \sigma(H-b)$, so that $\tau(H-b) < \sigma(H-b) = \sigma(f) = k$ and $H(z)-b$ can be rewritten as the form

$$H(z) - b = q(z)e^{\beta z^k}, \tag{3.21}$$

where $\beta(\neq 0)$ is a constant, $q(z)$ is an entire function of

$$\sigma(q) \le \max\{\tau(H-b), k-1\}. \tag{3.22}$$

By (3.18)–(3.20), we get

$$\Big(\lambda P(z)P(z+c)P_1(z) + \mu P^2(z)\Big)e^{2sz^k} - b = q(z)e^{\beta z^k}. \tag{3.23}$$

Since $P(z)P(z+c)P_1(z) \not\equiv 0$ and $q(z) \not\equiv 0$, by comparing growths of both sides of (3.23), we see that $\beta = 2s$. Thus, by (3.23), we have

$$\Big[\lambda P(z)P(z+c)P_1(z) + \mu P^2(z) - q(z)\Big]e^{2sz^k} - b = 0. \tag{3.24}$$

By Lemma 2.5 and (3.24), we get that $b = 0$. This contradicts our assumption that $b \neq 0$. Hence $\tau(H-b) = \sigma(f)$.

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. We suppose that $f(z)$ has infinitely many multiple zeros. If $a = 0$, then $H(z)$ has obviously infinitely many zeros. Now we suppose that $a \neq 0$. If $H(z) - a$ has only finitely many zeros, then $H(z) - a$ can be rewritten as the form

$$H(z) - a = f(z)\left[\lambda f(z+c) + \mu f(z)\right] - a = p(z)e^{q(z)}, \tag{3.25}$$

where $p(z)$, $q(z)$ are polynomials, and $p(z) \not\equiv 0$, $\deg q(z) \ge 1$.

Differentiating (3.25), we obtain

$$[f(z)(\lambda f(z+c)) + \mu f(z)]' = (p'(z) + P(z)q'(z))e^{q(z)}. \tag{3.26}$$

From (3.25), we get $e^{q(z)} = (f(z)[\lambda f(z+c) + \mu f(z)] - a)/p(z)$. Substituting $e^{q(z)} = (f(z)[\lambda f(z+c) + \mu f(z)] - a)/p(z)$ into (3.26), we have

$$\begin{aligned}\frac{[f(z)(\lambda f(z+c)) + \mu f(z)]'}{f(z)(\lambda f(z+c) + \mu f(z))} = {} & \frac{p'(z) + p(z)q'(z)}{p(z)} \\ & - a\frac{p'(z) + p(z)q'(z)}{p(z)} \frac{1}{f(z)(\lambda f(z+c)) + \mu f(z)}.\end{aligned} \tag{3.27}$$

Since $f(z)$ has infinitely many multiple zeros, there is a multiple zero z_0 such that $|z_0|$ is sufficiently large and $p(z_0) \neq 0$, $p'(z_0) + p(z_0)q'(z_0) \neq 0$. Thus, the right side of (3.27) has a multiple pole at z_0, but the left side of (3.27) has only a simple pole at z_0. This is a contradiction.

Hence $H(z)$ takes any value $a \in \mathbb{C}$ infinitely often.

The proof of Theorem 1.5 is complete. □

At last, for further study, we pose a question.

Question. If $n \leq 4$ in Theorem 1.2, what will happen?

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