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*Research Article*

# Hybrid Projective Synchronization for Two Identical Fractional-Order Chaotic Systems

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A hybrid projective synchronization scheme for two identical fractional-order chaotic systems is proposed in this paper. Based on the stability theory of fractional-order systems, a controller for the synchronization of two identical fractional-order chaotic systems is designed. This synchronization scheme needs not to absorb all the nonlinear terms of response system. Hybrid projective synchronization for the fractional-order Chen chaotic system and hybrid projective synchronization for the fractional-order hyperchaotic Lu system are used to demonstrate the validity and feasibility of the proposed scheme.

## 1. Introduction

Most recently, many authors begin to investigate the chaotic dynamics and synchronization for fractional-order dynamical systems [1–6]. Chaos synchronization of the fractional-order systems is just beginning to attract some attention due to its potential applications in secure communications and control processing [7–12]. Several types of chaos synchronization are well known, which include complete synchronization (CS), antisynchronization (AS), phase synchronization, generalized synchronization (GS), projective synchronization (PS), and modified projective synchronization (MPS). Among all patterns of synchronization, the most noticeable one may be projective synchronization (PS), which was first studied by Mainieri and Rehacek [13]. Projective synchronization (PS) has been extensively considered because it can obtain faster communication. The drive and response system could be synchronized up to a scaling factor in projective synchronization. In application to secure communications, this proportional feature can be used to extend

binary digital to $M$-nary digital communication for getting faster communication [14, 15].

However, most of projective synchronizations for the fractional-order systems have concentrated on studying the same scaling factor [16–19], and some projective synchronization schemes [16, 17] are suitable for a class of fractional-order systems or for some special fractional-order systems [19], and all the nonlinear terms of response system was absorbed in some previous works. Moreover, in order to increase the degree of secrecy for secure communications, the same scaling factor in PS can be replaced by vector function factor. Motivated by the above discussions, we propose a hybrid projective synchronization (HPS) scheme for two identical fractional-order chaotic systems in this paper. Hybrid projective synchronization (HPS) is a more general definition of projective synchronization, in which the drive system and response system could be synchronized up to a vector function factor. HPS is different from the PS. Furthermore, HPS could be used to get more secure communication than PS in application to secure communications, because it is obvious that the unpredictability of the vector function factor in HPS is more than that of the same scaling factor in PS. The main contribution of this paper is as follows: the HPS scheme in this paper is suitable for a large number of fractional-order systems (not for special fractional-order system), and this HPS scheme needs not to absorb all the nonlinear terms of response system. This is different from some previous works [16, 17, 19–21].

To illustrate the effectiveness of the proposed HPS scheme in this paper, the HPS for the fractional-order Chen system and HPS for the fractional-order hyperchaotic Lu system are investigated. Numerical simulations are used to demonstrate the effectiveness of the proposed schemes. The organization of this paper is as follows. In Section 2, the definition of HPS is given, and a HPS scheme for two identical fractional-order chaotic systems is presented. In Section 3, two groups of examples are used to verify the effectiveness of the proposed scheme. The conclusion is finally drawn in Section 4.

## 2. Hybrid Projective Synchronization Scheme

There are several definitions of fractional derivatives. In this paper, the Caputo-type fractional derivative defined will be used. The Caputo definition of the fractional derivative, which is sometimes called smooth fractional derivative, is described as

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, \quad m-1 < q < m, \tag{2.1}$$

where $0 < q \leq 1$ is fractional order and $d^q/dt^q$ denote the Caputo definition of the fractional derivative. $m$ is the smallest integer larger than $q$, and $f^{(m)}(t)$ is the $m$-order derivative in the usual sense. $\Gamma(\bullet)$ is the gamma function.

The fractional-order chaotic drive and response systems can be written as follows, respectively:

$$\frac{d^q x}{dt^q} = f(x), \tag{2.2}$$

$$\frac{d^q y}{dt^q} = g(y) + \Omega(x, y), \tag{2.3}$$

where $x \in R^n$, $y \in R^n$ are state vectors of the drive system (2.2) and the response system (2.3) and $f, g : R^n \to R^n$ are continuous vector functions, respectively. $\Omega(x, y)$ is a vector controller.

*Definition 2.1.* For the drive system (2.2) and the response system (2.3), it is said to be hybrid projective synchronization (HPS) if there exists an $n \times n$ reversible matrix $A$ such that

$$\lim_{t \to \infty} \|Ay - x\| = 0, \tag{2.4}$$

where $\|\cdot\|$ is the Euclidean norm.

*Remark 2.2.* If $A = I$, and $I$ is a unit matrix, then this synchronization is called complete synchronization (CS); if $A = -I$, then this synchronization is called antisynchronization (AS); if $A = aI$, and $a \neq \pm 1$ is a nonzero real constant, then this synchronization is called projective synchronization (PS); if $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, and $a_1, a_2, \ldots, a_n$ are not the same nonzero constant, then this synchronization is called modified projective synchronization (MPS). Therefore, CS, AS, PS, and MPS are the special cases of the hybrid projective synchronization scheme (HPS) in this paper.

In order to realize HPS for the fractional-order chaotic system (2.2), we take the fractional-order chaotic system (2.2) as drive system and construct a response system as follows:

$$\frac{d^q y}{dt^q} = A^{-1}\left[f(Ay) + \Omega(x, y)\right], \tag{2.5}$$

where $A^{-1}$ is the reverse matrix of the reversible matrix $A$, $y \in R^n$ are state vector of the response system (2.5), and $\Omega(x, y)$ is a controller which will be designed.

Define the HPS errors between the response system (2.5) and the drive system (2.2) as

$$e = Ay - x, \tag{2.6}$$

where

$$\begin{gathered} e = (e_1, e_2, \ldots, e_n)^{\mathrm{T}}, \\ e_i = \left(\sum_{j=1}^{n} a_{ij} y_j\right) - x_i \quad (i, j = 1, 2, \ldots n). \end{gathered} \tag{2.7}$$

Let

$$f(Ay) - f(x) = F(x, e). \tag{2.8}$$

Now, we assume that the errors vector $e$ can be divided into $e_\alpha = (e_{s_1}, \ldots, e_{s_m})^{\mathbf{T}}$ and $e_\beta = (e_{s_{m+1}}, \ldots, e_{s_n})^{\mathbf{T}}$, such that $F(x,e)$ has the form of

$$F(x,e) = \begin{pmatrix} B_\alpha e_\alpha + h_1(x, e_\alpha, e_\beta) \\ B_\beta e_\beta + h_{21}(x, e_\alpha, e_\beta) + h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{2.9}$$

where $h_1(x, e_\alpha, e_\beta) \in R^m$, $h_{21}(x, e_\alpha, e_\beta) \in R^{n-m}$, $h_{22}(x, e_\alpha, e_\beta) \in R^{n-m}$ and $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$, respectively. $B_\alpha \in R^{m\times m}$ and $B_\beta \in R^{(n-m)\times(n-m)}$ are real constant matrix.

Rewrite the controller $\Omega(x,y)$ as follows:

$$\Omega(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix}, \tag{2.10}$$

where $\mu_\alpha(x,e) \in R^m$ and $\mu_\beta(x,e) \in R^{n-m}$, respectively.

Now, Theorem 2.3 is given based on the previously mentioned conditions in order to achieve the HPS between the drive system (2.2) and the response system (2.5).

**Theorem 2.3.** *Choose the following controller:*

$$\Omega(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} \Lambda_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ \Lambda_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{2.11}$$

*where $\Lambda_\alpha \in R^{m\times m}$ and $\Lambda_\beta \in R^{(n-m)\times(n-m)}$ are suitable constant matrix, respectively.*

If all the eigenvalues of $B_\alpha + \Lambda_\alpha$ satisfy $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, \ldots, m)$ and all the eigenvalues of $B_\beta + \Lambda_\beta$ satisfy $|\arg \lambda_j| > 0.5\pi q$ $(j = 1, 2, \ldots, n-m)$, then hybrid projective synchronization between the drive system (2.2) and the response system (2.5) can be achieved.

*Proof.* According to the drive system (2.2) and the response system (2.5), the error dynamic system of hybrid projective synchronization can be obtained as follows:

$$\frac{d^q e}{dt^q} = \frac{A d^q y}{dt^q} - \frac{d^q x}{dt^q} = f(Ay) - f(x) + \Omega(x,y) = F(x,e) + \mu(x,e). \tag{2.12}$$

According to (2.9) and (2.10), the error dynamic system (2.12) can be rewritten as

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q} &= B_\alpha e_\alpha + h_1(x, e_\alpha, e_\beta) + \mu_\alpha(x,e), \\ \frac{d^q e_\beta}{dt^q} &= B_\beta e_\beta + h_{21}(x, e_\alpha, e_\beta) + h_{22}(x, e_\alpha, e_\beta) + \mu_\beta(x,e), \end{aligned} \tag{2.13}$$

because

$$\begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} \Lambda_\alpha e_\alpha - h_1(x,e_\alpha,e_\beta) \\ \Lambda_\beta e_\beta - h_{22}(x,e_\alpha,e_\beta) \end{pmatrix}. \tag{2.14}$$

So,

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q} &= (B_\alpha + \Lambda_\alpha) e_\alpha, \\ \frac{d^q e_\beta}{dt^q} &= (B_\beta + \Lambda_\beta) e_\beta + h_{21}(x,e_\alpha,e_\beta). \end{aligned} \tag{2.15}$$

Because all the eigenvalues of $B_\alpha + \Lambda_\alpha$ satisfy $|\arg \lambda_i| > 0.5\pi q$ $(i = 1,2,\ldots,m)$, according to the stability theory of linear fractional-order systems [22], the equilibrium point $e_i = 0$ $(i = 1,2,\ldots,m)$ in the first equation of system (2.15) is asymptotically stable, which indicates $\lim_{t\to+\infty} e_\alpha = 0$.

Since $\lim_{t\to+\infty} e_\alpha = 0$ and $\lim_{e_\alpha\to 0} h_{21}(x,e_\alpha,e_\beta) = 0$, therefore when time $t \to +\infty$, the second equation of system (2.15) can be changed as

$$\frac{d^q e_\beta}{dt^q} = (B_\beta + \Lambda_\beta) e_\beta, \tag{2.16}$$

because all the eigenvalues of $B_\beta + \Lambda_\beta$ satisfy $|\arg \lambda_j| > 0.5\pi q$ $(j = 1,2,\ldots,n-m)$. According to the stability theory of linear fractional-order systems [22], the equilibrium point $e_i = 0$ $(i = 1,2,\ldots,n-m)$ of system (2.16) is asymptotically stable, which indicates $\lim_{t\to+\infty} e_\beta = 0$.

According to $\lim_{t\to+\infty} e_\alpha = 0$ and $\lim_{t\to+\infty} e_\beta = 0$, the hybrid projective synchronization between the drive system (2.2) and the response system (2.5) can be achieved. This finishes the proof. □

*Remark 2.4.* In order to use the stability theory of linear fractional-order systems [22], the controller $\Omega(x,y)$ or $\mu(x,e)$ are chosen as $\begin{pmatrix} \Lambda_\alpha e_\alpha - h_1(x,e_\alpha,e_\beta) \\ \Lambda_\beta e_\beta - h_{22}(x,e_\alpha,e_\beta) \end{pmatrix}$. Moreover, the nonlinear term $h_{21}(x,e_\alpha,e_\beta) \in R^{n-m}$ in the error dynamic system (2.13) or response system (2.5) is preserved; this is different from some previous works [17, 19–21] which need to absorb all the nonlinear terms of response system or error dynamic system.

*Remark 2.5.* For the complex fractional-order multiscroll chaotic systems [23–25] and the complex dynamical network or the small-world dynamical networks [26–28], the hybrid projective synchronization would be much more complex. Further work on this issue is an ongoing research topic in our group.

## 3. Applications

In order to illustrate the effectiveness of the proposed hybrid projective synchronization scheme obtained in Section 2, two examples are considered in this section, which are HPS for the fractional-order Chen system and HPS for the fractional-order hyperchaotic Lu system.

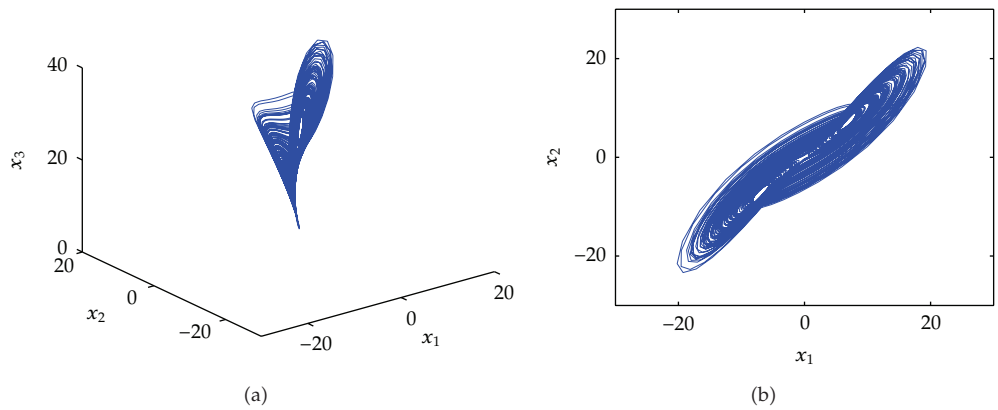


**Figure 1:** Chaotic attractor of fractional-order Chen chaotic system (3.1) for $q = 0.9$.

### *3.1. HPS for the Fractional-Order Chen System*

The fractional-order Chen system [23] is described as follows:

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= 35(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} &= -7x_1 + 28x_2 - x_1 x_3, \\ \frac{d^q x_3}{dt^q} &= x_1 x_2 - 3x_3. \end{aligned} \tag{3.1}$$

Tavazoei and Haeri pointed out that fractional-order Chen system (3.1) exhibits chaotic behavior for $q \geq 0.83$ [29]. The chaotic attractor of fractional-order Chen system for $q = 0.9$, is depicted in Figure 1.

According to the HPS scheme presented in the previous section, the response system is described by

$$\begin{pmatrix} \dfrac{d^q y_1}{dt^q} \\ \dfrac{d^q y_2}{dt^q} \\ \dfrac{d^q y_3}{dt^q} \end{pmatrix} = A^{-1} \left[ \begin{pmatrix} 35\left(\sum_{j=1}^{3} a_{2j} y_j - \sum_{j=1}^{3} a_{1j} y_j\right) \\ -7 \times \sum_{j=1}^{3} a_{1j} y_j + 28 \times \sum_{j=1}^{3} a_{2j} y_j - \sum_{j=1}^{3} a_{1j} y_j \times \sum_{j=1}^{3} a_{3j} y_j \\ \sum_{j=1}^{3} a_{1j} y_j \times \sum_{j=1}^{3} a_{2j} y_j - 3 \times \sum_{j=1}^{3} a_{3j} y_j \end{pmatrix} + \Omega(x, y) \right], \tag{3.2}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a reversible matrix and $A^{-1}$ is its reverse matrix.

Now, the term of $f(Ay) - f(x) = F(x, e)$ is yielded firstly:

$$f(Ay) - f(x) = F(x, e) = \begin{pmatrix} 35e_2 - 35e_1 \\ -7e_1 + 28e_2 - x_3 e_1 - x_1 e_3 - e_1 e_3 \\ x_2 e_1 + x_1 e_2 + e_1 e_2 - 3e_3 \end{pmatrix}. \tag{3.3}$$

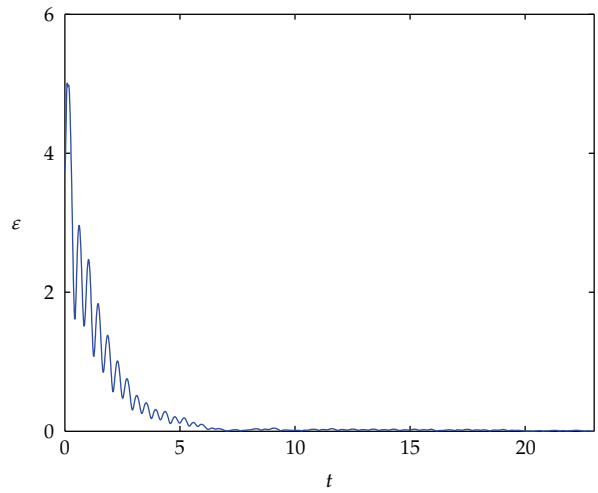


**Figure 2:** The HPS result between the drive system (3.1) and its response system (3.2) for $q = 0.9$.

So, we can choose $e_\alpha = e_1$ and $e_\beta = (e_2, e_3)^{\mathrm{T}}$. Therefore, $h_1(x, e_\alpha, e_\beta) = -35e_2$, $h_{21}(x, e_\alpha, e_\beta) = \left(\begin{smallmatrix} -7e_1 - x_3e_1 - e_1e_3 \\ x_2e_1 + e_1e_2 \end{smallmatrix}\right)$, $h_{22}(x, e_\alpha, e_\beta) = \left(\begin{smallmatrix} -x_1e_3 \\ x_1e_2 \end{smallmatrix}\right)$, $B_\alpha = -35$, and $B_\beta = \left(\begin{smallmatrix} 28 & 0 \\ 0 & -3 \end{smallmatrix}\right)$, respectively. Obviously, $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$.

According to Theorem 2.3 in Section 2, the controller $\Omega(x, y)$ can be chosen as follows:

$$\Omega(x, y) = \begin{pmatrix} \mu_\alpha(x, e) \\ \mu_\beta(x, e) \end{pmatrix} = \begin{pmatrix} \Lambda_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ \Lambda_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{3.4}$$

where $\mu_\alpha(x, e) = \Lambda_\alpha e_1 - 35e_2$, $\mu_\beta(x, e) = \Lambda_\beta\left(\begin{smallmatrix} e_2 \\ e_3 \end{smallmatrix}\right) - \left(\begin{smallmatrix} -x_1e_3 \\ x_1e_2 \end{smallmatrix}\right)$, $\Lambda_\alpha \in R^{1\times 1}$, and $\Lambda_\beta \in R^{2\times 2}$, respectively. Therefore, choose suitable matrix $\Lambda_\alpha \in R^{1\times 1}$ and $\Lambda_\beta \in R^{2\times 2}$. If the eigenvalues of $B_\alpha + \Lambda_\alpha$ satisfy $|\arg \lambda| > 0.5\pi q$ and if all the eigenvalues of $B_\beta + \Lambda_\beta$ satisfy $|\arg \lambda_j| > 0.5\pi q$ $(j = 1, 2)$, then hybrid projective synchronization between the drive system (3.1) and its response system (3.2) can be achieved.

For example, choose reversible matrix

$$A = \begin{pmatrix} -2 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -0.5 \end{pmatrix}, \qquad \Lambda_\alpha = 30, \qquad \Lambda_\beta = \begin{pmatrix} -24 & 10 \\ -5 & 0 \end{pmatrix}. \tag{3.5}$$

So, the eigenvalues of $B_\alpha + \Lambda_\alpha$ are $-5$, and all the eigenvalues of $B_\beta + \Lambda_\beta$ are $0.5 \pm 6.1441\,j$ and $|\arg \lambda(B_\beta + \Lambda_\beta)| = 0.9483 \times \pi/2 > 0.5\pi q$ $(q = 0.9)$, respectively. Therefore, the hybrid projective synchronization between drive system (3.1) and its response system (3.2) can be achieved. The corresponding numerical result is shown in Figure 2, in which the initial

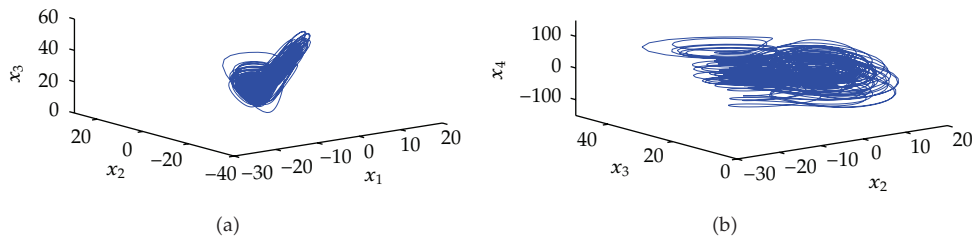


**Figure 3:** Chaotic attractor of fractional-order hyperchaotic Lu system (3.6) for $q = 0.95$.

conditions are $(x_{10}, x_{20}, x_{30}) = (3, 4, 5)$ for the drive system (3.1), and $(y_{10}, y_{20}, y_{30}) = (8, 10, 4)$ for the response system (3.2), respectively, and $\varepsilon = (\sum_{i=1}^{3} e_i^2)^{1/2}$.

### *3.2. HPS for Fractional-Order Hyperchaotic Lǔ System*

Min et al. reported a fractional-order hyperchaotic Lu system [30, 31] based on the hyperchaotic Lu system, which is described as follows:

$$\begin{aligned}
\frac{d^q x_1}{dt^q} &= 36(x_2 - x_1) + x_4, \\
\frac{d^q x_2}{dt^q} &= 20x_2 - x_1 x_3, \\
\frac{d^q x_3}{dt^q} &= x_1 x_2 - 3x_3, \\
\frac{d^q x_4}{dt^q} &= 1.3x_4 + x_1 x_3.
\end{aligned} \tag{3.6}$$

The chaotic attractor of fractional-order hyperchaotic Lu system for $q = 0.95$, is shown in Figure 3.

Taking system (3.6) as the drive system, according to the HPS scheme presented in Section 2, the response system is described by

$$\begin{pmatrix} \dfrac{d^q y_1}{dt^q} \\ \dfrac{d^q y_2}{dt^q} \\ \dfrac{d^q y_3}{dt^q} \\ \dfrac{d^q y_4}{dt^q} \end{pmatrix} = A^{-1} \left[ \begin{pmatrix} 36 \times \left( \sum_{j=1}^{4} a_{2j} y_j - \sum_{j=1}^{4} a_{1j} y_j \right) + \sum_{j=1}^{4} a_{4j} y_j \\ 20 \times \sum_{j=1}^{4} a_{2j} y_j - \sum_{j=1}^{4} a_{1j} y_j \times \sum_{j=1}^{4} a_{3j} y_j \\ \sum_{j=1}^{4} a_{1j} y_j \times \sum_{j=1}^{4} a_{2j} y_j - 3 \times \sum_{j=1}^{4} a_{3j} y_j \\ 1.3 \times \sum_{j=1}^{4} a_{4j} y_j + \sum_{j=1}^{4} a_{1j} y_j \times \sum_{j=1}^{4} a_{3j} y_j \end{pmatrix} + \Omega(x, y) \right], \tag{3.7}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ is a reversible matrix and $A^{-1}$ is its reverse matrix. As the same as what is mentioned previously, we can obtain

$$f(Ay) - f(x) = F(x,e) = \begin{pmatrix} 36e_2 - 36e_1 + e_4 \\ 20e_2 - x_3e_1 - x_1e_3 - e_1e_3 \\ x_2e_1 + x_1e_2 + e_1e_2 - 3e_3 \\ 1.3e_4 + x_3e_1 + x_1e_3 + e_1e_3 \end{pmatrix}. \tag{3.8}$$

Now, we can choose $e_\alpha = (e_1, e_2)^{\mathbf{T}}$ and $e_\beta = (e_3, e_4)^{\mathbf{T}}$. Therefore, $h_1(x, e_\alpha, e_\beta) = \left(\begin{smallmatrix} e_4 \\ -x_3e_1 - x_1e_3 - e_1e_3 \end{smallmatrix}\right)$, $h_{21}(x, e_\alpha, e_\beta) = \left(\begin{smallmatrix} x_2e_1 + x_1e_2 + e_1e_2 \\ x_3e_1 + e_1e_3 \end{smallmatrix}\right)$, $h_{22}(x, e_\alpha, e_\beta) = \left(\begin{smallmatrix} 0 \\ x_1e_3 \end{smallmatrix}\right)$, $B_\alpha = \left(\begin{smallmatrix} -36 & 36 \\ 0 & 20 \end{smallmatrix}\right)$, and $B_\beta = \left(\begin{smallmatrix} -3 & 0 \\ 0 & 1.3 \end{smallmatrix}\right)$, respectively. Obviously, $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$.

According to Theorem 2.3 in Section 2, the controller $\Omega(x, y)$ can be chosen as follows:

$$\Omega(x,y) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} \Lambda_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ \Lambda_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{3.9}$$

where $\mu_\alpha(x,e) = \Lambda_\alpha \left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} e_4 \\ -x_3e_1 - x_1e_3 - e_1e_3 \end{smallmatrix}\right)$, $\mu_\beta(x,e) = \Lambda_\beta \left(\begin{smallmatrix} e_2 \\ e_3 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ x_1e_3 \end{smallmatrix}\right)$, $\Lambda_\alpha \in R^{2\times 2}$ and $\Lambda_\beta \in R^{2\times 2}$, respectively. Therefore, choose suitable matrix $\Lambda_\alpha \in R^{2\times 2}$, and $\Lambda_\beta \in R^{2\times 2}$. If the eigenvalues of $B_\alpha + \Lambda_\alpha$ satisfy $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2)$ and if all the eigenvalues of $B_\beta + \Lambda_\beta$ satisfy $|\arg \lambda_j| > 0.5\pi q$ $(j = 1, 2)$, then the hybrid projective synchronization between the drive system (3.6) and its response system (3.7) can be achieved.

For example, choose reversible matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -0.5 \end{pmatrix}, \qquad \Lambda_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & -30 \end{pmatrix}, \qquad \Lambda_\beta = \begin{pmatrix} 0 & 0 \\ 0 & -2.3 \end{pmatrix}. \tag{3.10}$$

So, the eigenvalues of $B_\alpha + \Lambda_\alpha$ are $-36$ and $-10$, and the eigenvalues of $B_\beta + \Lambda_\beta$ are $-3$ and $-1$, respectively. Therefore, the hybrid projective synchronization between drive system (3.6) and its response system (3.7) can be achieved. The corresponding numerical result is shown in Figure 4, in which the initial conditions are $(x_{10}, x_{20}, x_{30}, x_{40}) = (3,4,5,6)$ for the drive system (3.6) and $(y_{10}, y_{20}, y_{30}, y_{40}) = (16,7,12,4)$ for the response system (3.7), respectively, and $\varepsilon = (\sum_{i=1}^{4} e_i^2)^{1/2}$.

## 4. Conclusions

We proposed a new synchronization scheme to achieve hybrid projective synchronization for two identical fractional-order chaotic systems in this paper. The drive system and response system could be synchronized up to a vector function factor, and this synchronization scheme needs not to absorb all the nonlinear terms of response system. The synchronization technique, based on stability theory of fractional-order systems, is simple and theoretically

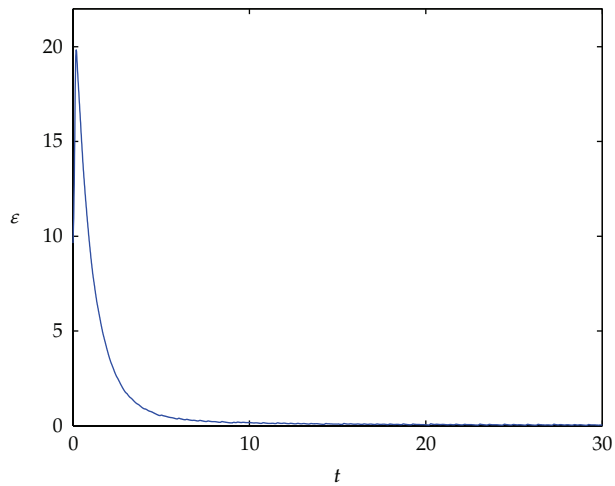


**Figure 4:** The HPS result between the drive system (3.6) and its response system (3.7) for $q = 0.95$.

rigorous. Numerical simulations are used to illustrate the effectiveness of the proposed synchronization method.

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