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Research Article

The Projection Pressure for Asymptotically Weak Separation Condition and Bowen's Equation

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Let $\{S_i\}_{i=1}^l$ be a weakly conformal iterated function system on $\mathbb{R}^d$ with attractor K. Let π be the canonical projection. In this paper we define a new concept called "projection pressure" $P_\pi(\varphi)$ for $\varphi \in C(\Sigma)$ and show the variational principle about the projection pressure under AWSC. Furthermore, we check that the zero of "projection pressure" still satisfies Bowen's equation. Using the root of Bowen's equation, we can get the Hausdorff dimension of the attractor K.

1. Introduction

Let $\{S_i : X \to X\}_{i=1}^l$ be a family of contractive maps on a nonempty closed set $X \subset \mathbb{R}^d$. Following Barnsley [1], we say that $\{S_i\}_{i=1}^l$ is an *iterated function system* (IFS) on X. Hutchinson [2] showed that there is a unique nonempty compact set $K \subset X$, called the *attractor* of $\{S_i\}_{i=1}^l$, such that $K = \cup_{i=1}^l S_i(K)$.

There are many references to compute the Hausdorff dimension of K or the Hausdorff dimension of multifractal spectrum, such as, [3–5]. Thermodynamic formalism played a significant role when we try to compute the Hausdorff dimension of K, especially the Bowen's equation. Usually, we call $P_J(t\psi) = 0$ the Bowen's equation, where P_J is the topological pressure of the map $f : J \to J$, and ψ is the geometric potential $\psi(z) = \log|f'(z)|$. The root of Bowen's equation always approaches the Hausdorff dimension of some sets. In [6], Bowen first discovered this equation while studying the Hausdorff dimension of quasicircles. Later Ruelle [7], Gatzouras and Peres [8] showed that Bowen's equation gives the Hausdorff dimension of J whenever f is a C^1 conformal map on a Riemannian manifold and J is a repeller. According to the method for calculating Hausdorff dimension of cookie-cutters presented by Bedford [9], Keller discussed the relation between classical pressure and dimension for IFS [10]. He concluded that if $\{S_i\}_{i=1}^{l}$ is a conformal IFS satisfying the disjointness condition with local energy function ψ, then the pressure function has a unique zero root $t_0 = \dim_H K$. In 2000, using the definition of Carathe' odory dimension characteristics, Barreira and Schmeling [11] introduced the notion of the u-dimension $\dim_u Z$ for positive functions u, showing that $\dim_u Z$ is the unique number t such that $P_Z(-tu) = 0$.

On the progress of calculating $\dim_H K$, [3–5] depend on the open set condition and separable condition. In fact, there are a lot of examples that do not satisfy this disjointness condition. Rao and Wen once discussed a kind of self-similar fractal with overlap structure called λ-Cantor set [12].

In order to study the Hausdorff dimension of an invariant measure μ for conformal and affine IFS with overlaps, Feng and Hu introduce a notion *projection entropy* (see [13]), which plays the similar role as the classical entropy of IFS satisfying the open set condition, and it becomes the classical entropy if the projection is finite to one.

Bedford pointed out that the Bowen's equation works if three elements are present: (i) conformal contractions, (ii) open set conditions, and (iii) subshift of finite-type (Markov) structure. Chen and Xiong [14] proved that subshift of finite-type (Markov) structure can be replaced by any subshift structure. In [15, 16], the authors defined projection pressure for two different types of IFS. In this paper, we consider projection pressure under asymptotically weak separation condition (AWSC) and check that Bowen's equation still holds.

2. The Projection Pressure for AWSC: Definition and Variational Principle

Let $\{S_i\}_{i=1}^{l}$ be an IFS on a closed set $X \subset \mathbb{R}^d$. Denote by K its attractor. Let $\Sigma = \{1,\dots,l\}^{\mathbb{N}}$ associated with the left shift σ. Let $M_\sigma(\Sigma)$ denote the space of σ-invariant measure on Σ, endowed with the weak-star topology, $C(X)$ the space of real-valued continuous functions of X, and $\pi : \Sigma \to K$ be the canonical projection defined by

$$\{\pi(x)\} = \bigcap_{n=1}^{\infty} S_{x_1} \circ S_{x_2} \circ \cdots \circ S_{x_n}(K), \quad \text{where } x = \{x_i\}_{i=1}^{\infty}. \tag{2.1}$$

A measure μ on K is called invariant (resp., ergodic) for the IFS if there is and invariant (resp., ergodic) measure ν on Σ such that $\mu = \nu \circ \pi^{-1}$.

Let $(\Omega, \mathcal{F}, \nu)$ be a probability space. For a sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$ and $f \in L^1(\Omega, \mathcal{F}, \nu)$, we denote by $E_\nu(f|\mathcal{A})$ the conditional expectation of f given $\mathcal{A}$. For countable $\mathcal{F}$-measurable partition ξ of Ω. We denote by $I_\nu(\xi|\mathcal{A})$ the conditional information of ξ given $\mathcal{A}$, which is given by the formular:

$$I_\nu(\xi \mid \mathcal{A}) = -\sum_{A \in \xi} \mathcal{X}_A \log E_\nu(\mathcal{X}_A \mid \mathcal{A}), \tag{2.2}$$

where $\mathcal{X}_A$ denote the characteristic function on $\mathcal{A}$.

The conditional entropy of ξ given $\mathcal{A}$, written $H_\nu(\xi|\mathcal{A})$ is defined by the formula $H_\nu(\xi|\mathcal{A}) = \int I_\nu(\xi|\mathcal{A}) d\nu$.

The above information and entropy are unconditional when $\mathcal{A} = \mathcal{N}$, the trivial σ-algebra consisting of sets of measure zero and one, and in this case we write

$$I_\nu(\xi \mid \mathcal{A}) = I_\nu(\xi), \qquad H_\nu(\xi \mid \mathcal{N}) = H_\nu(\xi). \tag{2.3}$$

Now we consider the space $(\Sigma, \mathcal{B}(\Sigma), m)$, where $\mathcal{B}(\Sigma)$ is the Borel σ-algebra on Σ and $m \in M_\sigma(\Sigma)$. Let $\mathcal{P}$ denote the Borel partition:

$$\mathcal{P} = \{ [j] : 1 \leq j \leq l \} \tag{2.4}$$

of Σ, where $[j] = \{(x_i)_{i=1}^{\infty} \in \Sigma,\ x_1 = j\}$. Let $\mathcal{I}$ denotes the σ-algebra:

$$\mathcal{I} = \left\{ B \in \mathcal{B}(\Sigma) : \sigma^{-1} B = B \right\}. \tag{2.5}$$

For convenience, we use γ to denote the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$. For $f \in C(X)$, denote $\|f\| = \sup_{x \in X} f(x)$ and $S_n f(x) = \sum_{i=0}^{n-1} f(\sigma^n x)$, for all $x \in X$. Let $\Sigma_n = \{[b] : [b] = (x_1, x_2, \ldots, x_n),\ x_i \in \Sigma,\ i = 1, \ldots, n\}$.

Definition 2.1. For any $m \in M_\sigma(\Sigma)$, we call

$$h_\pi(\sigma, m) = H_m\left(\mathcal{P} \mid \sigma^{-1}\pi^{-1}\gamma\right) - H_m\left(\mathcal{P} \mid \pi^{-1}\gamma\right) \tag{2.6}$$

the *projection entropy* of m under π w.r.t $\{S_i\}$, and we call

$$h_\pi(\sigma, m, x) = E_m(f \mid \mathcal{I})(x) \tag{2.7}$$

the *local projection entropy* of m at x under π w.r.t $\{S_i\}_{i=1}^{l}$, where f denote the function $I_m(\mathcal{P} | \sigma^{-1}\pi^{-1}\gamma) - I_m(\mathcal{P} | \sigma^{-1}\gamma)$.

It is clear that $h_\pi(\sigma, m) = \int h_\pi(\sigma, m, x) dm(x)$.

The following Lemma 2.2 gives the relation between the projection entropy and the classical entropy and the basic properties of the new entropy which are similar to the classical entropy's. For more details we can see Theorem 2.2 in [13].

Lemma 2.2. *Let $\{S_i\}_{i=1}^{l}$ be an IFS. Then*

(i) *For any $m \in M_\sigma(\Sigma)$, one has $0 \leq h_\pi(\sigma, m) \leq h(\sigma, m)$, where $h(\sigma, m)$ denotes the classical measure-theoretic entropy of m associated with σ.*

(ii) *The map $m \mapsto h_\pi(\sigma, m)$ is affine on $M_\sigma(\Sigma)$. Furthermore if $m = \int \nu d\mathbb{P}(\nu)$ is the ergodic decomposition of m, one has*

$$h_\pi(\sigma, m) = \int h_\pi(\sigma, \nu) d\mathbb{P}(\nu). \tag{2.8}$$

(iii) *For any $m \in M_\sigma(\Sigma)$, one has*

$$\lim_{n \to \infty} \frac{1}{n} I_m\left(\mathcal{P}_0^{n-1} \mid \pi^{-1}\gamma\right)(x) = h(\sigma, m, x) - h_\pi(\sigma, m, x), \tag{2.9}$$

for m-a.e. $x \in \Sigma$, where $h(\sigma, m, x)$ denotes the local entropy of m at x, that is, $h(\sigma, m, x) = I_m(\mathcal{P} \mid \sigma^{-1}\mathcal{B}(\Sigma))(x)$.

Definition 2.3. Let $k \in \mathbb{N}$ and $\nu \in M_{\sigma^k}(\Sigma)$. Define

$$h_\pi\left(\sigma^k, \nu\right) = H_\nu\left(\mathcal{P}_0^{k-1} \mid \sigma^{-k}\pi^{-1}\gamma\right) - H_\nu\left(\mathcal{P}_0^{k-1} \mid \pi^{-1}\gamma\right). \tag{2.10}$$

The term $h_\pi(\sigma^k, \nu)$ can be viewed as the projection measure-theoretic entropy of ν w.r.t. the IFS $\{S_{i_1} \circ \cdots \circ S_{i_k} : 1 \le i_j \le l \text{ for } 1 \le j \le k\}$. The following lemma exploits the connection between $h_\pi(\sigma^k, \nu)$ and $h_\pi(\sigma, m)$, where $m = (1/k)\sum_{i=0}^{k-1} \nu \circ \sigma^{-i}$.

Lemma 2.4. *Let $k \in \mathbb{N}$ and $\nu \in M_{\sigma^k}\Sigma$. Set $m = (1/k)\sum_{i=0}^{k-1} \nu \circ \sigma^{-i}$. Then m is σ-invariant, and $h_\pi(\sigma, \nu) = (1/k)h_\pi(\sigma^k, \nu) = (1/k)h_\pi(\sigma^k, m)$.*

Proof. See Proposition 4.3 in [13]. □

Definition 2.5. An IFS $\{S_i\}_{i=1}^l$ on a compact set $X \subset \mathbb{R}^d$ is said to satisfy the *asymptotically weak separation condition* (AWSC), if

$$\lim_{n\to\infty} \frac{1}{n} \log t_n = 0, \tag{2.11}$$

where t_n is given by

$$t_n = \sup_{x\in\mathbb{R}^d} \#\left\{S_u : u \in \{1, \ldots, l\}^n,\ x \in S_u(K)\right\}, \tag{2.12}$$

here K is the attractor of $\{S_i\}_{i=1}^l$.

Lemma 2.6. *Let $\{S_i\}_{i=1}^l$ be an IFS with attractor K. Suppose that Ω is a subset of $\{1, \ldots, l\}$ such that there is a map g: $\{1, \ldots, l\} \to \Omega$ so that*

$$S_i = S_{g(i)} \quad (i = 1, \ldots, l). \tag{2.13}$$

Let $(\Omega^{\mathbb{N}}, \widetilde{\sigma})$ denote the one-side full shift over Ω. Define G: $\Sigma \to \Omega^{\mathbb{N}}$ by $(x_j)_{j=1}^\infty \mapsto (g(x_j))_{j=1}^\infty$. Then

(i) *K is also the attractor of $\{S_i\}_{i\in\Omega}$. Moreover, if one lets $\widetilde{\pi} : \Omega^{\mathbb{N}} \to K$ denote the canonical projection w.r.t. $\{S_i\}_{i\in\Omega}$, then one has $\pi = \widetilde{\pi} \circ G$.*

(ii) *Let $m \in M_\sigma(\Sigma)$. Then $\nu = m \circ G^{-1} \in M_{\widetilde{\sigma}}(\Omega^{\mathbb{N}})$. Furthermore, $h_\pi(\sigma, m) = h_{\widetilde{\pi}}(\widetilde{\sigma}, \nu)$.*

Proof. See Lemma 4.23 in [13]. □

Lemma 2.7. *Let $\{S_i\}_{i=1}^l$ be an IFS with attractor$K \subset \mathbb{R}^d$. Assume that*

$$\#\{1 \le i \le l : x \in S_i(K)\} \le k \tag{2.14}$$

for some $k \in \mathbb{N}$ and each $x \in \mathbb{R}^d$. Then $h_\pi(\sigma, m) \ge h(\sigma, m) - \log k$ for any $m \in M_\sigma(\Sigma)$.

Proof. See Lemma 4.21 in [13]. □

Lemma 2.8. *Let $a_1, a_2, \ldots, a_k$ be given real numbers. If $p_i \geq 0$ and $\sum_{i=0}^{k} p_i = 1$, then $\sum_{i=0}^{k} p_i(a_i - \log p_i) \leq \log(\sum_{i=0}^{k} e^{a_i})$ and equality holds iff $p_i = e^{a_i} / \sum_{j=1}^{k} e^{a_j}$.*

Proof. See Lemma 9.9 in [17]. □

For convenience, for $n \in \mathbb{N}$, write $\Sigma_n = \{1, \ldots, l\}^n$. According to Lemma 2.6 there is a set $\Omega_n \subset \Sigma_n$ and a map $g : \Sigma_n \to \Omega_n$ such that $S_u = S_{g(u)}$ for $u \in \Sigma_n$. Let $(\Omega_n^{\mathbb{N}}, T)$ denote the one-sided full shift space over the alphabet $\Omega_n^{\mathbb{N}}$ and ξ_n denote the natural generator. Let $G : \Sigma \to \Omega_n^{\mathbb{N}}$ be defined by

$$(x_i)_{i=1}^{\infty} \longmapsto \left(g\left(x_{jn+1} \cdots x_{(j+1)n}\right)\right)_{j=0}^{\infty}. \tag{2.15}$$

Theorem 2.9. *Suppose an IFS $\{S_i\}_{i=1}^{l}$ satisfies the AWSC with attractor K and $f : \Sigma \to \mathbb{R}$ is continuous. Then*

$$\lim_{n\to\infty} \frac{1}{n}\left(\log \sum_{[u]\in\xi_n} \sup_{x\in G^{-1}[u]} e^{S_n f(x)}\right) = \sup\left\{h_\pi(\sigma, m) + \int f\,dm : m \in M_\sigma(\Sigma)\right\}. \tag{2.16}$$

Proof. We divided the proof into two steps.

Step 1.

$$\liminf_{n\to\infty} \frac{1}{n}\left(\log \sum_{[u]\in\xi_n} \sup_{x\in G^{-1}[u]} e^{S_n f(x)}\right) \geq \sup\left\{h_\pi(\sigma, m) + \int f\,dm : m \in M_\sigma(\Sigma)\right\}. \tag{2.17}$$

For arbitrary $n \in \mathbb{N}, m \in M_\sigma(\Sigma)$, then $m \in M_{\sigma^n}(\Sigma)$. By Lemma 2.8, we have

$$\begin{aligned}
\log \sum_{[u]\in\xi_n} \sup_{x\in G^{-1}[u]} e^{S_n f(x)} &\geq \sum_{[u]\in\xi_n} m\circ G^{-1}([u])\left(\sup_{x\in G^{-1}[u]} S_n f(x) - \log m\circ G^{-1}([u])\right) \\
&= H_{m\circ G^{-1}}(\xi_n) + \sum_{[u]\in\xi_n} m\circ G^{-1}([u]) \circ \sup_{x\in G^{-1}[u]} S_n f(x) \\
&\geq H_{m\circ G^{-1}}(\xi_n) + \int S_n f(x)\,dm \\
&= h\left(T, m\circ G^{-1}\right) + n\int f\,dm.
\end{aligned} \tag{2.18}$$

By Lemma 2.2(i) and Lemma 2.6(ii), divided by n yields

$$\begin{aligned}
\frac{1}{n}\log \sum_{[u]\in\xi_n} \sup_{x\in G^{-1}[u]} e^{S_n f(x)} &\geq \frac{h(T, m\circ G^{-1})}{n} + \int f\,dm \\
&\geq \frac{h_{\tilde{\pi}}(T, m\circ G^{-1})}{n} + \int f\,dm
\end{aligned}$$

$$= \frac{h_\pi(\sigma^n, m)}{n} + \int f\,dm$$
$$= h_\pi(\sigma, m) + \int f\,dm. \tag{2.19}$$

By the arbitrariness of m and n, we have Step 1.

Step 2.

$$\sup\left\{ h_\pi(\sigma, m) + \int f\,dm : m \in M_\sigma(\Sigma)\right\} \geq \limsup_{n\to\infty} \frac{1}{n}\left(\log \sum_{[u]\in\xi_n} \sup_{x\in G^{-1}([u])} e^{S_n f(\alpha)} \right). \tag{2.20}$$

By the continuity of f, for arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for arbitrary $a_N \in \Sigma_N$ and any $x, y \in a_N$, we have

$$|f(x) - f(y)| < \epsilon. \tag{2.21}$$

Now, for any $n \in \mathbb{N}$ and $a_{n+N} \in \Sigma_{n+N}$

$$|S_{n+N} f(x) - S_{n+N} f(y)| \leq n\epsilon + 2N\|f\|, \quad \forall x, y \in a_{n+N}. \tag{2.22}$$

Define a Bernoulli measure ν on $\Omega_{n+N}^{\mathbb{N}}$ by

$$\nu([u]) = \frac{\sup_{x\in G^{-1}[u]} e^{(S_{n+N} f)(x)}}{\sum_{[v]\in\xi_n} \sup_{y\in G^{-1}[v]} e^{(S_{n+N} f)(y)}},$$
$$\nu([w_1, \dots, w_k]) = \prod_{i=1}^{k} \nu([w_i]), \quad w_i \in \xi_n\ k \in \mathbb{N}. \tag{2.23}$$

Then ν can be viewed as a σ^{n+N}-invariant measure on Σ (by viewing $\Omega_n^{\mathbb{N}}$ as a subset of Σ). By Lemma 2.6, we have $h_\pi(\sigma^{n+N}, \nu) = h_{\widetilde{\pi}}(T, \nu)$. Define $\mu = (1/(n+N)) \sum_{i=0}^{n+N-1} \nu \circ \sigma^{-i} \in M_\sigma(\Sigma)$. We have

$$h_\pi(\sigma, \mu) + \int f\,d\mu$$
$$= \frac{h_\pi((\sigma^{n+N}, \nu))}{n+N} + \frac{\int S_{n+N} f\,d\nu}{n+N}$$
$$= \frac{1}{n+N}\left(h_{\widetilde{\pi}}(T, \nu) + \int S_{n+N} f\,d\nu \right)$$
$$\geq \frac{1}{n+N}\left(h(T, \nu) - \log t_{n+N} + \int S_{n+N} f\,d\nu \right)$$

$$
\begin{aligned}
&= \frac{1}{n+N}\left(H_\nu(\xi_{n+N}) - \log t_{n+N} + \int S_{n+N} f\, d\nu\right) \\
&\geq \frac{1}{n+N}\left(\sum_{[u]\in\xi_{n+N}}\left(-\nu([u])\log\nu([u]) + \nu([u])\inf_{x\in G^{-1}[u]} S_{n+N} f(x)\right)\right) - \frac{\log t_{n+N}}{n+N} \\
&\geq \frac{1}{n+N}\left(\sum_{[u]\in\xi_{n+N}}\left(-\nu([u])\log\nu([u]) + \nu([u])\left(\sup_{x\in G^{-1}[u]} S_{n+N} f(x) - n\epsilon - 2N\|f\|\right)\right)\right) \\
&\quad - \frac{\log t_{n+N}}{n+N} \\
&= \frac{1}{n+N}\log\sum_{[u]\in\xi_{n+N}}\sup_{x\in G^{-1}[u]} e^{(S_{n+N}f)(x)} - \frac{n\epsilon + 2N\|f\| + \log t_{n+N}}{n+N}.
\end{aligned}
\tag{2.24}
$$

Let $k = n+N$ and let $n \to \infty$, then $k \to \infty$. We have

$$
\sup\left\{h_\pi(\sigma,\mu) + \int f\, d\mu, m \in M_\sigma(\Sigma)\right\} \geq \limsup_{k\to\infty}\frac{1}{k}\log\sum_{[u]\in\xi_{n+N}}\sup_{x\in G^{-1}[u]} e^{S_k f(x)} - \epsilon. \tag{2.25}
$$

Since ϵ is arbitrary, we finish the proof of Step 2. □

Definition 2.10. If an IFS $\{S_i\}_{i=1}^{l}$ satisfies AWSC with attractor K and $f \in C(\Sigma)$. We call

$$
P_\pi(f) = \lim_{n\to\infty}\frac{1}{n}\left(\log\sum_{[u]\in\xi_n}\sup_{x\in G^{-1}[u]} e^{S_n f\pi(x)}\right) \tag{2.26}
$$

the projection pressure of f under π w.r.t. $\{S_i\}_{i=1}^{l}$.

It is clearly that, if $f = 0$, we have the same result of Lemma 9.1 in [13].

Corollary 2.11. $\lim_{n\to\infty}(\log\#\{S_u : u \in \Sigma_n\}/n) = \sup\{h_\pi(\sigma, m) : m \in M_\sigma(\Sigma)\}$.

3. Application for Projection Pressure

Definition 3.1. $\{S_i : X \to X\}_{i=1}^{l}$ is called a C^1 IFS on a compact set $X \subset \mathbb{R}^d$ if each S_i extends to a contracting C^1-diffeomorphism $S_i : U \to S_i(U) \subset U$ on an open set $U \supset X$.

For any $d \times d$ real matrix M, we use $\|M\|$ to denote the usual norm of M and $[\![M]\!]$ the smallest singular value of M, that is,

$$
\begin{aligned}
\|M\| &= \max\left\{|Mv| : v \in \mathbb{R}^d, |v| = 1\right\}, \\
[\![M]\!] &= \min\left\{|Mv| : v \in \mathbb{R}^d, |v| = 1\right\}.
\end{aligned}
\tag{3.1}
$$

Definition 3.2. The IFS $\{S_i\}_{i=1}^l$ is conformal if for every $i \in \{1,2,\ldots,l\}$, (1) $S_i : U \to S_i(U)$ is C^1, (2) $\|S_i'(x)\| \neq 0$ for all $x \in U$, and (3) $|S_i'(x)y| = \|S_i'(x)\||y|$ for all $x \in U$, $y \in \mathbb{R}^d$.

Definition 3.3. Let $\{S_i\}_{i=1}^l$ be a C^1 IFS. For $x = (x_j)_{j=1}^\infty \in \Sigma$, the *upper* and *lower Lyapunov exponents* of $\{S_i\}_{i=1}^l$ at x are defined, respectively, by

$$\begin{aligned}\overline{\lambda}(x) &= -\liminf_{n\to\infty} \frac{1}{n} \log [\![S'_{x_1,\ldots,x_n}(\pi\sigma^n x)]\!], \\ \underline{\lambda}(x) &= -\limsup_{n\to\infty} \frac{1}{n} \log \| S'_{x_1,\ldots,x_n}(\pi\sigma^n x) \|,\end{aligned} \tag{3.2}$$

where $S'_{x_1,\ldots,x_n}(\pi\sigma^n x)$ denote the differential of $S_{x_1,\ldots,x_n} := S_{x_1} \circ S_{x_2} \circ \cdots \circ S_{x_n}$ at $\pi\sigma^n x$. When $\overline{\lambda}(x) = \underline{\lambda}(x)$, the common value, denoted as $\lambda(x)$, is called the *Lyapunov exponents* of $\{S_i\}_{i=1}^l$ at x.

It is easy to check that both $\overline{\lambda}$ and $\underline{\lambda}$ are positive-valued σ-invariant functions on Σ (i.e., $\overline{\lambda} = \overline{\lambda} \circ \sigma$ and $\underline{\lambda} = \underline{\lambda} \circ \sigma$).

Definition 3.4. A C^1 IFS $\{S_i\}_{i=1}^l$ is said to be *weakly conformal* if

$$\frac{1}{n}\left(\log [\![S'_{x_1,\ldots,x_n}(\pi\sigma^n x)]\!] - \log \| S'_{x_1,\ldots,x_n}(\pi\sigma^n x) \|\right) \tag{3.3}$$

converges to 0 uniformly on Σ as n tends to ∞.

If IFS $\{S_i\}_{i=1}^l$ is weakly conformal, by Birkhoff's ergodic theorem, we can conclude $\int \lambda(x)dm = -\int \log \|S'_{x_1}(\pi\sigma x)\| dm = -\int \log [\![S'_{x_1}(\pi\sigma x)]\!]\, dm$.

Lemma 3.5. *Let K be the attractor of a weakly conformal IFS $\{S_i\}_{i=1}^l$. Then we have*

$$\begin{aligned}\dim_H K &= \dim_B K && (3.4)\\ &= \sup\left\{\dim_H \mu : \mu = m \circ \pi^{-1}, m \in M_\sigma(\Sigma),\ m \text{ is ergodic}\right\} && (3.5)\\ &= \max\left\{\dim_H \mu : \mu = m \circ \pi^{-1}, m \in M_\sigma(\Sigma)\right\} && (3.6)\\ &= \sup\left\{\frac{h_\pi(\sigma, m)}{\int \lambda\, dm} : m \in M_\sigma(\Sigma)\right\}. && (3.7)\end{aligned}$$

Proof. See Theorem 2.13 in [13]. □

Theorem 3.6. *Let $\{S_i(x)\}_{i=1}^l$ be a weakly conformal IFS satsifying AWSC. Let $\psi(x) = \log \|S'_{x_1}\pi\sigma(x)\| : \Sigma \to \mathbb{R}$ and $\pi : \Sigma \to K$ be the canonical projection. Then $\dim_H K$ is the unique root of $P_\pi(t\psi) = 0$.*

Proof. According to Theorem 2.9, we have

$$P_\pi(t\psi) = \sup\left\{ h_\pi(\sigma, m) + \int t\psi \, dm, m \in M_\sigma(\Sigma) \right\}. \tag{3.8}$$

Let $t_0 = \sup\{h_\pi(\sigma, m) / \int \lambda \, dm : m \in M_\sigma(\Sigma)\}$, according to (3.7) $t_0 = \dim_H K$.

First we show $P_\pi(t\psi)$ is decreased with respect to t. If $0 \leq t_1 \leq t_2$, then for any $m \in M_\sigma(\Sigma)$, we have $h_\pi(\sigma, m) + \int t_1\psi \, dm \geq h_\pi(\sigma, m) + \int t_2\psi \, dm$. Hence according to variational principle, we have $P_\pi(t_1\psi) \geq P_\pi(t_2\psi)$.

As $t_0 \geq h_\pi(\sigma, m) / \int \lambda \, dm$ for all $m \in M_\sigma(\Sigma)$, $h_\pi(\sigma, m) + t_0 \int \psi \, dm \leq 0$ for all $m \in M_\sigma(\Sigma)$, whence $P_\pi(t_0\psi) \leq 0$.

However, $P_\pi(0) > 0$, the existence of a positive zero t_1 for $t \mapsto P_\pi(t\psi)$ follows from the intermediate value theorem, that is, $\sup\{h_\pi(\sigma, m) + \int t_1\psi \, dm, \ m \in M_\sigma(\Sigma)\} = 0$.

For all $\epsilon > 0$ there is a $m \in M_\sigma(\Sigma)$ such that $h_\pi(\sigma, m) + \int t_1\psi \, dm \geq -\epsilon$. Thus, $t_1 \leq h_\pi(\sigma, m) / \int -\psi \, dm + \epsilon / \int -\psi \, dm \leq \sup\{h_\pi(\sigma, m) / \int -\psi \, dm, \ m \in M_\sigma(\Sigma)\} + \epsilon / \int -\psi \, dm$, let $\epsilon \to 0$ we have $t_1 \leq t_0$. And $t_0 \leq t_1$ as $P_\pi(t_1\psi) = 0$ implies $h_\pi(\sigma, m) + t_1 \int \psi \, d\mu \leq 0$ for all $\mu \in M_\sigma(\Sigma)$. So any root of $P_\pi(t\psi) = 0$ is equal to $\dim_H K$. □

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