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*Research Article*

# Differentiability Properties of the Pre-Image Pressure

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We study the differentiability properties of the pre-image pressure. For a TDS $(X, T)$ with finite topological pre-image entropy, we prove the pre-image pressure function $P_{\text{pre}}(T, \bullet)$ is Gateaux differentiable at $f \in C(X, \mathbb{R})$ if and only if $P_{\text{pre}}(T, \bullet)$ has a unique tangent functional at $f$. Also, we obtain some equivalent conditions for $P_{\text{pre}}(T, \bullet)$ to be Fréchet differentiable at $f$.

## 1. Introduction

By a topological dynamical system (for short TDS), we mean a pair $(X, T)$ where $X$ is a compact metric space and $T : X \to X$ is a continuous surjection from $X$ to itself. Entropies are fundamental to our current understanding of dynamical systems. The classical measure-theoretic entropy for an invariant measure and the topological entropy were introduced in [1, 2], respectively, and the classical variational principle was completed in [3, 4]. Topological entropy measures the maximal exponential growth rate of orbits for arbitrary topological dynamical systems, and measure-theoretic entropy measures the maximal loss of information for the iteration of finite partitions in a measure-preserving transformation.

Topological pressure is a generalization of topological entropy for a dynamical system. The notion was first introduced by Ruelle [5] in 1973 for an expansive dynamical system and later by Walters [6] for the general case. The theory related to the topological pressure, variational principle, and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory, and dynamical systems (see, e.g., the books [7–12]). Since the works of Bowen [13] and Ruelle [14], the topological pressure has become a basic tool in the dimension theory related to dynamical systems. One of the basic questions of physical interest in that of differentiability of the pressure. The differentiability of the pressure was considered by many people (see, e.g., [15–18]).

Recently, the pre-image structure of maps has become deeply characterized via entropies. In several papers (see [19–24]), some important pre-image entropy invariants of dynamical systems have been introduced and their relationships with topological entropy have been established. In a certain sense, these new invariants give a quantitative estimate of how "noninvertible" a system is. In [25], we defined the topological pre-image pressure of topological dynamical systems, which is a generalization of the Cheng-Newhouse pre-image entropy (see [19]), and proved a variational principle for it. We gave some applications of the pre-image pressure to equilibrium states (see [25, 26]). Under the assumption that $h_{\text{pre}}(T) < \infty$ and the metric pre-image entropy function $h_{\{\text{pre},\bullet\}}(T)$ is upper semicontinuous, we obtained a way to describe a kind of continuous dependence of equilibrium states. Also, we proved that the set of all continuous functions with unique equilibrium states is a dense $G_\delta$-set of $C(X,\mathbb{R})$, and for any finite collection of ergodic measures, we can find some continuous function such that its set of equilibrium states contains the given set (see [26]).

The purpose of this paper is to study the differentiability properties of the pre-image pressure of the TDS $(X,T)$ with finite topological pre-image entropy. In Section 2, we concentrate on reviewing some basic definitions and give some basic properties of tangent functionals to the pre-image pressure.

In Section 3, the Gateaux differentiability of the pre-image pressure is discussed. We show that the pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is Gateaux differentiable at $f \in C(X,\mathbb{R})$ if and only if it has a unique tangent functional at $f$.

In Section 4, we discuss the Fréchet differentiability of the pre-image pressure. We obtain some equivalent conditions for $P_{\text{pre}}(T,\bullet)$ to be Fréchet differentiable at $f$. Also, we show that the pre-image function $P_{\text{pre}}(T,\bullet)$ is Fréchet differentiable if and only if $(X,T)$ is uniquely ergodic, and hence the pre-pressure is linear.

## 2. Preliminaries

Throughout the paper, let $(X,T)$ be a TDS with finite topological pre-image entropy $h_{\text{pre}}(T)$ (see [19] for definition). In this section, we will recall some basic definitions and give some useful properties.

Let $(X,T)$ be a TDS and let $\mathcal{B}(X)$ be the collection of all Borel subsets of $X$. Recall that a *cover* of $X$ is a family of Borel subsets of $X$ whose union is $X$. An *open cover* is one that consists of open sets. A *partition* of $X$ is a cover of $X$ consisting of pairwise disjoint sets. We denote the set of finite covers, finite open covers, and finite partition of $X$ by $\mathcal{C}_X$, $\mathcal{C}_X^o$, and $\mathcal{P}_X$, respectively. Given two covers $\mathcal{U},\mathcal{V}$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denoted by $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. We set $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

Denote by $C(X,\mathbb{R})$ the Banach space of all continuous, real-valued functions on $X$ endowed with the supremum norm. For $f \in C(X,R)$ and $n \in \mathbb{N}$, we denote $\sum_{i=0}^{n-1} f(T^i(x))$ by $(S_n f)(x)$.

### *2.1. Topological Pre-Image Pressure*

In an early paper with Zeng et al. [25], following the idea of topological pressure (see Chapter 9, [12]), we defined a new notion of pre-image pressure, which extends the Cheng-Newhouse pre-image entropy $h_{\text{pre}}(T)$ [19]. For a given TDS $(X,T)$, the pre-image pressure of

$T$ is a map $P_{\mathrm{pre}}(T,\bullet) : C(X,\mathbb{R}) \to \mathbb{R}$ which is convex, Lipschitz continuous, increasing, with $P_{\mathrm{pre}}(T,0) = h_{\mathrm{pre}}(T)$. More precisely, let $\mathcal{U} \in \mathcal{C}_X^o$. For $x \in X$ and $k \in \mathbb{N}$, we put

$$P_n\left(T, f, \mathcal{U}, T^{-k}(x)\right) := \inf_{\mathcal{V}} \sum_{B\in\mathcal{V}} \sup_{y\in B} \mathrm{e}^{(S_n f)(y)}, \tag{2.1}$$

where the infimum is taken over all finite subcovers $\mathcal{V}$ of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$ with respect to $T^{-k}(x)$. We define the *pre-image pressure of $T$ related to $\mathcal{U}$ at $f$* as

$$P_{\mathrm{pre}}(T, f, \mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \log P_{\mathrm{pre},n}(T, f, \mathcal{U}), \tag{2.2}$$

where $P_{\mathrm{pre},n}(T, f, \mathcal{U}) := \sup_{x\in X,\, k\geqslant n} P_n(T, f, \mathcal{U}, T^{-k}(x))$. The *pre-image pressure of $T$ at $f$* is defined by

$$P_{\mathrm{pre}}(T, f) := \sup_{\mathcal{U}\in\mathcal{C}_X^o} P_{\mathrm{pre}}(T, f, \mathcal{U}). \tag{2.3}$$

It is clear that $P_{\mathrm{pre}}(T, f) \leqslant P(T, f)$ (topological pressure, see [12]) and $P_{\mathrm{pre}}(T, 0) = h_{\mathrm{pre}}(T)$. $P_{\mathrm{pre}}(T, f) \leqslant \|f\|$ if $T$ is a homeomorphism.

### *2.2. Measure-Theoretic Pre-Image Entropy*

Denote by $\mathcal{M}(X)$, $\mathcal{M}(X,T)$, and $\mathcal{M}^e(X,T)$ the set of all Borel probability measures, $T$-invariant Borel probability measures and $T$-invariant ergodic measures, on $X$, respectively. Note that $\mathcal{M}(X,T) \supseteq \mathcal{M}^e(X,T) \neq \emptyset$, and both $\mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are convex compact metric spaces when endowed with the weak*-topology; $\mathcal{M}^e(X,T)$ is a $G_\delta$ subset of $\mathcal{M}(X,T)$ (see [12, Chapter 6]). Beside the weak*-topology on $\mathcal{M}(X,T)$, we also have the norm topology arising from the metric:

$$\|\mu - \nu\| := \sup\left\{ \left| \int g\, \mathrm{d}\mu - \int g \mathrm{d}\nu \right| : g \in C(X,\mathbb{R}),\ \|g\| \leqslant 1 \right\}. \tag{2.4}$$

Note that $\|\mu - \nu\| = 2$ if $\mu \neq \nu \in \mathcal{M}^e(X,T)$.

Given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and a sub-$\sigma$-algebra $\mathcal{A} \subseteq \mathcal{B}(X)$, define

$$H_\mu(\alpha \mid \mathcal{A}) := \sum_{A\in\alpha} \int_X -\mathbb{E}(1_A \mid \mathcal{A}) \log \mathbb{E}(1_A \mid \mathcal{A}) \mathrm{d}\mu, \tag{2.5}$$

where $\mathbb{E}(1_A \mid \mathcal{A})$ is the conditional expectation of $1_A$ with respect to $\mathcal{A}$. It is a standard fact that $H_\mu(\alpha \mid \mathcal{A})$ increases with respect to $\alpha$ and decreases with respect to $\mathcal{A}$. Note that $\beta \in \mathcal{P}_X$ naturally generates a sub-$\sigma$-algebra $\mathcal{F}(\beta)$ of $\mathcal{B}(X)$; where there is no ambiguity, we write $\mathcal{F}(\beta)$ as $\beta$. It is easy to check, for $\alpha, \beta \in \mathcal{P}_X$, that $H_\mu(\alpha \mid \beta) = H_\mu(\alpha \vee \beta) - H_\mu(\beta)$. More generally, for a sub-$\sigma$-algebra $\mathcal{A} \subseteq B(X)$, we have

$$H_\mu(\alpha \vee \beta \mid \mathcal{A}) = H_\mu(\beta \mid \mathcal{A}) + H_\mu(\alpha \mid \beta \vee \mathcal{A}). \tag{2.6}$$

When $\mu \in \mathcal{M}(X,T)$ and $\mathcal{A} \subseteq \mathcal{B}(X)$ is a $T$-invariant sub-$\sigma$-algebra, that is, $T^{-1}\mathcal{A} = \mathcal{A}$ (up to $\mu$-null sets), it is not hard to see that $a_n = H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A})$ is a nonnegative subadditive sequence for a given $\alpha \in \mathcal{P}_X$, that is, $a_{n+m} \le a_n + a_m$ for all positive integers $n$ and $m$. It is well known that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge 1} \frac{a_n}{n}. \tag{2.7}$$

The *conditional entropy of* $\alpha$ *with respect to* $\mathcal{A}$ is then defined by

$$h_\mu(T,\alpha \mid \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A}\right) = \inf_{n\ge 1} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A}\right). \tag{2.8}$$

Moreover, the *metric conditional entropy of* $(X,T)$ *with respect to* $\mathcal{A}$ is defined by

$$h_\mu(T,X \mid \mathcal{A}) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha \mid \mathcal{A}). \tag{2.9}$$

Note that if $\mathcal{N} = \{\emptyset, X\}$ is a trivial sub-$\sigma$-algebra, we recover the metric entropy, and we write $h_\mu(T,\alpha \mid \mathcal{N})$ and $h_\mu(T,X \mid \mathcal{N})$ simple by $h_\mu(T,\alpha)$ and $h_\mu(T)$.

Particularly, set $\mathcal{B}^- = \bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}(X)$, then $\mathcal{B}^-$ is a $T$-invariant sub-$\sigma$ algebra. We call $\mathcal{B}^-$ the *infinite past* $\sigma$*-algebra related to* $\mathcal{B}(X)$. We define the *measure-theoretic (or metric) pre-image entropy of* $\alpha$ *with respect to* $(X,T)$ by

$$h_{\mathrm{pre},\mu}(T,\alpha) := h_\mu(T,\alpha \mid \mathcal{B}^-) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{B}^-\right). \tag{2.10}$$

Moreover, we define the *metric pre-image entropy of* $(X,T)$ by

$$h_{\mathrm{pre},\mu}(T) := \sup_{\alpha\in\mathcal{P}_X} h_{\mathrm{pre},\mu}(T,\alpha). \tag{2.11}$$

### *2.3. A Variational Principle for Pre-Image Pressure*

The following variational relationship for topological pre-image pressure and measure-theoretic pre-image entropy is established in [25].

**Theorem 2.1.** *Let* $(X,T)$ *be a TDS and* $f \in C(X,\mathbb{R})$*. Then,*

$$P_{\mathrm{pre}}(T,f) = \sup_{\mu\in\mathcal{M}(X,T)} \left\{ h_{\mathrm{pre},\mu}(T) + \int_X f \,\mathrm{d}\mu \right\}. \tag{2.12}$$

We also have (see, e.g., [26]) the following proposition.

**Proposition 2.2.** *Let* $(X,T)$ *be a TDS,* $\alpha \in \mathcal{P}_X$ *and* $\mu \in \mathcal{M}(X,T)$*. Then,* $\mu \mapsto h_{\mathrm{pre},\mu}(T,\alpha)$ *and* $\mu \mapsto h_{\mathrm{pre},\mu}(T)$ *are both affine functions on* $\mathcal{M}(X,T)$*. Moreover, if the ergodic decomposition of* $\mu$ *is*

$\mu = \int_{\mathcal{M}^e(X,T)} \theta \mathrm{d}\lambda(\theta)$, *then*

$$h_{\text{pre},\mu}(T,\alpha) = \int_{\mathcal{M}^e(X,T)} h_{\text{pre},\theta}(T,\alpha)\mathrm{d}\lambda(\theta), \qquad h_{\text{pre},\mu}(T) = \int_{\mathcal{M}^e(X,T)} h_{\text{pre},\theta}(T)\mathrm{d}\lambda(\theta). \tag{2.13}$$

### 2.4. Equilibrium States and Tangent Functionals to Pre-Image Pressure

Given $f \in C(X,\mathbb{R})$. A finite signed Borel measure $\mu$ on $(X,\mathcal{B}(X))$ is called a *tangent functional* to $P_{\text{pre}}(T,\bullet)$ at $f$ if

$$P_{\text{pre}}(T, f+g) - P_{\text{pre}}(T,f) \geqslant \int g\,\mathrm{d}\mu, \quad \forall g \in C(X,\mathbb{R}). \tag{2.14}$$

Let $\mathcal{T}_f(X,T)$ denote the collection of all tangent functionals to $P_{\text{pre}}(T,\bullet)$ at $f$. An application of the Hahn-Banach theorem gives $\mathcal{T}_f(X,T) \neq \emptyset$. It is easy to see that $\mu \in \mathcal{T}_f(X,T)$ if and only if

$$P_{\text{pre}}(T,f) - \int f\,\mathrm{d}\mu = \inf\left\{ P_{\text{pre}}(T,g) - \int g\,\mathrm{d}\mu : g \in C(X,\mathbb{R}) \right\}. \tag{2.15}$$

Also, we have $\mathcal{T}_f(X,T) \subseteq \mathcal{M}(X,T)$ (see [25] for details).

**Theorem 2.3.** *The following holds.*

(1) *For* $f \in C(X,\mathbb{R})$,

$$\mathcal{T}_f(X,T) = \bigcap_{n=1}^{\infty} \overline{\left\{ \mu \in \mathcal{M}(X,T) : h_{\text{pre},\mu}(T) + \int f\,\mathrm{d}\mu > P_{\text{pre}}(T,f) - \frac{1}{n} \right\}}. \tag{2.16}$$

(2) *If* $f_1, f_2 \in C(X,\mathbb{R})$ *and* $\mu \in \mathcal{T}_{f_1}(X,T) \cap \mathcal{T}_{f_2}(X,T)$, *then*

$$P_{\text{pre}}(T, pf_1 + (1-p)f_2) = P_{\text{pre}}(T,f_1) + (1-p)\int (f_1 - f_1)\mathrm{d}\mu, \quad \forall p \in [0,1], \tag{2.17}$$

*and* $\mathcal{T}_{pf_1+(1-p)f_2}(X,T) \subseteq \mathcal{T}_{f_1}(X,T) \cap \mathcal{T}_{f_1}(X,T)$.

*Proof.* (1) Let $\mu \in I_f \equiv \bigcap_{n=1}^{\infty} \overline{\{\mu \in \mathcal{M}(X,T) : h_{\text{pre},\mu}(T) + \int f\,\mathrm{d}\mu > P_{\text{pre}}(T,f) - 1/n\}}$. By Theorem 2.1, there is $\mu_n \in \mathcal{M}(X,T)$ with $\mu_n \to \mu$ and $h_{\text{pre},\mu_n}(T) + \int f\,\mathrm{d}\mu_n \to P_{\text{pre}}(T,f)$. Hence, for each $g \in C(X,\mathbb{R})$,

$$P_{\text{pre}}(T, f+g) - P_{\text{pre}}(T,f) \geqslant \lim_{n\to\infty}\left( h_{\text{pre},\mu_n}(T) + \int (f+g)\mathrm{d}\mu_n - P_{\text{pre}}(T,f) \right) = \int g\,\mathrm{d}\mu, \tag{2.18}$$

which follows that $\mu \in \mathcal{C}_f(X,T)$. Now suppose there is $\mu_0 \in \mathcal{C}_f(X,T) \setminus I_f$. Since $I_f$ is convex, the standard separation theorem [27, page 417] follows that there exists $g \in C(X,\mathbb{R})$ with

$$\int g\,\mathrm{d}\mu_0 > \sup\left\{\int g\,\mathrm{d}\mu : \mu \in I_f\right\}. \tag{2.19}$$

By Theorem 2.1, we can choose $\mu_n \in \mathcal{M}(X,T)$ such that

$$h_{\mathrm{pre},\mu_n}(T) + \int \left(f + \frac{g}{n}\right)\mathrm{d}\mu_n > P_{\mathrm{pre}}\left(T, f + \frac{g}{n}\right) - \frac{1}{n^2}. \tag{2.20}$$

Without loss of generality, we can assume $\mu_n \to \mu^*$. Then,

$$\begin{aligned}\int g\,\mathrm{d}\mu_0 &= n\int \frac{g}{n}\mathrm{d}\mu_0 \leqslant n\left(P_{\mathrm{pre}}\left(T, f + \frac{g}{n}\right) - P_{\mathrm{pre}}(T,f)\right)\\ &< n\left(h_{\mathrm{pre},\mu_n}(T) + \int \left(f + \frac{g}{n}\right)\mathrm{d}\mu_n + \frac{1}{n^2} - h_{\mathrm{pre},\mu_n}(T) - \int f\,\mathrm{d}\mu_n\right)\\ &= \int g\,\mathrm{d}\mu_n + \frac{1}{n} \to \int g\,\mathrm{d}\mu^*.\end{aligned} \tag{2.21}$$

However, $\mu^* \in I_f$ follows from the fact:

$$\begin{aligned}h_{\mathrm{pre},\mu_n}(T) + \int f\,\mathrm{d}\mu_n &> P_{\mathrm{pre}}\left(T, f + \frac{g}{n}\right) - \int \frac{g}{n}\,\mathrm{d}\mu_n - \frac{1}{n^2} \quad (\text{by } (2.20))\\ &> P_{\mathrm{pre}}(T,f) - \frac{2\cdot\|g\|}{n} - \frac{1}{n^2}, \quad (\text{by } [25,\ \text{Lemma } 4.1(3)]),\end{aligned} \tag{2.22}$$

which is a contradiction.

(2) If $0 < p < 1$, then

$$\begin{aligned}&pP_{\mathrm{pre}}(T,f_1) + (1-p)P_{\mathrm{pre}}(T,f_2)\\ &\quad\geqslant P_{\mathrm{pre}}(T, pf_1 + (1-p)f_2) \quad (\text{by } [25, \text{Lemma } 4.1(3)])\\ &\quad= P_{\mathrm{pre}}(T, f_1 + (1-p)(f_2 - f_1))\\ &\quad\geqslant P_{\mathrm{pre}}(T,f_1) + (1-p)\int (f_2 - f_1)\,\mathrm{d}\mu \quad (\text{since } \mu \in \mathcal{C}_{f_1}(X,T)).\end{aligned} \tag{2.23}$$

Hence,

$$P_{\mathrm{pre}}(T,f_2) - \int f_2\,\mathrm{d}\mu \geqslant P_{\mathrm{pre}}(T,f_1) - \int f_1\,\mathrm{d}\mu. \tag{2.24}$$

By symmetry, $P_{\text{pre}}(T, f_2) - \int f_2\, d\mu = P_{\text{pre}}(T, f_1) - \int f_1\, d\mu$, which implies

$$P_{\text{pre}}(T, f_1) + (1-p)\int (f_2 - f_1)\, d\mu = pP_{\text{pre}}(T, f_1) + (1-p)P_{\text{pre}}(T, f_2) \geqslant P_{\text{pre}}(T, pf_1 + (1-p)f_2). \tag{2.25}$$

□

A member $\mu \in \mathcal{M}(X, T)$ is called an *equilibrium state* for $P_{\text{pre}}(T, \bullet)$ at $f$ if

$$P_{\text{pre}}(T, f) = h_{\text{pre},\mu}(T) + \int f\, d\mu. \tag{2.26}$$

Let $\mathcal{M}_f(X, T)$ denote the collection of all equilibrium states for $P_{\text{pre}}(T, \bullet)$ at $f$.

The set $\mathcal{T}_f(X, T)$ is convex and compact in the weak*-topology. The set $\mathcal{M}_f(X, T)$ is convex but it may be not closed in the weak*-topology. Note that $\mathcal{M}_f(X, T) \subseteq \mathcal{T}_f(X, T) \subseteq \mathcal{M}(X, T)$ and $\mathcal{M}_f(X, T)$ could be empty (see Example 5.1, [25]). We also have $\mathcal{M}_f(X, T) = \mathcal{T}_f(X, T)$ if and only if the metric pre-image entropy map $h_{\text{pre},\bullet}(T)$ is upper semicontinuous at every element of $\mathcal{T}_f(X, T)$, Theorem 5.2 [25]. The extreme points of $\mathcal{M}_f(X, T)$ are precisely the ergodic members of $\mathcal{M}_f(X, T)$ and if $\mu \in \mathcal{M}_f(X, T)$, then almost every measure in the ergodic decomposition of $\mu$ is a member of $\mathcal{M}_f(X, T)$ (see Proposition 2.1, [26]). When the metric pre-image entropy map $h_{\text{pre},\bullet}(T)$ is upper semicontinuous on $\mathcal{M}(X, T)$, then $\bigcup_{f \in C(X,\mathbb{R})} \mathcal{M}_f(X, T)$ is dense in $\mathcal{M}(X, T)$ in the norm topology, and given any finite collection of ergodic measures $\{\mu_1, \dots, \mu_n\}$, there is some $f \in C(X, \mathbb{R})$ such that $\{\mu_1, \dots, \mu_n\} \subseteq \mathcal{M}_f(X, T)$ [26, Theorem 4.2].

The following theorem shows when tangent functionals to pre-image pressure are not equilibrium states.

**Theorem 2.4.** *Let $(X, T)$ be a TDS and $f \in C(X, \mathbb{R})$. The following statements are mutually equivalent:*

(1) $\mu \in \mathcal{T}_f(X, T) \setminus \mathcal{M}_f(X, T)$;

(2) $h_{\text{pre},\mu}(T) + \int f\, d\mu < P_{\text{pre}}(T, f)$ *and there exist* $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}(X, T)$ *with* $\mu_n \to \mu$ *and* $h_{\text{pre},\mu_n}(T) + \int f\, d\mu_n \to P_{\text{pre}}(T, f)$;

(3) $\mu \in \mathcal{T}_f(X, T)$ *and* $h_{\text{pre},\bullet}(T)$ *is not upper semicontinuous at* $\mu$.

*Proof.* (1)⇒(2) Follows from the variational principle and Theorem 2.3(1).

(2)⇒(3) By Theorem 2.3(1), $\mu \in \mathcal{T}_f(X, T)$. If $h_{\text{pre},\bullet}(T)$ is upper semicontinuous at $\mu$, then

$$P_{\text{pre}}(T, f) = \lim_{n\to\infty}\left(h_{\text{pre},\mu_n}(T) + \int f\, d\mu_n\right) \leqslant h_{\text{pre},\mu}(T) + \int f\, d\mu. \tag{2.27}$$

Hence, $\mu \in \mathcal{M}_f(X, T)$ by the variational principle.

(3)⇒(1) If (3) holds, then there are $\mu_n \in \mathcal{M}(X, T)$ with $\mu_n \to \mu$ and $\lim_{n\to\infty} h_{\text{pre},\mu_n}(T) > h_{\text{pre},\mu}(T)$. Hence,

$$P_{\text{pre}}(T, f) \geqslant \lim_{n\to\infty}\left(h_{\text{pre},\mu_n}(T) + \int f\, d\mu_n\right) > h_{\text{pre},\mu}(T) + \int f\, d\mu. \tag{2.28}$$

Therefore, $\mu \in \mathcal{T}_f(X, T) \setminus \mathcal{M}_f(X, T)$. □

## 3. Gateaux Differentiability of the Pre-Image Pressure

In [26], we studied the uniqueness of the equilibrium state for the pre-image pressure. We showed that when the metric pre-image entropy map $h_{\text{pre},\bullet}(T)$ is upper semicontinuous on $\mathcal{M}(X,T)$, then the set of all functions with unique equilibrium state is dense in $C(X,\mathbb{R})$. Without the upper semicontinuity assumption, one can show that all functions with unique tangent functional are dense in $C(X,\mathbb{R})$ (can see [27, page 450] or [11, Appendix A.3.6]). In this section, we will show a continuous function with unique tangent functional to pre-image pressure if and only if it is Gateaux differentiable.

Given $f,g \in C(X,\mathbb{R})$. Since $P_{\text{pre}}(T,\bullet)$ is convex, the map $t \mapsto (P_{\text{pre}}(T,f+tg) - P_{\text{pre}}(T,f))/t$ is increasing and hence

$$d^{+}P_{\text{pre}}(T,f)(g) = \lim_{t\to 0+}\frac{P_{\text{pre}}(T,f+tg)-P_{\text{pre}}(T,f)}{t},$$
$$d^{-}P_{\text{pre}}(T,f)(g) = \lim_{t\to 0-}\frac{P_{\text{pre}}(T,f+tg)-P_{\text{pre}}(T,f)}{t} \tag{3.1}$$

exist. Note that $d^{+}P_{\text{pre}}(T,f)(g) = -d^{-}P_{\text{pre}}(T,f)(-g)$. The pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is said to be *Gateaux differentiable at* $f$ if, for all $g \in C(X,\mathbb{R})$,

$$\lim_{t\to 0}\frac{P_{\text{pre}}(T,f+tg)-P_{\text{pre}}(T,f)}{t} \tag{3.2}$$

exist. It is easy to check that $P_{\text{pre}}(T,\bullet)$ is Gateaux differentiable at $f$ if and only if $g \mapsto d^{+}P_{\text{pre}}(T,f)(g)$ is linear.

**Lemma 3.1.** *Let $(X,T)$ be a TDS and $f,g \in C(X,\mathbb{R})$. Then,*

$$d^{+}P_{\text{pre}}(T,f)(g) = \sup\left\{\int g\,\mathrm{d}\mu : \mu \in \mathcal{T}_f(X,T)\right\}. \tag{3.3}$$

*Proof.* If $\mu \in \mathcal{T}_f(X,T)$, then, for $g \in C(X,\mathbb{R})$,

$$\int g\,\mathrm{d}\mu \leqslant \frac{P_{\text{pre}}(T,f+tg)-P_{\text{pre}}(T,f)}{t}, \quad \forall t > 0. \tag{3.4}$$

Hence,

$$\sup\left\{\int g\,\mathrm{d}\mu : \mu \in \mathcal{T}_f(X,T)\right\} \leqslant d^{+}P_{\text{pre}}(T,f)(g). \tag{3.5}$$

Next, we prove the converse inequality. Set $a = d^{+}P_{\text{pre}}(T,f)(g)$. Define a continuous linear functional $\gamma : \{tg : t \in \mathbb{R}\} \to \mathbb{R}$ by

$$\gamma(tg) = ta, \quad t \in \mathbb{R}. \tag{3.6}$$

The convexity of $P_{\text{pre}}(T,\bullet)$ implies

$$\gamma(tg) = t \cdot d^{+}P_{\text{pre}}(T,f)(g) \leqslant P_{\text{pre}}(T,f+tg) - P_{\text{pre}}(T,f), \quad \forall t \in \mathbb{R}. \tag{3.7}$$

By the Hahn-Banach theorem, $\gamma$ can be extended to a continuous linear functional on $C(X,\mathbb{R})$ such that

$$\gamma(h) \leqslant P_{\text{pre}}(T,f+h) - P_{\text{pre}}(T,f), \quad \forall h \in C(X,\mathbb{R}). \tag{3.8}$$

By the Riesz representation theorem, there is $\mu \in \mathcal{M}(X)$ with

$$\gamma(h) = \int h \, \mathrm{d}\mu, \quad \forall g \in C(X,\mathbb{R}). \tag{3.9}$$

Combining (3.6), (3.8), and (3.9), we have $\mu \in \mathfrak{T}_f(X,T)$, and

$$\int g \, \mathrm{d}\mu = \gamma(g) = a = d^{+}P_{\text{pre}}(T,f)(g). \tag{3.10}$$

The lemma is proved. □

**Theorem 3.2** (Uniqueness of tangent functional and Gateaux differentiability). *The following statements are mutually equivalent:*

(1) *the pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is Gateaux differentiable at $f \in C(X,\mathbb{R})$;*

(2) *the unique tangent functional to $P_{\text{pre}}(T,\bullet)$ at $f$ is $\mu$;*

(3) *for each $g \in C(X,\mathbb{R})$,*

$$\lim_{t\to 0} \frac{P_{\text{pre}}(T,f+tg) - P_{\text{pre}}(T,f)}{t} = \int g \, \mathrm{d}\mu. \tag{3.11}$$

*Proof.* (1)⇒(2) If the pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is Gateaux differentiable at $f$, then the function $g \mapsto d^{+}P_{\text{pre}}(T,f)(g)$ is linear. By Lemma 3.1,

$$\begin{aligned} \sup\left\{\int g \, \mathrm{d}\mu : \mu \in \mathfrak{T}_f(X,T)\right\} &= d^{+}P_{\text{pre}}(T,f)(\mathrm{g}) \\ &= -d^{+}P_{\text{pre}}(T,f)(-g) \\ &= -\sup\left\{\int -g \, \mathrm{d}\mu : \mu \in \mathfrak{T}_f(X,T)\right\} \\ &= \inf\left\{\int g \, \mathrm{d}\mu : \mu \in \mathfrak{T}_f(X,T)\right\} \end{aligned} \tag{3.12}$$

for each $g \in C(X,\mathbb{R})$. This implies there is a unique tangent functional to $P_{\text{pre}}(T,\bullet)$ at $f$.

(2)⇒(3) It directly follows from Lemma 3.1.

(3)⇒(1) It follows from the definition. □

## 4. Fréchet Differentiability of the Pre-Image Pressure

In this section, we will study the Fréchet differentiability of pre-image pressure. The pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is said to be *Fréchet differentiable at* $f$ if there is $\gamma \in C(X,\mathbb{R})^*$ such that

$$\lim_{g\to 0}\frac{|P_{\text{pre}}(T,f+g)-P_{\text{pre}}(T,f)-\gamma(g)|}{\|g\|}=0. \tag{4.1}$$

The pre-image pressure function $P_{\text{pre}}(T,\bullet)$ is said to be *Fréchet differentiable* if it is Fréchet differentiable at each $f\in C(X,\mathbb{R})$.

Note that if $P_{\text{pre}}(T,\bullet)$ is Fréchet differentiable at $f$, then it is Gateaux differentiable at $f$ and $\gamma(g)=\int g\,d\mu$, where $\mu$ is the unique tangent functional to $P_{\text{pre}}(T,\bullet)$ at $f$.

**Theorem 4.1.** *The following conditions are mutually equivalent:*

(1) $P_{\text{pre}}(T,\bullet)$ *if Fréchet differentiable at* $f$*;*

(2) $\mathcal{T}_f(X,T)=\{\mu_f\}$ *and* $\|\mu_n-\mu_f\|\to 0$ *for each* $\{\mu_n\}\subseteq \mathcal{M}(X,T)$ *with* $h_{\text{pre},\mu_n}(T)+\int f\,d\mu_n \to P_{\text{pre}}(T,f)$*;*

(3) $P_{\text{pre}}(T,\bullet)$ *is locally affine at* $f$*;*

(4) $\mathcal{T}_f(X,T)=\{\mu_f\}$ *and*

$$\lim_{g\to 0}\sup\{\|\mu-\mu_f\| : \mu\in\mathcal{T}_{f+g}(X,T)\}=0. \tag{4.2}$$

*Proof.* (1)⇒(2) Suppose $P_{\text{pre}}(T,\bullet)$ is Fréchet differentiable at $f$. Then, $f$ has a unique tangent functional $\mu_f$ to $P_{\text{pre}}(T,\bullet)$ at $f$. Let $\{\mu_n\}_{n=1}^{\infty}\subseteq\mathcal{M}(X,T)$ with $h_{\text{pre},\mu_n}(T)+\int f\,d\mu_n\to P_{\text{pre}}(T,f)$. Given $\epsilon>0$, there are $N\in\mathbb{N}$ and $\delta>0$ such that

$$P_{\text{pre}}(T,f)\leqslant h_{\text{pre},\mu_n}(T)+\int f\,d\mu_n+\epsilon\delta,\quad \forall n\geqslant N, \tag{4.3}$$

$$0\leqslant P_{\text{pre}}(T,f+g)-P_{\text{pre}}(T,f)-\int g\,d\mu_f\leqslant \epsilon\cdot\|g\|\quad\text{whenever } \|g\|<\delta. \tag{4.4}$$

Hence, if $n\geqslant N$ and $\|g\|<\delta$, then

$$\begin{aligned}\int g\,d\mu_n-\int g\,d\mu &= P_{\text{pre}}(T,f)+\int g\,d\mu_n-P_{\text{pre}}(T,f)-\int g\,d\mu_f\\ &\leqslant h_{\text{pre},\mu_n}(T)+\int(f+g)\,d\mu_n+\epsilon\delta-P_{\text{pre}}(T,f)-\int g\,d\mu_f\quad(\text{by 4.4})\end{aligned}$$

$$\begin{aligned}
&\leqslant P_{\text{pre}}(T, f+g) - P_{\text{pre}}(T, f) - \int g \,\mathrm{d}\mu_f + \epsilon\delta \quad (\text{by Theorem 2.1}) \\
&\leqslant \epsilon \cdot (\|g\| + \delta) \leqslant 2\epsilon\delta \quad (\text{by 4.4}).
\end{aligned} \tag{4.5}$$

Note that (4.5) is also true when $-g$ instead of $g$. So,

$$\left|\int g \,\mathrm{d}\mu_n - \int g \,\mathrm{d}\mu\right| \leqslant 2\epsilon\delta \quad \text{whenever } \|g\| < \delta, \quad n \geqslant N. \tag{4.6}$$

Therefore,

$$\begin{aligned}
\|\mu_n - \mu_f\| &= \sup\left\{\left|\int g \,\mathrm{d}\mu_n - \int g \,\mathrm{d}\mu_f\right| : \|g\| < 1\right\} \\
&= \frac{1}{\delta}\sup\left\{\left|\int g \,\mathrm{d}\mu_n - \int g \,\mathrm{d}\mu_f\right| : \|g\| < \delta\right\} \\
&\leqslant 2\epsilon \quad (\text{by 4.6}).
\end{aligned} \tag{4.7}$$

By arbitrary of $\epsilon$, $\|\mu_n - \mu_f\| \to 0$.

(2)⇒(3) By the variational principle of pre-image pressure, we can pick ergodic measures $\mu_n$ such that $h_{\text{pre},\mu_n}(T) + \int f \mathrm{d}\mu_n \to P_{\text{pre}}(T, f)$, Then $\|\mu_n - \mu_f\| \to 0$. Note that two distinct ergodic measures have norm-distance 2. So there is $N$ such that $\mu_n = \mu_f$ for all $n \geqslant N$. Hence, $\mu_f \in \mathcal{M}^e(X, T)$ and

$$\epsilon := P_{\text{pre}}(T, f) - \sup\left\{h_{\text{pre},\mu}(T) + \int f \,\mathrm{d}\mu : \mu \in \mathcal{M}^e(X, T),\ \mu \neq \mu_f\right\} > 0. \tag{4.8}$$

This implies for each $g \in C(X, \mathbb{R})$, with $\|f - g\| < \epsilon/2$, that

$$\begin{aligned}
&\sup\left\{h_{\text{pre},\mu}(T) + \int g \,\mathrm{d}\mu : \mu \in \mathcal{M}^e(X, T),\ \mu \neq \mu_f\right\} \\
&\quad\leqslant P_{\text{pre}}(T, f) - \epsilon + \|f - g\| \\
&\quad\leqslant P_{\text{pre}}(T, g) - \epsilon + 2\|f - g\| \\
&\quad< P_{\text{pre}}(T, g).
\end{aligned} \tag{4.9}$$

By the variational principle of pre-image pressure again, we have

$$P_{\text{pre}}(T, g) = h_{\text{pre},\mu_f}(T) + \int g \,\mathrm{d}\mu_f \quad \text{whenever } \|f - g\| < \frac{\epsilon}{2}. \tag{4.10}$$

Hence, $P_{\text{pre}}(T, \bullet)$ is affine on the neighborhood $B(f, \epsilon/2) = \{g \in C(X, \mathbb{R}) : \|f - g\| < \epsilon/2\}$ of $f$.

(3)⇒(4) Is obvious.

(4)⇒(1) Let $g \in C(X,\mathbb{R})$ and $\mu \in \mathcal{T}_{f+g}(X,T)$. By definition, we have

$$P_{\mathrm{pre}}(T,f) - P_{\mathrm{pre}}(T,f+g) \geqslant -\int g\,\mathrm{d}\mu. \tag{4.11}$$

Hence,

$$\begin{aligned} 0 &\leqslant P_{\mathrm{pre}}(T,f+g) - P_{\mathrm{pre}}(T,f) - \int g\,\mathrm{d}\mu_f \\ &\leqslant \int g\,\mathrm{d}\mu - \int g\,\mathrm{d}\mu_f \quad (\text{by } (4.11)) \\ &\leqslant \|g\| \cdot \|\mu - \mu_f\|. \end{aligned} \tag{4.12}$$

Therefore,

$$0 \leqslant \frac{P_{\mathrm{pre}}(T,f+g) - P_{\mathrm{pre}}(T,f) - \int g\,\mathrm{d}\mu_f}{\|g\|} \leqslant \sup\{\|\mu - \mu_f\| : \mu \in \mathcal{T}_{f+g}(X,T)\} \longrightarrow 0 \tag{4.13}$$

as $g \to 0$. That is $P_{\mathrm{pre}}(T,\bullet)$ is Fréchet differentiable at $f$. □

**Corollary 4.2.** *Let $(X,T)$ be a TDS with finite pre-image entropy. Then, $P_{\mathrm{pre}}(T,\bullet)$ is Fréchet differentiable if and only if $T$ is uniquely ergodic.*

*Proof.* Using Theorem 4.1, $P_{\mathrm{pre}}(T,\bullet)$ is locally affine whenever $P_{\mathrm{pre}}(T,\bullet)$ is Fréchet differentiable. Hence, the map $f \in C(X,\mathbb{R}) \mapsto \mu_f \in \mathcal{M}(X,T)$ is locally constant, where $\mathcal{T}_f(X,T) = \{\mu_f\}$ for each $f \in C(X,\mathbb{R})$. Since $C(X,\mathbb{R})$ is connected, the map is constant. So $\mu_f = \mu_0$ for all $f \in C(X,\mathbb{R})$. If $\mu \in \mathcal{M}(X,T) \setminus \{\mu_0\}$, then we can choose $f \in C(X,\mathbb{R})$ such that $\int f\,\mathrm{d}\mu > \int f\,\mathrm{d}\mu_0$. Then for sufficiently large $k$, we have

$$P_{\mathrm{pre}}(T,kf) \geqslant h_{\mathrm{pre},\mu}(T) + \int kf\,\mathrm{d}\mu > h_{\mathrm{pre},\mu_0}(T) + \int kf\,\mathrm{d}\mu_0 = P_{\mathrm{pre}}(T,kf), \tag{4.14}$$

which is a contradiction. Therefore, $\mathcal{M}(X,T) = \{\mu_0\}$. □

*Remark 4.3.* In the situation of Corollary 4.2, there is only one invariant measure, and the pre-image pressure is the expectation with respect to this measure, hence, it is linear.

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