Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 930410, 11 pages
doi:10.1155/2012/930410

Research Article

Global Attractivity of a Higher-Order Difference Equation

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Received 7 May 2012; Accepted 8 July 2012

Academic Editor: M. De la Sen

The aim of this work is to investigate the global stability, periodic nature, oscillation, and the boundedness of all admissible solutions of the difference equation $x_{n+1} = Ax_{n-2r-1}/(B - C\prod_{i=l}^{k} x_{n-2i})$, $n = 0,1,2,\ldots$ where A,B,C are positive real numbers and l,r,k are nonnegative integers, such that $l \leq k$.

1. Introduction and Preliminaries

Although some difference equations look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1, 2]. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations [2–4], relatively a large number of difference equations have not been thoroughly understood yet [5–8].

Aloqeili in [9] discussed the stability properties and semicycle behavior of the solutions of the difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}, \quad n = 0,1,2,\ldots, \tag{1.1}$$

with real initial conditions and positive real number a.

In [10], the authors investigated the global asymptotic stability of the difference equation:

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, 2, \dots, \tag{1.2}$$

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, such that $l \leq k$.

Also in [11], they discussed the existence of unbounded solutions under certain conditions of the difference equation:

$$x_{n+1} = \frac{A\prod_{i=l}^{k} x_{n-2i-1}}{B + C\prod_{i=l}^{k-1} x_{n-2i}}, \quad n = 0, 1, 2, \dots, \tag{1.3}$$

where A, B, C are nonnegative real numbers and l, k are nonnegative integers, $l < k$

In [12], the global asymptotic stability of the difference equation:

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}}, \quad n = 0, 1, 2, \dots \tag{1.4}$$

was discussed, where A, B, C are nonnegative real numbers and r, l, k are nonnegative integers such that $l \leq k$ and $r \leq k$.

In [13], the global stability and periodic nature of the solutions of the difference equations:

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, 2, \dots \tag{1.5}$$

were discussed, where the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

In [14], we discussed the oscillation, boundedness, and the global behavior of all admissible solutions of the difference equation:

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_n x_{n-2}}, \quad n = 0, 1, 2, \dots, \tag{1.6}$$

where A, B, C are positive real numbers.

In this paper, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B - C\prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, 2, \dots, \tag{1.7}$$

where A, B, C are nonnegative real numbers and l, r, k are nonnegative integers, such that $l \leq k$.

Consider the difference equation:

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2 \ldots, \tag{1.8}$$

where $f : R^{k+1} \to R$. An equilibrium point for (1.8) is a point $\overline{x} \in R$ such that $\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x})$.

(1) An equilibrium point $\overline{x}$ for (1.8) is called locally stable if for every $\epsilon > 0, \exists \delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\overline{x} - \delta, \overline{x} + \delta[$ is such that $x_n \in]\overline{x} - \epsilon, \overline{x} + \epsilon[$, $\forall n \in N$. Otherwise $\overline{x}$ is said to be unstable.

(2) The equilibrium point $\overline{x}$ of (1.8) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\overline{x} - \gamma, \overline{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\overline{x}$.

(3) An equilibrium point $\overline{x}$ for (1.8) is called global attractor if every solution $\{x_n\}$ converges to $\overline{x}$ as $n \to \infty$.

(4) The equilibrium point $\overline{x}$ for (1.8) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with (1.8) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \ldots, \overline{x}) y_{n-i}, \quad n = 0, 1, 2, \ldots. \tag{1.9}$$

The characteristic equation associated with (1.9) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \ldots, \overline{x}) \lambda^{k-i} = 0. \tag{1.10}$$

Theorem 1.1 (see [2]). *Assume that f is a C^1 function and let $\overline{x}$ be an equilibrium point of* (1.8). *Then the following statements are true.*

(1) *If all roots of* (1.10) *lie in the open disk $|\lambda| < 1$, then $\overline{x}$ is locally asymptotically stable.*

(2) *If at least one root of* (1.10) *has absolute value greater than one, then $\overline{x}$ is unstable.*

2. Linearized Stability Analysis

Consider the difference equation:

$$x_{n+1} = \frac{A x_{n-2r-1}}{B - C \prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where A, B, C are nonnegative real numbers and l, r, k are nonnegative integers, such that $l \leq k$.

The change of variables $x_n = \sqrt[k-l+1]{B/C}\,y_n$ reduces (1.7) to the difference equation:

$$y_{n+1} = \frac{\gamma y_{n-2r-1}}{1 - \prod_{i=l}^{k} y_{n-2i}}, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

where $\gamma = A/B$.

Now we determine the equilibrium points of (2.2) and discuss their local asymptotic behavior. It is clear that the values of the equilibrium points depend on whether $k - l$ is even or odd.

When $k - l$ is odd, we have the equilibrium points $\overline{y} = 0$ and $\overline{y} = \pm \sqrt[k-l+1]{1-\gamma}$ if $\gamma < 1$ and $\overline{y} = 0$ only if $\gamma \geq 1$.

When $k - l$ is even, we have the equilibrium points $\overline{y} = 0$ and $\overline{y} = \sqrt[k-l+1]{1-\gamma}$.

Now assume that $\mathfrak{K} = \max\{2k, 2r+1\}$.

The linearized equation associated with (2.2) about $\overline{y}$ is

$$z_{n+1} - \frac{\gamma}{1 - \overline{y}^{k-l+1}} z_{n-2r-1} - \frac{\gamma \overline{y}^{k-l+1}}{\left(1 - \overline{y}^{k-l+1}\right)^2} \sum_{i=l}^{k} z_{n-2i} = 0, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

The characteristic equation associated with this equation is

$$\lambda^{\mathfrak{K}+1} - \frac{\gamma}{1 - \overline{y}^{k-l+1}} \lambda^{\mathfrak{K}-2r-1} - \frac{\gamma \overline{y}^{k-l+1}}{\left(1 - \overline{y}^{k-l+1}\right)^2} \sum_{i=l}^{k} \lambda^{\mathfrak{K}-2i} = 0. \tag{2.4}$$

We summarize the results of this section in the following two theorems.

Theorem 2.1. *Assume that $\mathfrak{K} = 2k > 2r + 1$. Then the following statements are true.*

(1) *The zero equilibrium point is locally asymptotically stable if $\gamma < 1$ and unstable (saddle point) if $\gamma > 1$.*

(2) *When $k - l$ is even, the equilibrium point $\overline{y} = \sqrt[k-l+1]{1-\gamma}$ is unstable if $\gamma < 1$ and unstable (saddle point) if $\gamma > 1$.*

(3) *When $k - l$ is odd, the equilibrium points $\overline{y} = \pm \sqrt[k-l+1]{1-\gamma}$ are unstable.*

Proof. (1) The linearized equation (2.3) about $\overline{y} = 0$ is

$$z_{n+1} - \gamma z_{n-2r-1} = 0, \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

The characteristic equation associated with this equation is

$$\lambda^{2k+1} - \gamma \lambda^{2k-2r-1} = 0. \tag{2.6}$$

So $\lambda = 0$, $\lambda = \pm \sqrt[2r+2]{\gamma}$. Therefore the result follows.

(2) Suppose that $k-l$ is even. The linearized equation (2.3) about $\overline{y} = \sqrt[k-l+1]{1-\gamma}$ is

$$z_{n+1} - z_{n-2r-1} - \frac{1}{\gamma}(1-\gamma)\sum_{i=l}^{k} z_{n-2i} = 0, \quad n = 0,1,2,\ldots. \tag{2.7}$$

The associated characteristic equation (2.4) becomes

$$\lambda^{2k+1} - \lambda^{2k-2r-1} - \frac{1}{\gamma}(1-\gamma)\sum_{i=l}^{k} \lambda^{2k-2i} = 0. \tag{2.8}$$

Let

$$f(\lambda) = \lambda^{2k+1} - \lambda^{2k-2r-1} - \frac{1}{\gamma}(1-\gamma)\sum_{i=l}^{k} \lambda^{2k-2i} = 0. \tag{2.9}$$

We can see that $f(\lambda)$ has a real root in $(1,\infty)$ if $\gamma < 1$ and when $\gamma > 1$, $f(\lambda)$ has a root in $(1,\infty)$ and some roots with $|\lambda| < 1$. Therefore the result follows.

(3) When $k-l$ is odd, $f(\lambda)$ has a root in $(1,\infty)$ and some roots with $|\lambda| < 1$, if $\gamma < 1$. Therefore $\overline{y} = \pm\sqrt{1-\gamma}$ are unstable. □

Theorem 2.2. *Assume that $\mathfrak{K} = 2r+1 > 2k$. Then the following statements are true.*

(1) *The zero equilibrium point is locally asymptotically stable if $\gamma < 1$ and a source if $\gamma > 1$.*

(2) *If $k-l$ is even, then the equilibrium point $\overline{y} = \sqrt[k-l+1]{1-\gamma}$ is unstable (saddle point).*

(3) *If $k-l$ is odd, then the equilibrium points $\overline{y} = \pm\sqrt[k-l+1]{1-\gamma}$ are unstable (saddle points).*

Proof. It is sufficient to consider the linearized equation about $\overline{y}$:

$$z_{n+1} - \frac{\gamma}{1-\overline{y}^{k-l+1}} z_{n-2r-1} - \frac{\gamma\overline{y}^{k-l+1}}{\left(1-\overline{y}^{k-l+1}\right)^2}\sum_{i=l}^{k} z_{n-2i} = 0, \quad n = 0,1,2,\ldots \tag{2.10}$$

and its associated characteristic equation:

$$\lambda^{2r+2} - \frac{\gamma}{1-\overline{y}^{k-l+1}} - \frac{\gamma\overline{y}^{k-l+1}}{\left(1-\overline{y}^{k-l+1}\right)^2}\sum_{i=l}^{k} \lambda^{2r+1-2i} = 0. \tag{2.11}$$

□

3. Oscillation

Let t be the largest nonnegative integer such that $0 < 2t+1 \leq \mathfrak{K}$ and let s be the largest nonnegative integer such that $0 \leq 2s \leq \mathfrak{K}$.

Theorem 3.1. *Assume that $\gamma < 1$. Then the interval $(-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})$ is an invariant interval for* (2.2).

Proof. The proof is by induction. Suppose that $y_{-i} \in (-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})$, $i = 0, 1, \ldots, \mathfrak{K}$. Hence $|y_{-i}| < \sqrt[k-l+1]{1-\gamma}$, $i = 0, 1, \ldots, \mathfrak{K}$.

This implies that $|\prod_{i=l}^{k} y_{-2i}| < 1-\gamma$. Then

$$
\begin{aligned}
|y_1| &= \frac{\gamma|y_{-2r-1}|}{\left|1-\prod_{i=l}^{k} y_{-2i}\right|} \le \frac{\gamma|y_{-2r-1}|}{\left|1-\left|\prod_{i=l}^{k} y_{-2i}\right|\right|} < |y_{-2r-1}|, \\
|y_2| &= \frac{\gamma|y_{-2r}|}{\left|1-\prod_{i=l}^{k} y_{-2i+1}\right|} \le \frac{\gamma|y_{-2r}|}{\left|1-\left|\prod_{i=l}^{k} y_{-2i+1}\right|\right|} < |y_{-2r}|.
\end{aligned}
\tag{3.1}
$$

If for a certain $n_0 \in \mathbb{N}$ we have $y_{n_0-\mathfrak{K}}, y_{n_0-\mathfrak{K}+1}, \ldots, y_{n_0} \in (-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})$, then

$$
|y_{n_0+1}| = \frac{\gamma|y_{n_0-2r-1}|}{\left|1-\prod_{i=l}^{k} y_{-2i}\right|} \le \frac{\gamma|y_{n\_2r-1}|}{\left|1-\left|\prod_{i=l}^{k} y_{-2i}\right|\right|} < |y_{n_0-2r-1}| < \sqrt[k-l+1]{1-\gamma}. \tag{3.2}
$$

This completes the proof. □

Corollary 3.2. *Assume that $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ be a solution of* (2.2) *such that either $y_{-\mathfrak{K}}, y_{-\mathfrak{K}+1}, \ldots, y_{-1}, y_0 \in (0, \sqrt[k-l+1]{1-\gamma})$ (or$(-\sqrt[k-l+1]{1-\gamma}, 0)$). Then $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ is positive (or negative). Moreover, $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ converges to the zero equilibrium point.*

Theorem 3.3. *Let $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ be a nontrivial solution of* (2.2) *such that either*

(C_1) $-\sqrt[k-l+1]{1-\gamma} < y_{-2t-1}, y_{-2t+1}, \ldots, y_{-1} < 0 < y_{-2s}, y_{-2s+2}, \ldots, y_0 < \sqrt[k-l+1]{1-\gamma}$

or

(C_2) $-\sqrt[k-l+1]{1-\gamma} < y_{-2s}, y_{-2s+2}, \ldots, y_0 < 0 < y_{-2t-1}, y_{-2t+1}, \ldots, y_{-1} < \sqrt[k-l+1]{1-\gamma}$ *is satisfied. Then $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ oscillates about $\overline{y} = 0$ with semicycles of length one. Moreover $y_{2(r+1)n+2j} <$ (or $>$)$y_{2(r+1)(n-1)+2j}$ and $y_{2(r+1)n+2j+1} >$ (or $<$) $y_{2(r+1)(n-1)+2j+1}$, $j = 1, 2, \ldots, r+1$ and $n = 0, 1, 2, \ldots$.*

Proof. Assume that condition (C_1) is satisfied. Then we have $y_1 = \gamma y_{-2r-1}/(1-\prod_{i=l}^{k} y_{-2i}) > y_{-2r-1}$, and $y_2 = \gamma y_{-2r}/(1-\prod_{i=l}^{k} y_{-2i+1}) < y_{-2r}$.

By induction we get $0 > y_{2(r+1)n+2j+1} > y_{2(r+1)(n-1)+2j+1}$, and $0 < y_{2(r+1)n+2j} < y_{2(r+1)(n-1)+2j}$, $n = 0, 1, 2, \ldots$.

If condition (C_2) is satisfied, the result is similar and will be omitted. □

4. Global Behavior of (2.2)

Theorem 4.1. *The following statements are true.*

(1) *If $\gamma < 1$, then the zero equilibrium point is a global attractor with basin $(-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})^{\mathfrak{K}+1}$.*

(2) *If $\gamma = 1$, then* (2.2) *has prime period two solutions of the form $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$, where $\varphi \in \mathbb{R}$.*

(3) *If $\gamma > 1$, then there exist solutions which are neither bounded nor persist.*

Proof. (1) Suppose that $y_{-\mathfrak{K}}, y_{-\mathfrak{K}+1}, \ldots, y_{-1}, y_0 \in (-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})$. Then using Theorem 3.1, we have that $y_n \in (-\sqrt[k-l+1]{1-\gamma}, \sqrt[k-l+1]{1-\gamma})$, $n \geq 1$.

Moreover, we have $|y_{n+1}| < |y_{n-2r-1}|$, $n = 0, 1, 2, \ldots$.

That is, the subsequences $\{|y_{2(r+1)n+j}|\}_{n=-1}^{\infty}$, $j = 1, 2, \ldots, 2r+2$ are decreasing.

From (2.2) we have

$$\left|y_{2(r+1)n+j}\right| = \frac{\gamma\left|y_{2(r+1)(n-1)+j}\right|}{\left|1-\prod_{i=l}^{k} y_{2(r+1)n+j-2i-1}\right|} \leq \frac{\gamma\left|y_{2(r+1)(n-1)+j}\right|}{\left|1-\left|\prod_{i=l}^{k} y_{2(r+1)n+j-2i-1}\right|\right|}. \tag{4.1}$$

Now suppose that $|y_{2(r+1)n+j}| \to L_j$ as $n \to \infty$, $j = 1, 2, \ldots, 2r+2$. Then the last inequality implies that

$$L_j \leq \frac{\gamma L_j}{\left|1-\prod_{i=l}^{k} L_{j-2i-1}\right|}, \quad j = 1, 2, \ldots, 2r+2. \tag{4.2}$$

If for a certain $j \in \{1, 2, \ldots, 2r+2\}$ we have $L_j \neq 0$, then $|1-\prod_{i=l}^{k} L_{j-2i-1}| \leq \gamma$. This implies that

$$1+\gamma \geq \prod_{i=l}^{k} L_{j-2i-1} \geq 1-\gamma. \tag{4.3}$$

This is a contradiction as the the subsequences $\{|y_{2(r+1)n+j}|\}_{n=-1}^{\infty}$, $j = 1, 2, \ldots, 2r+2$ are decreasing. Therefore, $L_j = 0$, $j = 1, 2, \ldots, 2r+2$, and $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ converges to zero.

(2) Clear!

(3) Let $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ be a solution of (2.2) with initial conditions, $|y_{-i}| < \sqrt[k-l+1]{\gamma-1}$ $(> \sqrt[k-l+1]{\gamma+1})$, $i = 2s, 2s-2, \ldots, 2, 0$, and $|y_{-i}| > \sqrt[k-l+1]{\gamma+1}$ $(< \sqrt[k-l+1]{\gamma-1})$, $i = 2t-1, 2t-3, \ldots, 1$.

We consider only the case $|y_{-i}| < \sqrt[k-l+1]{\gamma-1}$, $i = 2s, 2s-2, \ldots 2, 0$, and $|y_{-i}| > \sqrt[k-l+1]{\gamma+1}$, $i = 2t-1, 2t-3, \ldots, 1$.

It follows that $|\prod_{i=l}^{k} y_{-2i}| = |y_{-2k}||y_{-2k+2}| \cdots |y_0| < \gamma - 1$.

That is,

$$-\gamma+1 < \prod_{i=l}^{k} y_{-2i} < \gamma-1. \tag{4.4}$$

This implies that $-\gamma + 2 < 1 - \prod_{i=l}^{k} y_{-2i} < \gamma$ and so $|1 - \prod_{i=l}^{k} y_{-2i}| < \gamma$. Hence we have

$$\left|y_1\right| = \frac{\left|\gamma y_{-2r-1}\right|}{\left|1 - \prod_{i=l}^{k} y_{-2i}\right|} > \frac{\gamma\left|y_{-2r-1}\right|}{\gamma} = \left|y_{-2r-1}\right| > \sqrt[k-l+1]{\gamma + 1}. \tag{4.5}$$

Also $|\prod_{i=l}^{k} y_{-2i+1}| > \gamma+1$ implies that $\gamma+2 < 1-\prod_{i=l}^{k} y_{-2i+1} < -\gamma$ and so $|1-\prod_{i=l}^{k} y_{-2i+1}| > \gamma$. Hence we have

$$\left|y_2\right| = \frac{\left|\gamma y_{-2r}\right|}{\left|1 - \prod_{i=l}^{k} y_{-2i+1}\right|} < \frac{\gamma\left|y_{-2r}\right|}{\gamma} = \left|y_{-2r}\right| < \sqrt[k-l+1]{\gamma - 1}. \tag{4.6}$$

By induction we get

$$\begin{gathered} \left|y_{2(r+1)n+2j+1}\right| > \left|y_{2(r+1)(n-1)+2j+1}\right| > \sqrt[k-l+1]{\gamma + 1}, \\ \left|y_{2(r+1)n+2j}\right| < \left|y_{2(r+1)(n-1)+2j}\right| < \sqrt[k-l+1]{\gamma - 1}, \end{gathered} \tag{4.7}$$

$n \geq 0$ and $j = 0, 1, \ldots, r$. Now suppose that

$$\begin{gathered} \left|y_{2(r+1)(n-1)+2j+1}\right| \longrightarrow L_{2j+1} \in \left(\sqrt[k-l+1]{\gamma + 1}, \infty\right], \\ \left|y_{2(r+1)(n-1)+2j}\right| \longrightarrow L_{2j} \in \left[0, \sqrt[k-l+1]{\gamma - 1}\right), \end{gathered} \tag{4.8}$$

as $n \to \infty$, $j = 0, 1, \ldots, r$.

But as

$$\left|y_{2(r+1)n+2j}\right| = \frac{\gamma\left|y_{2(r+1)(n-1)+2j}\right|}{\left|1 - \prod_{i=l}^{k} y_{2(r+1)n+2j-2i-1}\right|} \leq \frac{\gamma\left|y_{2(r+1)(n-1)+2j}\right|}{\left|1 - \left|\prod_{i=l}^{k} y_{2(r+1)n+j-2i-1}\right|\right|}, \tag{4.9}$$

then

$$L_{2j} \leq \frac{\gamma L_{2j}}{\left|1 - \prod_{i=l}^{k} L_{2j-2i-1}\right|}, \quad j = 0, 1, \ldots, r. \tag{4.10}$$

We claim that for each $j = 0, 1, \ldots, r$, $L_{2j} = 0$.

For the sake of contradiction suppose that there exists $j \in \{0, 1, \ldots, r\}$ with $L_{2j} \in (0, \sqrt[k-l+1]{\gamma - 1})$.

Then (4.10) gives

$$\left|1 - \prod_{i=l}^{k} L_{2j-2i-1}\right| \leq \gamma. \tag{4.11}$$

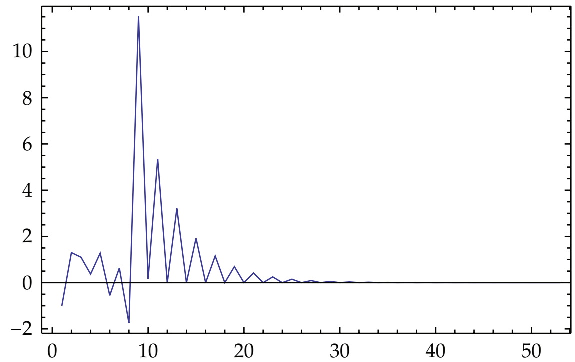

Figure 1: The difference equation $y_{n+1} = 0.6y_{n-1}/(1 - y_n y_{n-2})$.

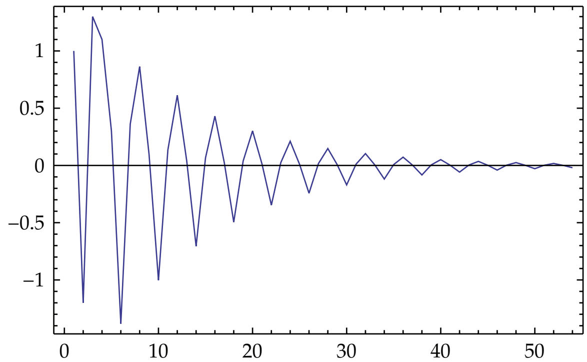

Figure 2: The difference equation $y_{n+1} = 0.7y_{n-3}/(1 - y_n y_{n-2})$.

This implies that

$$-\gamma + 1 \le \prod_{i=l}^{k} L_{2j-2i-1} \le \gamma + 1. \tag{4.12}$$

As $L_{2j+1} \in (\sqrt[k-l+1]{\gamma + 1}, \infty]$, $j = 0, 1, \ldots, r$, we have a contradiction.

Thus it is true that for each $j = 0, 1, \ldots, r$ we have $L_{2j} = 0$ and so $\lim_{n\to\infty} y_{2n} = 0$.

We now claim that for each $j = 0, 1, \ldots, r$, $L_{2j} = \infty$.

For the sake of contradiction, suppose that there exists $j \in \{0, 1, \ldots, r\}$ with $L_{2j+1} \in (\sqrt[k-l+1]{\gamma + 1}, \infty)$. Then

$$\begin{aligned} L_{2j+1} &= \lim_{n\to\infty} \left| y_{2(r+1)(n-1)+2j+1} \right| = \frac{\gamma \lim_{n\to\infty} \left| y_{2(r+1)(n-1)+2j+1} \right|}{\left| 1 - \prod_{i=l}^{k} \lim_{n\to\infty} y_{2(r+1)n+2j-2i} \right|} \\ &\ge \frac{\gamma \lim_{n\to\infty} \left| y_{2(r+1)(n-1)+2j+1} \right|}{1 + \prod_{i=l}^{k} \lim_{n\to\infty} \left| y_{2(r+1)n+2j-2i} \right|} = \frac{\gamma L_{2j+1}}{1 + \prod_{i=l}^{k} L_{2j}} = \gamma L_{2j+1}. \end{aligned} \tag{4.13}$$

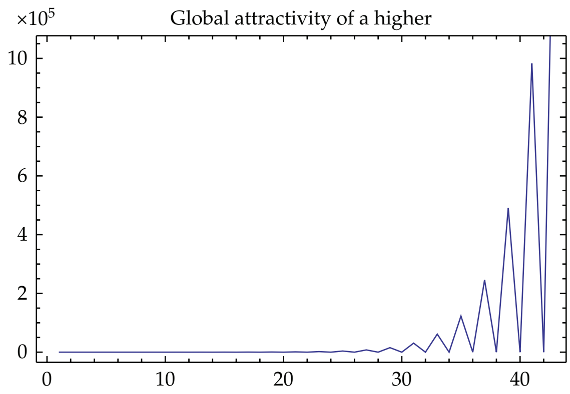

Figure 3: The difference equation $y_{n+1} = 2y_{n-1}/(1 - y_n y_{n-2})$.

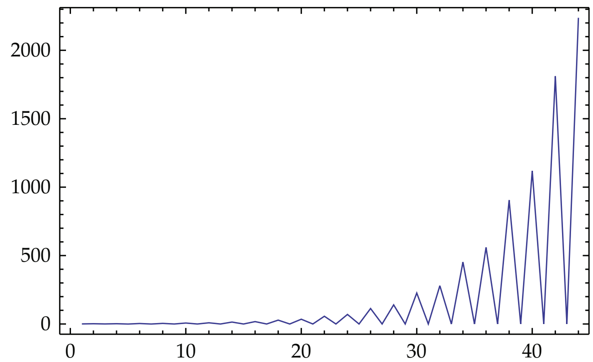

Figure 4: The difference equation $y_{n+1} = 2y_{n-3}/(1 - y_n y_{n-2})$.

This is a contradiction. Therefore for each $j = 0, 1, \ldots, r$ we have $L_{2j+1} = \infty$ and so $\lim_{n\to\infty} y_{2n+1} = \infty$.

The case when $|y_{-i}| > \sqrt[k-l+1]{\gamma + 1}$, $i = 2s, 2s-2, \ldots, 2, 0$ and $|y_{-i}| < \sqrt[k-l+1]{\gamma - 1}$, $i = 2t - 1, 2t-3, \ldots, 1$ is similar and will be omitted. □

5. Numerical Examples

Example 5.1. Figure 1 shows that if $r = 0$, $l = 0$, $k = 1$ ($\mathfrak{K} = \max\{2k, 2r+1\} = 2$) and $\gamma = 0.6$, then the solution $\{y_n\}_{n=-2}^{\infty}$ with initial conditions $y_{-2} = -1$, $y_{-1} = 1.3$, $y_0 = 1.1$ converges to zero.

Example 5.2. Figure 2 shows that if $r = 1$, $l = 0$, $k = 1$ ($\mathfrak{K} = \max\{2k, 2r+1\} = 3$) and $\gamma = 0.7$, then the solution $\{y_n\}_{n=-3}^{\infty}$ with initial conditions $y_{-3} = -1$, $y_{-2} = -1.2$, $y_{-1} = 1.3$, $y_0 = 1.1$ converges to zero.

Example 5.3. Figure 3 shows that if $r = 0$, $l = 0$, $k = 1$ ($\mathfrak{K} = \max\{2k, 2r+1\} = 2$) and $\gamma = 2$, then the solution $\{y_n\}_{n=-3}^{\infty}$ with initial conditions $y_{-2} = 2$, $y_{-1} = 0.4$, $y_0 = 2.1$ is unbounded.

Example 5.4. Figure 4 shows that if $r = 1$, $l = 0$, $k = 1$ ($\mathfrak{K} = \max\{2k, 2r + 1\} = 3$) and $\gamma = 2$, then the solution $\{y_n\}_{n=-3}^{\infty}$ with initial conditions $y_{-3} = 0.5$, $y_{-2} = 2$, $y_{-1} = 0.4$, $y_0 = 2.1$ is unbounded.

Acknowledgment

This paper was funded by the Deanship of the Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. (15-662-D1432). The author, therefore, acknowledge with thanks DSR technical and financial support.

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