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Research Article

Non-Archimedean Hyers-Ulam Stability of an Additive-Quadratic Mapping

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Using fixed point method and direct method, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation $r^2 f((x+y+z)/r)+r^2 f((x-y+z)/r)+r^2 f((x+y-z)/r)+r^2 f((-x+y+z)/r)=4f(x)+4f(y)+4f(z)$, where r is a positive real number, in non-Archimedean normed spaces.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Th. M. Rassias [3] proved a generalization of Hyers' theorem for linear mappings. Furthermore, in 1994, a generalization of the Th. M. Rassias' theorem was obtained by Găvruţa [4] by replacing the bound $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\phi(x,y)$.

In 1983, the Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. The reader is referred to [2–4, 6–48] and references therein for detailed information on stability of functional equations.

In 1897, Hensel [24] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [16, 26–28, 37]).

Definition 1.1. By a *non-Archimedean field,* one means a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0,\infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$;

(2) $|rs| = |r||s|$;

(3) $|r + s| \leq \max\{|r|, |s|\}$.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ $(r \in \mathbb{K}, x \in X)$;

(3) the strong triangle inequality (ultrametric), namely,

$$\left\|x + y\right\| \leq \max\{\|x\|, \left\|y\right\|\}, \quad x, y \in X. \tag{1.1}$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space, due to the fact that

$$\|x_n - x_m\| \leq \max\{\left\|x_{j+1} - x_j\right\| : m \leq j \leq n-1\} \quad (n > m). \tag{1.2}$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, one means one in which every Cauchy sequence is convergent.

Definition 1.4. Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

One recalls a fundamental result in fixed point theory.

Theorem 1.5 (see [7, 17]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.3}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \geq n_0$*;*

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$*;*

(4) $d(y, y^*) \leq (1/(1-\alpha)) d(y, Jy)$ *for all* $y \in Y$.

In 1998, D. H. Hyers, G. Isac and Th. M. Rassias [25] provided applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [15, 39, 40, 42]).

This paper is organized as follows. In Section 2, using fixed point method, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation:

$$\begin{aligned} r^2 f\left(\frac{x+y+z}{r}\right) &+ r^2 f\left(\frac{x-y+z}{r}\right) + r^2 f\left(\frac{x+y-z}{r}\right) + r^2 f\left(\frac{-x+y+z}{r}\right) \\ &= 4f(x) + 4f(y) + 4f(z), \end{aligned} \tag{1.4}$$

where $x, y, z \in X$, in non-Archimedean normed space. In Section 3, using direct method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.4) in non-Archimedean normed spaces.

2. Stability of the Functional Equation (1.4): A Fixed Point Approach

In this section, we deal with the stability problem for the quadratic functional equation (1.4).

Theorem 2.1. *Let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\frac{1}{|2|}\varphi(2x, 2y, 2z) \leq \alpha\varphi(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\begin{aligned} \Bigg\| r^2 f\left(\frac{x+y+z}{r}\right) &+ r^2 f\left(\frac{x-y+z}{r}\right) + r^2 f\left(\frac{x+y-z}{r}\right) \\ &+ r^2 f\left(\frac{-x+y+z}{r}\right) - 4f(x) - 4f(y) - 4f(z) \Bigg\|_Y \leq \varphi(x, y, z) \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{1}{1-\alpha} \max\left\{\frac{\varphi(2x,0,0)}{|8|}, \frac{\varphi(x,x,0)}{|8|}\right\} \tag{2.3}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Putting $y = z = 0$ in (2.2) and replacing x by $2x$, we get

$$\left\|r^2 f\left(\frac{2x}{r}\right) - 2f(2x)\right\|_Y \leq \frac{1}{|2|}\varphi(2x,0,0) \tag{2.4}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (2.2), we have

$$\left\|r^2 f\left(\frac{2x}{r}\right) - 4f(x)\right\|_Y \leq \frac{1}{|2|}\varphi(x,x,0) \tag{2.5}$$

for all $x \in X$. By (2.4) and (2.5), we get

$$\begin{aligned}\left\|\frac{f(2x)}{2} - f(x)\right\|_Y &\leq \frac{1}{|4|}\max\left\{\left\|r^2 f\left(\frac{2x}{r}\right) - 2f(2x)\right\|_Y, \left\|r^2 f\left(\frac{2x}{r}\right) - 4f(x)\right\|_Y\right\} \\ &\leq \frac{1}{|8|}\max\{\varphi(2x,0,0), \varphi(x,x,0)\}.\end{aligned} \tag{2.6}$$

Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in (0,+\infty) : \|g(x) - h(x)\|_Y \leq \mu \ \max\{\varphi(2x,0,0), \varphi(x,x,0)\},\ \forall x \in X\}, \tag{2.7}$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [31]). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{2.8}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \lambda$. Then

$$\|g(x) - h(x)\|_Y \leq \lambda \max\{\varphi(2x,0,0), \varphi(x,x,0)\} \tag{2.9}$$

for all $x \in X$. Hence

$$\begin{aligned}
\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y &= \frac{1}{|2|}\|g(2x) - h(2x)\|_Y \\
&\leq \frac{\lambda \max\{\varphi(4x,0,0), \varphi(2x,2x,0)\}}{|2|} \\
&\leq \alpha\lambda \max\{\varphi(2x,0,0), \varphi(x,x,0)\}
\end{aligned} \tag{2.10}$$

for all $x \in X$. So $d(g,h) = \lambda$ implies that $d(Jg, Jh) \leq \alpha\lambda$. This means that $d(Jg, Jh) \leq \alpha d(g,h)$ for all $g, h \in S$. It follows from (2.6) that

$$d(f, Jf) \leq \frac{1}{|8|}. \tag{2.11}$$

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following.

(1) A is a fixed point of J, that is,

$$2A(x) = A(2x) \tag{2.12}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $M = \{g \in S : d(h,g) < \infty\}$. This implies that A is the unique mapping satisfying (2.12) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x) - A(x)\|_Y \leq \mu \max\{\varphi(2x,0,0), \varphi(x,x,0)\}$ for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x) \tag{2.13}$$

for all $x \in X$.

(3) $d(f, A) \leq (1/(1-\alpha))d(f, Jf)$, which implies the inequality $d(f, A) \leq 1/(|8|(1-\alpha))$. This implies that the inequalities (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\| r^2 A\left(\frac{x+y+z}{r}\right) + r^2 A\left(\frac{x-y+z}{r}\right) + r^2 A\left(\frac{x+y-z}{r}\right) \right. \\
&\quad \left. + r^2 A\left(\frac{-x+y+z}{r}\right) - 4A(x) - 4A(y) - 4A(z) \right\|_Y \\
&\quad = \lim_{n\to\infty} \frac{1}{|2|^n} \left\| r^2 f\left(\frac{2^n(x+y+z)}{r}\right) + r^2 f\left(\frac{2^n(x-y+z)}{r}\right) + r^2 f\left(\frac{2^n(x+y-z)}{r}\right) \right. \\
&\qquad\qquad \left. + r^2 f\left(\frac{2^n(-x+y+z)}{r}\right) - 4f(2^n x) - 4f(2^n y) - 4f(2^n z) \right\|_Y \\
&\quad \le \lim_{n\to\infty} \frac{1}{|2|^n} \varphi(2^n x, 2^n y, 2^n z) \\
&\quad \le \lim_{n\to\infty} \alpha^n \varphi(x,y,z) \\
&\quad = 0
\end{aligned} \tag{2.14}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
r^2 A\left(\frac{x+y+z}{r}\right) + r^2 A\left(\frac{x-y+z}{r}\right) + r^2 A\left(\frac{x+y-z}{r}\right) + r^2 A\left(\frac{-x+y+z}{s}\right) \\
= 4A(x) + 4A(y) + 4A(z)
\end{aligned} \tag{2.15}$$

for all $x, y, z \in X$. Hence $A : X \to Y$ satisfying (1.4).

It follows from (2.1) and (2.6) that

$$\begin{aligned}
\|A(2x) - 2A(x)\|_Y &= \lim_{n\to\infty} \left\| \frac{f(2^{n+1}x)}{2^n} - \frac{2f(2^n x)}{2^n} \right\|_Y \\
&\le \lim_{n\to\infty} \frac{|2|}{|8||2|^n} \max\left\{\varphi\left(2^{n+1}x, 0, 0\right), \varphi(2^n x, 2^n x, 0)\right\} \\
&\le \lim_{n\to\infty} \frac{1}{|4|} \alpha^n \max\{\varphi(2x,0,0), \varphi(x,x,0)\} = 0
\end{aligned} \tag{2.16}$$

for all $x \in X$. So $A(2x) = 2A(x)$ for all $x \in X$. Hence $A : X \to Y$ is additive and we get the desired result. □

Corollary 2.2. *Let θ be a positive real number and q a real number with $0 < q < 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}&\left\| r^2 f\left(\frac{x+y+z}{r}\right) + r^2 f\left(\frac{x-y+z}{r}\right) + r^2 f\left(\frac{x+y-z}{r}\right)\right.\\&\quad\left. + r^2 f\left(\frac{-x+y+z}{r}\right) - 4f(x) - 4f(y) - 4f(z)\right\|_Y \le \theta\left(\|x\|^q + \|y\|^q + \|z\|^q\right)\end{aligned} \tag{2.17}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{2|2|^q \theta \|x\|^q}{|2|^{q+3} - |2|^4} \tag{2.18}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y, z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{1-q}$ and we get the desired result. □

Theorem 2.3. *Let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$|2|\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \alpha\varphi(x, y, z) \tag{2.19}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{\alpha}{1-\alpha} \max\left\{\frac{\varphi(2x, 0, 0)}{|8|}, \frac{\varphi(x, x, 0)}{|8|}\right\} \tag{2.20}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := 2g(x/2)$ for all $x \in X$.

Replacing x by $x/2$ in (2.6) and using (2.19), we have

$$\begin{aligned}\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y &\le \frac{1}{|4|} \max\left\{\varphi(x, 0, 0), \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)\right\}\\&\le \frac{\alpha}{|8|} \max\{\varphi(2x, 0, 0), \varphi(x, x, 0)\}.\end{aligned} \tag{2.21}$$

So $d(f, Jf) \le \alpha/|8|$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let θ be a positive real number and q a real number with $q > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.17). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_Y \leq \frac{2|2|^q \theta \|x\|^q}{|2|^4 - |2|^{q+3}} \tag{2.22}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y, z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$Then we can choose $\alpha = |2|^{q-1}$ and we get the desired result. □

Theorem 2.5. *Let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\frac{1}{|4|}\varphi(2x, 2y, 2z) \leq \alpha\varphi(x, y, z) \tag{2.23}$$

for all x, y, $z \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{|4|(1-\alpha)} \max\left\{\frac{\varphi(2x, 0, 0)}{|4|}, \frac{\varphi(x, x, 0)}{|2|}\right\} \tag{2.24}$$

for all $x \in X$.

Proof. Putting $y = x$ and $z = 0$ in (2.2), we have

$$\left\|r^2 f\left(\frac{2x}{r}\right) - 4f(x)\right\|_Y \leq \frac{1}{|2|}\varphi(x, x, 0) \tag{2.25}$$

for all $x \in X$.

Substituting $y = z = 0$ and then replacing x by $2x$ in (2.2), we obtain

$$\left\|r^2 f\left(\frac{2x}{r}\right) - f(2x)\right\|_Y \leq \frac{1}{|4|}\varphi(2x, 0, 0). \tag{2.26}$$

By (2.25) and (2.26), we get

$$\begin{aligned}\left\|\frac{f(2x)}{4} - f(x)\right\|_Y &= \frac{1}{|4|}\left\|r^2 f\left(\frac{2x}{r}\right) - 4f(x) - r^2 f\left(\frac{2x}{r}\right) + f(2x)\right\|_Y \\ &\leq \frac{1}{|4|}\max\left\{\left\|r^2 f\left(\frac{2x}{r}\right) - 4f(x)\right\|_Y, \left\|r^2 f\left(\frac{2x}{r}\right) - f(2x)\right\|_Y\right\} \\ &\leq \frac{1}{|4|}\max\left\{\frac{\varphi(2x, 0, 0)}{|4|}, \frac{\varphi(x, x, 0)}{|2|}\right\}.\end{aligned} \tag{2.27}$$

Consider the set $S^* = \{g : X \to Y; g(0) = 0\}$ and the generalized metric d^* in S^* defined by

$$d(g,h) = \inf\left\{\mu \in (0,+\infty) : \|g(x) - h(x)\|_Y \le \mu \max\left\{\frac{\varphi(2x,0,0)}{|4|}, \frac{\varphi(x,x,0)}{|2|}\right\}, \forall x \in X\right\}, \tag{2.28}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S^*, d^*) is complete (see [31]).

Now we consider the linear mapping $J : (S^*, d^*) \to (S^*, d^*)$ such that $Jg(x) := (1/4)g(2x)$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let θ be a positive real number and q a real number with $q > 1$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.17)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \le \frac{1}{|4| - |4|^q} \max\left\{\frac{2\theta\|x\|^q}{|2|}, \frac{|2|^q\theta\|x\|^q}{|4|}\right\} \tag{2.29}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x,y,z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |4|^{q-1}$ and we get the desired result. □

Theorem 2.7. *Let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$|4|\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \alpha\varphi(x,y,z) \tag{2.30}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \le \frac{\alpha}{|4|(1-\alpha)} \max\left\{\frac{\varphi(2x,0,0)}{|4|}, \frac{\varphi(x,x,0)}{|2|}\right\} \tag{2.31}$$

for all $x \in X$.

Proof. It follows from (2.27) that

$$\begin{aligned}\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\|_Y &\le \max\left\{\frac{1}{|4|}\varphi(x,0,0), \frac{1}{|2|}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)\right\} \\ &\le \frac{\alpha}{|4|}\max\left\{\frac{\varphi(2x,0,0)}{|4|}, \frac{\varphi(x,x,0)}{|2|}\right\}.\end{aligned} \tag{2.32}$$

The rest of the proof is similar to the proof of Theorems 2.1 and 2.5. □

Corollary 2.8. *Let θ be a positive real number and q a real number with $0 < q < 1$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.17). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|4|^{1-q}}{|4| - |4|^{2-q}} \max\left\{\frac{2\theta\|x\|^q}{|2|}, \frac{|2|^q\theta\|x\|^q}{|4|}\right\} \tag{2.33}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y, z) = \theta(\|x\|^q + \|y\|^q + \|z\|^q)$ for all $x, y, z \in X$. Then we can choose $\alpha = |4|^{1-q}$ and we get the desired result. □

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.4). Let $f_e(x) := (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Then f_e is an even mapping satisfying (1.4) and f_o is an odd mapping satisfying (1.4) such that $f(x) = f_e(x) + f_o(x)$. So we obtain the following.

Theorem 2.9. *Let X be a non-Archimedean normed space and Y a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\frac{1}{|4|}\varphi(2x, 2y, 2z) \leq \alpha\varphi(x, y, z) \tag{2.34}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned} &\|f(x) - A(x) - Q(x)\|_Y \\ &\quad\leq \max\left\{\left\|\frac{f(x) - f(-x)}{2} - A(x)\right\|_Y, \left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\|_Y\right\} \\ &\quad\leq \max\left\{\frac{1}{1-\alpha}\max\left\{\frac{\varphi(2x,0,0)}{|8|}, \frac{\varphi(x,x,0)}{|8|}\right\}, \frac{1}{|4|(1-\alpha)}\max\left\{\frac{\varphi(2x,0,0)}{|4|}, \frac{\varphi(x,x,0)}{|2|}\right\}\right\} \end{aligned} \tag{2.35}$$

for all $x \in X$.

3. Stability of the Functional Equation (1.4): A Direct Method

In this section, using direct method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.4) in non-Archimedean space.

Theorem 3.1. *Let G be an additive semigroup and X a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.1}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Omega(x) = \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right) \right\};\ 0 \le k < n \right\} \tag{3.2}$$

exists and $f : G \to X$ is an odd mapping satisfying

$$\begin{aligned} &\left\| r^2 f\left(\frac{x+y+z}{r}\right) + r^2 f\left(\frac{x-y+z}{r}\right) + r^2 f\left(\frac{x+y-z}{r}\right)\right. \\ &\quad \left. + r^2 f\left(\frac{-x+y+z}{r}\right) - 4f(x) - 4f(y) - 4f(z) \right\|_X \le \zeta(x,y,z). \end{aligned} \tag{3.3}$$

Then the limit

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.4}$$

exists for all $x \in G$ and defines an additive mapping $A : G \to X$ such that

$$\| f(x) - A(x) \| \le \frac{1}{|4|} \Omega(x). \tag{3.5}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right) \right\};\ j \le k < n+j \right\} = 0, \tag{3.6}$$

then A is the unique additive mapping satisfying (3.5).

Proof. By (2.21), we know that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_X \le \frac{1}{|4|} \max\left\{ \varphi(x,0,0), \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \right\} \tag{3.7}$$

for all $x \in G$. Replacing x by $x/2^n$ in (3.7), we obtain

$$\left\| 2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\|_X \le \frac{|2|^n}{|4|} \max\left\{ \varphi\left(\frac{x}{2^n}, 0, 0\right), \varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0\right) \right\}. \tag{3.8}$$

Thus, it follows from (3.1) and (3.8) that the sequence $\{2^n f(x/2^n)\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, it follows that $\{2^n f(x/2^n)\}_{n\geq 1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right). \tag{3.9}$$

By induction on n, one can show that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|_X \leq \frac{1}{|4|}\max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right),\varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\};\ 0\leq k<n\right\} \tag{3.10}$$

for all $n \geq 1$ and $x \in G$. Indeed, (3.10) holds for $n = 1$ by (3.7). Now, if (3.10) holds for n, then by (3.8), we have

$$\begin{aligned}
&\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - f(x)\right\| \\
&\quad= \left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) \pm 2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \\
&\quad\leq \max\left\{\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|, \left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|\right\} \\
&\quad\leq \frac{1}{|4|}\max\left\{|2|^n \max\left\{\varphi\left(\frac{x}{2^n},0,0\right),\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0\right)\right\},\right. \\
&\qquad\qquad \left.\max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right),\varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\};\ 0\leq k<n\right\}\right\} \\
&\quad= \frac{1}{|4|}\max\left\{|2|^k \max\left\{\varphi\left(\frac{x}{2^k},0,0\right),\varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\};\ 0\leq k<n+1\right\}.
\end{aligned} \tag{3.11}$$

By taking $n \to \infty$ in (3.10) and using (3.2), one obtains (3.5). By (3.1) and (3.3), we get

$$\begin{aligned}
&\left\|r^2 A\left(\frac{x+y+z}{r}\right) + r^2 A\left(\frac{x-y+z}{r}\right) + r^2 A\left(\frac{x+y-z}{r}\right)\right. \\
&\quad\left. + r^2 A\left(\frac{-x+y+z}{r}\right) - 4A(x) - 4A(y) - 4A(z)\right\|_X \\
&\quad= \lim_{n\to\infty} |2|^n \left\|r^2 f\left(\frac{x+y+z}{2^n r}\right) + r^2 f\left(\frac{x-y+z}{2^n r}\right) + r^2 f\left(\frac{x+y-z}{2^n r}\right)\right. \\
&\qquad\qquad \left. + r^2 f\left(\frac{-x+y+z}{2^n r}\right) - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\|_X \\
&\quad\leq \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \\
&\quad= 0
\end{aligned} \tag{3.12}$$

for all $x, y, z \in X$. Therefore, the mapping $A : G \to X$ satisfies (1.4).

To prove the uniqueness property of A, let L be another mapping satisfying (3.5). Then we have

$$\begin{aligned}
&\|A(x) - L(x)\|_X \\
&\quad = \lim_{j\to\infty} |2|^j \left\| A\left(\frac{x}{2^j}\right) - L\left(\frac{x}{2^j}\right) \right\|_X \\
&\quad \le \lim_{j\to\infty} |2|^j \max\left\{ \left\| A\left(\frac{x}{2^j}\right) - f\left(\frac{x}{2^j}\right) \right\|_X, \left\| f\left(\frac{x}{2^j}\right) - L\left(\frac{x}{2^j}\right) \right\|_X \right\} \\
&\quad \le \lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right) \right\};\ j \le k < n + j \right\} \\
&\quad = 0
\end{aligned} \tag{3.13}$$

for all $x \in G$. Therefore, $A = L$. This completes the proof. □

Corollary 3.2. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi\left(|2|^{-1}t\right) \le \xi\left(|2|^{-1}\right)\xi(t), \qquad \xi\left(|2|^{-1}\right) < |2|^{-1} \tag{3.14}$$

for all $t \ge 0$. Assume that $\kappa > 0$ and $f : G \to X$ is a mapping with $f(0) = 0$ such that

$$\begin{aligned}
&\left\| r^2 f\left(\frac{x+y+z}{r}\right) + r^2 f\left(\frac{x-y+z}{r}\right) + r^2 f\left(\frac{x+y-z}{r}\right) \right. \\
&\quad \left. + r^2 f\left(\frac{-x+y+z}{r}\right) - 4f(x) - 4f(y) - 4f(z) \right\|_X \le \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|))
\end{aligned} \tag{3.15}$$

for all $x, y, z \in G$. Then there exists a unique additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\|_X \le \frac{1}{|4|} \max\left\{ \kappa\zeta(\|x\|), \frac{2}{|2|}\kappa\zeta(\|x\|) \right\}. \tag{3.16}$$

Proof. One can define $\zeta : G^3 \to [0,\infty)$ by $\zeta(x,y,z) := \kappa(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|))$. Then we have

$$\lim_{n\to\infty} |2|^n \zeta\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \le \lim_{n\to\infty} \left(|2|\xi\left(|2|^{-1}\right)\right)^n \zeta(x,y,z) = 0 \tag{3.17}$$

for all $x, y, z \in G$. The last equality comes from the fact that $|2|\xi(|2|^{-1}) < 1$. On the other hand, it follows that

$$\begin{aligned}
\Omega(x) &= \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \varphi\left(\frac{x}{2^k}, 0, 0\right), \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0\right) \right\};\ 0 \le k < n \right\} \\
&\le \max\left\{ \varphi(x,0,0), \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \right\} = \max\left\{ \kappa\zeta(\|x\|), \frac{2}{|2|}\kappa\zeta(\|x\|) \right\}
\end{aligned} \tag{3.18}$$

exists for all $x \in G$. Also, we have

$$\begin{aligned}&\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{|2|^k\max\left\{\varphi\left(\frac{x}{2^k},0,0\right),\varphi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},0\right)\right\};\ j\le k<n+j\right\}\\&=\lim_{j\to\infty}|2|^j\max\left\{\varphi\left(\frac{x}{2^j},0,0\right),\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\right)\right\}=0.\end{aligned}\tag{3.19}$$

Thus, applying Theorem 3.1, we have the conclusion. This completes the proof. □

Theorem 3.3. *Let G be an additive semigroup and X a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty}\frac{\zeta(2^n x,2^n y,2^n z)}{|2|^n}=0\tag{3.20}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Omega(x)=\lim_{n\to\infty}\max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|2|^k};\ 0\le k<n\right\}\tag{3.21}$$

exists and $f : G \to X$ is an odd mapping satisfying (3.3). *Then the limit $A(x) := \lim_{n\to\infty}(f(2^n x)/2^n)$ exists for all $x \in G$ and*

$$\|f(x)-A(x)\|_X\le\frac{1}{|8|}\Omega(x)\tag{3.22}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{\frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|2|^k};\ j\le k<n+j\right\}=0,\tag{3.23}$$

then A is the unique mapping satisfying (3.22).

Proof. By (2.6), we get

$$\left\|\frac{f(2x)}{2}-f(x)\right\|_X\le\frac{\max\{\varphi(2x,0,0),\varphi(x,x,0)\}}{|8|}\tag{3.24}$$

for all $x \in G$. Replacing x by $2^n x$ in (3.24), we obtain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}\right\|_X\le\frac{\max\{\varphi(2^{n+1}x,0,0),\varphi(2^n x,2^n x,0)\}}{|2|^{n+3}}.\tag{3.25}$$

Thus it follows from (3.20) and (3.25) that the sequence $\{f(2^n x)/2^n\}_{n\geq 1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}. \tag{3.26}$$

On the other hand, it follows from (3.25) that

$$\begin{aligned} \left\| \frac{f(2^p x)}{2^q} - \frac{f(2^q x)}{2^q} \right\|_X &= \left\| \sum_{k=p}^{q-1} \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\|_X \\ &\leq \max\left\{ \left\| \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\|_X ;\ p \leq k < q-1 \right\} \\ &\leq \frac{1}{|8|} \max\left\{ \frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|2|^k};\ p \leq k < q \right\} \end{aligned} \tag{3.27}$$

for all $x \in G$ and $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality, and using (3.21), we obtain (3.22).

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

Theorem 3.4. *Let G be an additive semigroup and X a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.28}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Theta(x) = \lim_{n\to\infty} \max\left\{ |4|^k \max\left\{ \frac{1}{|4|}\varphi\left(\frac{x}{2^k},0,0\right), \frac{1}{|2|}\varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}},0\right) \right\};\ 0 \leq k < n \right\} \tag{3.29}$$

exists and $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and (3.3). Then the limit $A(x) := \lim_{n\to\infty} 4^n f(x/2^n)$ exists for all $x \in G$ and defines a quadratic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\|_X \leq \Theta(x). \tag{3.30}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{ |4|^k \max\left\{ \frac{1}{|4|}\varphi\left(\frac{x}{2^k},0,0\right), \frac{1}{|2|}\varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}},0\right) \right\};\ j \leq k < n+j \right\} = 0, \tag{3.31}$$

then Q is the unique additive mapping satisfying (3.30).

Proof. It follows from (2.27) that

$$\left\| f(x) - 4f\left(\frac{x}{2}\right)\right\|_X \le \max\left\{\frac{1}{|4|}\varphi(x,0,0), \frac{1}{|2|}\varphi\left(\frac{x}{2},\frac{x}{2},0\right)\right\}. \tag{3.32}$$

Replacing x by $x/2^n$ in (3.32), we have

$$\left\| 4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\|_X \le |4|^n \max\left\{\frac{1}{|4|}\varphi\left(\frac{x}{2^n},0,0\right), \frac{1}{|2|}\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},0\right)\right\}. \tag{3.33}$$

It follows from (3.28) and (3.32) that the sequence $\{4^n f(x/2^n)\}_{n\ge1}$ is Cauchy sequence.

The rest of the proof is similar to the proof of Theorem 3.1. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 3.5. *Let G be an additive semigroup and X a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty} \frac{\zeta(2^n x, 2^n y, 2^n z)}{|4|^n} = 0 \tag{3.34}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$\Theta(x) = \lim_{n\to\infty} \max\left\{\frac{1}{|4|^k}\max\left\{\frac{1}{|4|}\varphi\left(2^{k+1}x,0,0\right), \frac{1}{|2|}\varphi\left(2^k x, 2^k x, 0\right)\right\};\ 0\le k<n\right\} \tag{3.35}$$

exists and $f : G \to X$ is an even mapping satisfying $f(0) = 0$ and (3.3)*. Then the limit $Q(x) := \lim_{n\to\infty}(f(2^n x)/4^n)$ exists for all $x \in G$ and*

$$\|f(x) - Q(x)\|_X \le \frac{1}{|4|}\Theta(x) \tag{3.36}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{1}{|4|^k}\max\left\{\frac{1}{|4|}\varphi\left(2^{k+1}x,0,0\right), \frac{1}{|2|}\varphi\left(2^k x, 2^k x, 0\right)\right\};\ j\le k<n+j\right\} = 0, \tag{3.37}$$

then Q is the unique mapping satisfying (3.36).

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.4). Let $f_e(x) := (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Then f_e is an even mapping satisfying (1.4) and f_o is an odd mapping satisfying (1.4) such that $f(x) = f_e(x) + f_o(x)$. So we obtain the following.

Theorem 3.6. *Let G be an additive semigroup and X a non-Archimedean Banach space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty} \frac{\zeta(2^n x, 2^n y, 2^n z)}{|4|^n} = 0 \tag{3.38}$$

for all x, y, $z \in G$. Suppose that the limits

$$\begin{aligned}\Omega(x) &= \lim_{n\to\infty} \max\left\{ \frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|2|^k};\ 0 \le k < n \right\},\\ \Theta(x) &= \lim_{n\to\infty} \max\left\{ \frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|4|^k};\ 0 \le k < n \right\}\end{aligned} \tag{3.39}$$

exist for all $x \in G$ and $f : G \to X$ is a mapping satisfying $f(0) = 0$ and (3.3). *Then there exist an additive mapping $A : G \to X$ and a quadratic mapping $Q : G \to X$ such that*

$$\begin{aligned}&\|f(x) - A(x) - Q(x)\|_X\\ &\quad\le \max\left\{ \left\| \frac{f(x)+f(-x)}{2} - Q(x) \right\|_X, \left\| \frac{f(x)-f(-x)}{2} - A(x) \right\|_X \right\}\\ &\quad\le \max\left\{ \frac{1}{|8|}\Omega(x), \frac{1}{|4|}\Theta(x) \right\}\end{aligned} \tag{3.40}$$

for all $x \in G$. Moreover, if

$$\begin{aligned}\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{ \frac{\max\{\varphi(2^{k+1}x,0,0),\varphi(2^k x,2^k x,0)\}}{|2|^k};\ j \le k < n+j \right\} = 0,\\ \lim_{j\to\infty}\lim_{n\to\infty} \max\left\{ \frac{1}{|4|^k} \max\left\{ \frac{1}{|4|}\varphi\left(2^{k+1}x,0,0\right), \frac{1}{|2|}\varphi\left(2^k x,2^k x,0\right) \right\};\ j \le k < n+j \right\} = 0,\end{aligned} \tag{3.41}$$

then A and Q are the unique mappings satisfying (3.40).

4. Conclusion

We linked here three different disciplines, namely, the non-Archimedean normed spaces, functional equations, and fixed point theory. We established the generalized Hyers-Ulam stability of the functional equation (1.4) in non-Archimedean normed spaces.

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