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*Research Article*

# Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model

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We study the optimal investment strategies of DC pension, with the stochastic interest rate (including the CIR model and the Vasicek model) and stochastic salary. In our model, the plan member is allowed to invest in a risk-free asset, a zero-coupon bond, and a single risky asset. By applying the Hamilton-Jacobi-Bellman equation, Legendre transform, and dual theory, we find the explicit solutions for the CRRA and CARA utility functions, respectively.

## 1. Introduction

There are two radically different methods to design a pension fund: defined-benefit plan (hereinafter DB) and defined-contribution plan (hereinafter DC). In DB, the benefits are fixed in advance by the sponsor and the contributions are adjusted in order to maintain the fund in balance, where the associated financial risks are assumed by the sponsor agent; in DC, the contributions are fixed and the benefits depend on the returns on the assets of the fund, where the associated financial risks are borne by the beneficiary. Historically, DB is the more popular. However, in recent years, owing to the demographic evolution and the development of the equity markets, DC plays a crucial role in the social pension systems.

Our main objective in this paper is to find the optimal investment strategies for DC, which is a common model in the employment system. The paper extends the previous works of Cairns et al. [1] and Gao [2]. In particular, we consider the following framework: (i) the optimal investment strategies are derived with CARA and CRRA utility functions; (ii) the interest rate is affine (including the CIR model and the Vasicek model); (iii) the salary follows a general stochastic process.

Because the member of DC has some freedom in choosing the investment allocation of her pension fund in the accumulation phase, she has to solve an optimal investment strategies' problem. Traditionally, the usual method to deal with it has been the maximization of expected utility of final wealth. Consistently with the economics and financial literature, the most widely used utility function exhibits constant relative risk aversion (CRRA), that is, the power or logarithmic utility function (e.g., [1–5]). Some papers use the utility function that exhibits constant absolute risk aversion (CARA), that is, the exponential utility function (e.g., [6]). Some papers also adopt the CRRA and CARA utility functions simultaneously (e.g., [7, 8]). In this paper, we show the optimal investment strategies for DC pension with the CRRA and CARA utility functions.

The optimal portfolios for DC with stochastic interest rate have been widely discussed in the literatures. Some of them are by Boulier et al. [3], Battocchio and Menoncin [6], and Cairns et al. [1], where the interest rate is assumed to be of the Vasicek model. However, in the works of Deelstra et al. [4] and Gao [2], the interest rate has an affine structure, which includes the Cox-Ingersoll-Ross (CIR) model and the Vasicek model. In the Vasicek model, the volatility of interest rate is only a constant. It can generate a negative interest rate, which is not in accord with the facts. But in the CIR model, the volatility of interest rate is modified by the square of interest rate, which more tallies with practice. Obviously,

the affine interest rate model does not only contain the Cox-Ingersoll-Ross (CIR) model and the Vasicek model, but also more accords with practice.

Meanwhile, Deelstra et al. [4] assumed that the stochastic interest rates followed the affine dynamics, described the contribution flow by a nonnegative, progressive measurable and square-integrable process, and then studied optimal investment strategies for different examples of guarantees and contributions. Battocchio and Menoncin [6] took into account two background risks (the salary risk and the inflation) in the Vasicek framework and analyzed in detail the behavior of the optimal portfolio with respect to salary and inflation. Cairns et al. [1] incorporated asset, salary (labor-income), and interest-rate risk (the Vasicek model), used the member's final salary as a numeraire, and then discussed various properties and characteristics of the optimal asset-allocation strategy both with and without the presence of nonhedgeable salary risk. However, except for them, the studies related with DC generally suppose that the salary is a constant, but the assumption is difficult to be accepted for the pension investment. In fact, the optimal investment for a pension fund involves quite a long period, generally from 20 to 40 years. The pension investment is considered to be a long-term investment problem. During the period, the salary switches violently; so it becomes crucial to take into account the salary risk. As a result, we consider the salary risk and use the member's final salary as a numeraire based on the work of Cairns et al. [1].

In addition, under the logarithmic utility function, Gao [2] just studied the portfolio problem of DC with the affine interest rate but did not consider the stochastic salary. The contribution of this paper: (i) extends the research of Gao [2] to the case of the power (CRRA) and exponential (CARA) utility functions under the stochastic salary; (ii) extends the research of Cairns et al. [1] to the case of the plan member with the CRRA and CARA utility functions under the affine interest rate model (including the CIR model and the Vasicek model). We consider that the financial market consists of three assets: a risk-less asset (i.e., cash), a zero-coupon bond, and a single risky asset (i.e., stock). Applying the maximum principle, we derive a nonlinear second-order partial differential equation (PDE) for the value function of the optimization problem. However, it is difficult to characterize the solution structure, especially under the framework of stochastic interest rates and stochastic salary. But the primary problem can be changed into a dual one by applying a Legendre transform. The transform methods can be found from the works of Xiao et al. [5] and Gao [2, 8].

The most novel feature of our research is the application of affine interest rate model and stochastic salary under the CRRA and CARA utility functions, which has not been reported in the existing literature. We assume that the term structure of the interest rates is affine, not a constant and the salary volatility is a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. Consequently, a complicated nonlinear second-order partial differential equation is derived by using the methods of stochastic optimal control. However, we find that it is difficult to determine an explicit solution, and then we transform the primary problem into the dual one by applying a Legendre transform and derive a linear partial differential equation. Furthermore, we obtain the explicit solutions for the optimal strategies under the CRRA or CARA utility functions.

The rest of the paper is organized as follows. In Section 2, we introduce the mathematical model including the financial market, the stochastic salary, and the wealth process. In Section 3, we propose the optimization problems. In Section 4, we transform the nonlinear second partial differential equation into a linear partial differential equation by the Legendre transform and dual theory. In Section 5, we obtain the explicit solutions for the CRRA and CARA utility functions, respectively. In Section 6, we draw the conclusions.

## 2. Mathematical Model

In this section, we introduce the market structure and define the stochastic dynamics of the asset values and the salary.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where $T > 0$ denotes the retirement time of a representative shareholder.

*2.1. The Financial Market.* We suppose that the market is composed of three kinds of financial assets: a risk-free asset, a zero-coupon bond, and a single risky asset, and the investor can buy or sell continuously without incurring any restriction as short sales constraint or any trading cost. For the sake of simplicity, we will only consider a risky asset which can indeed represent the index of the stock market.

Let us begin with a complete probability space $(\Omega, F, P)$, where $\Omega$ is the real space, and $P$ is the probability measure. $\{W_r(t), W_s(t) : t \geq 0\}$ is a standard, two-dimensional Brownian motion defined on a complete probability space $(\Omega, F, P)$. The filtration $F = \{F_t\}_{t\in[0,T]}$ is a right continuous filtration of sigma-algebras on this space and denotes the information structure generated by the Brownian motions.

We denote the price of the risk-free asset (i.e., cash) at time $t$ by $S_0(t)$, which evolves according to the following equation:

$$dS_0(t) = r(t) S_0(t)\, dt, \quad S_0(0) = 1, \tag{1}$$

where the dynamics of the short interest rate process $r(t)$ are described by the following stochastic differential equation:

$$\begin{aligned} dr(t) &= (a - br(t))\, dt - \sigma_r dW_r(t), \\ \sigma_r &= \sqrt{k_1 r(t) + k_2}, \quad t \geq 0, \end{aligned} \tag{2}$$

with the coefficients $a, b, r(0)$, $k_1$, and $k_2$ being positive real constants.

Notes that the dynamics recover, as a special case, the Vasicek [9] (resp., Cox et al. [10]) dynamics, when $k_1$ (resp., $k_2$) is equal to zero. So under these dynamics, the term structure of the interest rates is affine, which has been studied by Duffie and Kan [11], Deelstra et al. [4], and Gao [2].

We assume that the price of the risky asset is a continuous time stochastic process. We denote the price of the risky asset

(i.e., stock) at time $t$ by $S(t), t \geq 0$. The dynamics of $S(t)$ are given by

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma_s\left(dW_s(t) + \lambda_1 dt\right) + \eta_1\sigma_r\left(dW_r(t) + \lambda_2\sigma_r dt\right), \quad S(0) = S_0, \tag{3}$$

with $\lambda_1, \lambda_2$ (resp., $\sigma_s, \eta_1$) being constants (resp., positive constants) (see Deelstra et al. [4] and Gao [2]). Here, the two Brownian motions, $W_r(t)$ and $W_s(t)$, are supposed to be orthogonal.

The last asset is a zero-coupon bond with maturity $T$, whose price at time $t$ is denoted by $B(t,T), t \geq 0$, which is described by the following stochastic differential equation (c.f. [2, 4]):

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + \sigma_B(T-t, r(t)) \times \left(dW_r(t) + \lambda_2\sigma_r dt\right), \quad B(T,T) = 1, \tag{4}$$

where $\sigma_B(T-t, r(t)) = f(T-t)\sigma_r$ with

$$f(t) = \frac{2\left(e^{mt}-1\right)}{m - (b - k_1\lambda_2) + e^{mt}\left(m + b - k_1\lambda_2\right)}, \quad m = \sqrt{(b-k_1\lambda_2)^2 + 2k_1}. \tag{5}$$

*2.2. The Stochastic Salary.* Based on the works of Deelstra et al. [4], Battocchio and Menoncin [6], and Cairns et al. [1], we denote the salary at time $t$ by $L(t)$ which is described by

$$\frac{dL(t)}{L(t)} = \mu_L(t, r(t))\,dt + \eta_2\sigma_r dW_r(t) + \eta_3\sigma_s dW_s(t), \quad L(0) = L_0, \tag{6}$$

where $\eta_2, \eta_3$ are real constants, which are two volatility scale factors measuring how the risk sources of interest rate and stock affect the salary. That is to say, the salary volatility is supposed to a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. This assumption is in accordance with that of Deelstra et al. [4], but is different from those of Battocchio and Menoncin [6] and Cairns et al. [1] who also assumed that the salary was affected by nonhedgeable risk source (i.e., non-financial market). Moreover, we assume that the instantaneous mean of the salary is such that $\mu_L(t, r(t)) = r(t) + m_L$, where $m_L$ is a real constant.

*2.3. Pension Wealth Process.* According to the viewpoint of Cairns et al. [1], we consider that the contributions are continuously into the pension fund at the rate of $kL(t)$. Let $V_t$ denote the wealth of pension fund at time $t \in [0,T]$. $\pi_B(t)$ and $\pi_S(t)$ are denoted, respectively, by the proportion of the pension fund invested in the bond and the stock; so $\pi_0(t) = 1 - \pi_B(t) - \pi_S(t)$ is the proportion of the pension fund invested in the risk-free asset. The dynamics of the pension wealth are given by

$$dV(t) = \left(1 - \pi_B - \pi_S\right) V(t)\frac{dS_0(t)}{S_0(t)} + \pi_B V(t)\frac{dB(t,T)}{B(t,T)} + \pi_s V(t)\frac{dS(t)}{S(t)} + kL(t)\,dt, \tag{7}$$

where $V(0) = V_0$ stands for an initial wealth.

Taking into (1), (3), and (4), the evolution of pension wealth can be rewritten as

$$\begin{aligned} dV(t) = {} & V(t)\left(r(t) + \pi_B\lambda_2\sigma_r\sigma_B + \pi_S\left(\lambda_1\sigma_S + \lambda_2\eta_1\sigma_r^2\right)\right)dt \\ & + kL(t)\,dt + V(t)\left(\pi_B\sigma_B + \pi_S\eta_1\sigma_r\right)dW_r(t) \\ & + V(t)\pi_S\sigma_S dW_s(t). \end{aligned} \tag{8}$$

At the time of retirement, the plan member will be concerned about the preservation of his standard of living so he will be interested in his retirement income relative to his preretirement salary [1]. Considering the plan member's salary as a numeraire, we define a new state variable $X(t) = V(t)/L(t)$ (i.e., the relative wealth).

Taking into (6) and (8), by applying product law and Ito's formula, the stochastic differential equation for $X(t)$ is

$$\begin{aligned} dX(t) = {} & X(t)\Big[r(t) - \mu_L + \eta_2^2\sigma_r^2 + \eta_3^2\sigma_S^2 \\ & + \pi_B\sigma_r\sigma_B\left(\lambda_2 - \eta_2\right) \\ & + \pi_S\left(\lambda_1\sigma_S + \eta_3\sigma_S^2 + \lambda_2\eta_1\sigma_r^2 - \eta_1\eta_2\sigma_r^2\right)\Big]dt + kdt \\ & + X(t)\left(\pi_B\sigma_B + \pi_S\eta_1\sigma_r - \eta_2\sigma_r\right)dW_r(t) \\ & + X(t)\left(\pi_S - \eta_3\right)\sigma_S dW_s(t), \\ & X(0) = \frac{V(0)}{L(0)} = \frac{V_0}{L_0}. \end{aligned} \tag{9}$$

In the remainder, therefore, we will focus on $X(t)$ alone.

## 3. The Optimization Program

The plan member will retire at time $T$ and is risk averse; so the utility function $U(x)$ is typically increasing and concave ($U''(x) < 0$). In this section, we are interested in maximizing the utility of the plan member's terminal relative wealth.

Let us denote a strategy $\pi_t$ which is described by a dynamic process $(\pi_B(t), \pi_S(t))$. For a strategy $\pi_t$, we define the utility attained by the plan member from state $x$ at time $t$ as

$$H_{\pi_t}(t, r, x) = E_{\pi_t}\left[U(X(T)) \mid r(t) = r, X(t) = x\right]. \tag{10}$$

Our objective is to find the optimal value function:

$$H(t,r,x) = \sup_{\pi_t \in \pi} H_{\pi_t}(t,r,x), \tag{11}$$

and the optimal strategy is $\pi_t^* = (\pi_B^*(t), \pi_S^*(t))$ such that $H_{\pi_t^*}(t,r,x) = H(t,r,x)$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$\begin{aligned} &H_t + (a-br)H_r + \frac{1}{2}\sigma_r^2 H_{rr} \\ &\quad + \max_{\pi_t \in \pi}\{x(\alpha_1 + \pi_B\alpha_2 + \pi_S\alpha_3)H_x + kH_x \\ &\qquad + \frac{1}{2}x^2(\pi_B\sigma_B + \pi_S\eta_1\sigma_r - \eta_2\sigma_r)^2 H_{xx} \\ &\qquad + \frac{1}{2}x^2(\pi_S - \eta_3)^2\sigma_S^2 H_{xx} \\ &\qquad - x\sigma_r(\pi_B\sigma_B + \pi_S\eta_1\sigma_r - \eta_2\sigma_r)H_{rx}\} = 0, \end{aligned} \tag{12}$$

with

$$\begin{aligned} \alpha_1 &= r - \mu_L + \eta_2^2\sigma_r^2 + \eta_3^2\sigma_S^2, \\ \alpha_2 &= \sigma_r\sigma_B(\lambda_2 - \eta_2), \\ \alpha_3 &= \lambda_1\sigma_S + \eta_3\sigma_S^2 + \lambda_2\eta_1\sigma_r^2 - \eta_1\eta_2\sigma_r^2, \\ H(T,r,x) &= U(x), \end{aligned} \tag{13}$$

where $H_t, H_r, H_x, H_{rx}, H_{rr}$, and $H_{xx}$ denote partial derivatives of first and second orders with respect to time, short interest rate, and relative wealth.

The first-order maximizing conditions for the optimal strategies $\pi_B^*$ and $\pi_S^*$ are

$$\begin{aligned} &\alpha_2 H_x + x\sigma_B(\pi_B^*\sigma_B + \pi_S^*\eta_1\sigma_r - \eta_2\sigma_r)H_{xx} - \sigma_r\sigma_B H_{rx} = 0, \\ &\alpha_3 H_x + x\eta_1\sigma_r(\pi_B^*\sigma_B + \pi_S^*\eta_1\sigma_r - \eta_2\sigma_r)H_{xx} \\ &\quad + x\sigma_S(\pi_S^* - \eta_3)H_{xx} - \eta_1\sigma_r^2 H_{rx} = 0. \end{aligned} \tag{14}$$

We have

$$\begin{aligned} \pi_S^* &= \eta_3 - \frac{\lambda_1 + \eta_3\sigma_S^2}{x\sigma_S}\frac{H_x}{H_{xx}}, \\ \pi_B^* &= \frac{\sigma_r(\eta_2 - \eta_1\eta_3)}{\sigma_B} + \frac{\alpha_4\sigma_r}{x\sigma_B}\frac{H_x}{H_{xx}} + \frac{\sigma_r}{x\sigma_B}\frac{H_{rx}}{H_{xx}}, \\ \alpha_4 &= \frac{(\eta_2\sigma_S + \lambda_1\eta_1 + \eta_1\eta_3\sigma_S^2 - \lambda_2\sigma_S)}{\sigma_S}, \end{aligned} \tag{15}$$

Putting this in (12), we obtain a partial differential equation (PDE) for the value function $H$:

$$\begin{aligned} &H_t + (a-br)H_r + \frac{1}{2}\sigma_r^2 H_{rr} + (k + x\beta_0)H_x \\ &\quad + \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2\sigma_r^2\right)\frac{H_x^2}{H_{xx}} \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2\frac{H_x H_{rx}}{H_{xx}} - \frac{1}{2}\sigma_r^2\frac{H_{rx}^2}{H_{xx}} = 0, \\ &\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L, \\ &\beta_1 = \eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3 - \eta_3\sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{16}$$

with $H(T,r,x) = U(x)$.

Here, we notice that the stochastic control problem described in the previous section has been transformed into a PDE. The problem is now to solve (16) for the value function $H$ and replace it in (15) in order to obtain the optimal investment strategies.

## 4. The Legendre Transform

In this section, we transform the non-linear second partial differential equation into a linear partial differential equation via the Legendre transform and dual theory.

**Theorem 1.** *Let $f : R^n \to R$ be a convex function, for $z > 0$, define the Legendre transform:*

$$L(z) = \max_x\{f(x) - zx\}. \tag{17}$$

The function $L(z)$ is called the Legendre dual of the function $f(x)$ (c.f. [12]).

If $f(x)$ is strictly convex, the maximum in the above equation will be attained at just one point, which we denote by $x_0$. It is attained at the unique solution to the first-order condition, namely, $df(x)/dx - z = 0$.

So, we may rewrite $L(z) = f(x_0) - zx_0$.

According to Theorem 1, we can take advantage of the assumed convexity of the value function $H(t,r,x)$ to define the Legendre transform:

$$\begin{aligned} &\widehat{H}(t,r,z) \\ &\quad = \sup_{x>0}\{H(t,r,x) - zx \mid 0 < x < \infty\}, \quad 0 < t < T, \end{aligned} \tag{18}$$

where $z > 0$ denotes the dual variable to $x$, which is the same as those of Xiao et al. [5] and Gao [2, 8].

The value of $x$ where this optimum is attained is denoted by $g(t,r,z)$, so that

$$\begin{aligned} &g(t,r,z) \\ &\quad = \inf_{x>0}\left\{x \mid H(t,r,x) \geq zx + \widehat{H}(t,r,z)\right\}, \quad 0 < t < T. \end{aligned} \tag{19}$$

The two functions $g(t,r,z)$ and $\widehat{H}(t,r,z)$ are closely related, and we will refer to either one of them as the dual of $H$. In this paper, we will work mainly with the function $g$, as it is easier to compute numerically and suffices for the purpose of computing optimal investment strategies.

This leads to

$$\begin{aligned} \widehat{H}(t,r,z) &= H(t,r,g) - zg, \\ g(t,r,z) &= x, \qquad H_x = z. \end{aligned} \tag{20}$$

So the function $\widehat{H}$ is related to $g$ by $g = -\widehat{H}_z$.

At the terminal time, we denote

$$\begin{aligned} \widehat{U}(z) &= \sup_{v>0} \{U(v) - zv \mid 0 < v < \infty\}, \\ G(z) &= \sup_{v>0} \left\{v \mid U(v) \geq zv + \widehat{U}(z)\right\}. \end{aligned} \tag{21}$$

As a result, $G(z) = (U')^{-1}(z)$.

Generally speaking, $G$ is referred to as the inverse of marginal utility. Note that $H(T,r,x) = U(x)$, and then at the terminal time $T$, we can define

$$\begin{aligned} g(T,r,z) &= \inf_{x>0} \left\{x \mid U(x) \geq zx + \widehat{H}(T,r,z)\right\}, \\ \widehat{H}(T,r,z) &= \sup_{x>0} \{U(x) - zx\}. \end{aligned} \tag{22}$$

So $g(T,r,z) = (U')^{-1}(z)$.

By differentiating (20) with respect to $t$, $r$, and $z$, the transformation rules for the derivatives of the value function $H$ and the dual function $\widehat{H}$ can be given by (e.g., [2, 5, 8, 12]):

$$\begin{aligned} &H_x = z, \qquad H_t = \widehat{H}_t, \\ &H_r = \widehat{H}_r, \qquad H_{rr} = \widehat{H}_{rr} - \frac{\widehat{H}_{rz}^2}{\widehat{H}_{zz}}, \\ &H_{rx} = -\frac{\widehat{H}_{rz}}{\widehat{H}_{zz}}, \qquad H_{xx} = -\frac{1}{\widehat{H}_{zz}}. \end{aligned} \tag{23}$$

Substituting the expression (23), we rewrite (16) and obtain the following partial differential equation:

$$\begin{aligned} &\widehat{H}_t + (a - br)\widehat{H}_r + \frac{1}{2}\sigma_r^2 \widehat{H}_{rr} + (k + x\beta_0) z \\ &\quad - \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z^2 \widehat{H}_{zz} \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 z \widehat{H}_{rz} = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{24}$$

Combining with $x = g = -\widehat{H}_z$ and differentiating the above equation for $\widehat{H}$ with respect to $z$, we derive

$$\begin{aligned} &g_t + (a - br) g_r + \frac{1}{2}\sigma_r^2 g_{rr} - k - \beta_0 g - \beta_0 z g_z \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 g_r + (\lambda_2 - \eta_2)\sigma_r^2 z g_{rz} \\ &\quad - 2\left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z g_z \\ &\quad - \left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) z^2 g_{zz} = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{25}$$

Here, we notice that the non-linear second-order partial differential equation (16) has been transformed into a linear partial differential equation (25) by using the Legendre transform and dual theory. Under the given utility function, it is easy to find the solution of (25) by the classical variable decomposition approach.

Similarly, we can compute the optimal investment strategies as the feedback formulas in terms of derivatives of the value function. In terms of the dual function $g$, they are given by

$$\begin{aligned} \pi_0(t) &= 1 - \pi_B(t) - \pi_S(t), \\ \pi_S^* &= \eta_3 + \frac{\lambda_1 + \eta_3 \sigma_S^2}{x\sigma_S} z \widehat{H}_{zz} = \eta_3 - \frac{\lambda_1 + \eta_3 \sigma_S^2}{x\sigma_S} z g_z, \\ \pi_B^* &= \frac{(\eta_2 - \eta_1 \eta_3)}{f(T-t)} - \frac{\alpha_4 z \widehat{H}_{zz}}{x f(T-t)} + \frac{\widehat{H}_{rz}}{x f(T-t)} \\ &= \frac{(\eta_2 - \eta_1 \eta_3)}{f(T-t)} + \frac{\alpha_4}{x} z g_z - \frac{g_r}{x f(T-t)}, \\ \alpha_4 &= \frac{\left(\eta_2 \sigma_S + \lambda_1 \eta_1 + \eta_1 \eta_3 \sigma_S^2 - \lambda_2 \sigma_S\right)}{\sigma_S}, \\ f(t) &= \frac{2\left(e^{mt} - 1\right)}{m - (b - k_1 \lambda_2) + e^{mt}(m + b - k_1 \lambda_2)} \\ m &= \sqrt{(b - k_1 \lambda_2)^2 + 2k_1}. \end{aligned} \tag{26}$$

The problem is now to solve the linear partial differential equation (25) for $g$ and to replace these solutions in (26) in order to obtain the optimal strategies.

## 5. Optimal Investment Strategies with Some Specific Utilities

This section provides the explicit solutions for the CRRA and CARA utility functions.

*5.1. The Explicit Solution for The CRRA Utility Function.* Assume that the plan member takes a power utility function

$$U(x) = \frac{x^p}{p}, \quad (\text{with } p < 1, p \neq 0). \tag{27}$$

The relative risk aversion of a decision maker with the utility described in (27) is constant, and (27) is a CRRA utility.

According to $g(T, r, z) = (U')^{-1}(z)$ and the CRRA utility function, we obtain

$$g(T, r, z) = z^{1/(p-1)}. \tag{28}$$

Therefore, we conjecture a solution to (25) with the following form:

$$g(t, r, z) = z^{1/(p-1)} h(t, r) + a(t), \tag{29}$$

with the boundary conditions given by $a(T) = 0$, $h(T, r) = 1$.

Then,

$$\begin{aligned} g_t &= h_t z^{1/(p-1)} + a'(t), \qquad g_r = h_r z^{1/(p-1)}, \\ g_z &= -\frac{h}{1-p} z^{(1/(p-1))-1}, \qquad g_{rr} = h_{rr} z^{1/(p-1)}, \\ g_{rz} &= -\frac{h_r}{1-p} z^{(1/(p-1))-1}, \\ g_{zz} &= \frac{(2-p)h}{(1-p)^2} z^{(1/(p-1))-2}. \end{aligned} \tag{30}$$

Introducing these derivatives in (25), we derive

$$\begin{aligned} &\left\{ h_t + (a - br) h_r - \frac{(\lambda_2 - \eta_2) p \sigma_r^2}{1-p} h_r + \frac{1}{2}\sigma_r^2 h_{rr} \right. \\ &\quad \left. + \frac{\beta_0 p}{1-p} h - \frac{ph}{(1-p)^2}\left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) \right\} z^{1/(p-1)} \\ &\quad + a'(t) - \beta_0 a(t) - k = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{31}$$

We can split (31) into two equations in order to eliminate the dependence on $z^{1/(p-1)}$:

$$a'(t) - \beta_0 a(t) - k = 0, \tag{32}$$

$$\begin{aligned} &h_t + (a - br) h_r - \frac{(\lambda_2 - \eta_2) p \sigma_r^2}{1-p} h_r + \frac{1}{2}\sigma_r^2 h_{rr} \\ &\quad + \frac{\beta_0 p}{1-p} h - \frac{ph}{(1-p)^2}\left(\beta_1 - \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2\right) = 0. \end{aligned} \tag{33}$$

Taking into account the boundary condition $a(T) = 0$, the solution to (32) is

$$\begin{aligned} a(t) &= -k\left(\frac{1 - e^{-\beta_0 (T-t)}}{\beta_0}\right), \\ \beta_0 &= \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \end{aligned} \tag{34}$$

where $\bar{a}_{\overline{T-t}|} = (1 - e^{-\beta_0 (T-t)})/\beta_0$ is a continuous annuity of duration $T - t$, and $\beta_0$ is the continuous technical rate.

Noting that (33) is a linear second-order PDE, we find the solution by the classical variable decomposition approach.

Let

$$h(t, r) = A(t) e^{B(t) r} \tag{35}$$

with the boundary conditions: $A(T) = 1$, $B(T) = 0$. Introducing this in (33), we obtain

$$\begin{aligned} &\frac{A_t}{A} + \frac{a - ((\lambda_2 - \eta_2) k_1 + a) p}{1-p} B + \frac{1}{2} k_2 B^2 \\ &\quad + \frac{p(\beta_0 - \beta_1 - p\beta_0)}{(1-p)^2} + \frac{(\lambda_2 - \eta_2)^2 p k_2}{2(1-p)^2} \\ &\quad + r\left(B_t - \frac{b + ((\lambda_2 - \eta_2) k_1 - b) p}{1-p} B \right. \\ &\qquad \left. + \frac{1}{2} k_1 B^2 + \frac{(\lambda_2 - \eta_2)^2 p k_1}{2(1-p)^2}\right) = 0, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{36}$$

We can decompose (36) into two conditions in order to eliminate the dependence on $r$ and $t$:

$$\begin{aligned} &\frac{A_t}{A} + \frac{a - ((\lambda_2 - \eta_2) k_1 + a) p}{1-p} B + \frac{1}{2} k_2 B^2 \\ &\quad + \frac{p(\beta_0 - \beta_1 - p\beta_0)}{(1-p)^2} + \frac{(\lambda_2 - \eta_2)^2 p k_2}{2(1-p)^2} = 0, \\ &B_t - \frac{b + ((\lambda_2 - \eta_2) k_1 - b) p}{1-p} B \\ &\quad + \frac{1}{2} k_1 B^2 + \frac{(\lambda_2 - \eta_2)^2 p k_1}{2(1-p)^2} = 0. \end{aligned} \tag{37}$$

Taking into account the boundary conditions, the solutions to (37) are

$$B(t) = \frac{m_1 - m_1 e^{(1/2)k_1(m_1-m_2)(T-t)}}{1-(m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}},$$
$$A(t) = \exp\left\{\frac{((\lambda_2-\eta_2)k_1+a)p-a}{1-p}\int B(t)\,dt - \frac{1}{2}k_2\int B^2(t)\,dt - \frac{p(\beta_0-\beta_1-p\beta_0)}{(1-p)^2}t + C\right\}, \quad A(T)=1, \tag{38}$$

where

$$m_{1,2} = \left(b + ((\lambda_2-\eta_2)k_1 - b)p \pm \sqrt{(b+((\lambda_2-\eta_2)k_1-b)p)^2 - (\lambda_2-\eta_2)^2k_1^2p}\right) \times ((1-p)k_1)^{-1}. \tag{39}$$

From the above calculation, we finally obtain the optimal investment strategies under the CRRA utility.

**Proposition 2.** *The optimal investment strategies are given by*

$$\pi_0(t) = 1-\pi_B(t)-\pi_S(t),$$
$$\pi_S^* = \eta_3 + \frac{\lambda_1+\eta_3\sigma_S^2}{(1-p)\sigma_S}I(t,r), \tag{40}$$
$$\pi_B^* = \frac{1}{f(T-t)}\left\{(\eta_2-\eta_1\eta_3) - \frac{\alpha_4}{1-p}I(t,r)J(t)\right\},$$

*where*

$$I(t,r) = 1 + \frac{kl}{\nu}\bar{a}_{\overline{T-t}|}, \tag{41}$$
$$\bar{a}_{\overline{T-t}|} = \frac{1-e^{-\beta_0(T-t)}}{\beta_0},$$
$$J(t) = 1 + \frac{(1-p)B(t)}{\alpha_4},$$
$$B(t) = \frac{m_1 - m_1 e^{(1/2)k_1(m_1-m_2)(T-t)}}{1-(m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}},$$
$$f(t) = \frac{2(e^{mt}-1)}{m-(b-k_1\lambda_2)+e^{mt}(m+b-k_1\lambda_2)},$$
$$m = \sqrt{(b-k_1\lambda_2)^2+2k_1},$$
$$\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L,$$
$$\alpha_4 = \frac{(\eta_2\sigma_S + \lambda_1\eta_1 + \eta_1\eta_3\sigma_S^2 - \lambda_2\sigma_S)}{\sigma_S}, \tag{42}$$
$$m_{1,2} = \left(b + ((\lambda_2-\eta_2)k_1 - b)p \pm \sqrt{(b+((\lambda_2-\eta_2)k_1-b)p)^2 - (\lambda_2-\eta_2)^2k_1^2p}\right) \times ((1-p)k_1)^{-1}. \tag{43}$$

*Remark 3.* Note that the power utility function (27) will degenerate into a logarithmic utility function $U(x) = \ln x$ as the limit $p \to 0$ (e.g., [7, 13, 14]). Meanwhile, in (6), if $\eta_2 = 0, \eta_3 = 0$, the salary is not stochastic; so the contributions are not stochastic. If we further assume that $l = 1$, the model is the same as the model of Gao [2]. From Proposition 2, we find that as the limit $p \to 0$, the coefficients $m_{1,2}$ will reduce to $2b/k_1$ and zero, respectively. In this case, the coefficients $B(t)$ and $J(t)$ will, respectively, reduce to zero and one. As a result, the optimal investment strategies for a logarithmic utility function are

$$\pi_S^* = \frac{\lambda_1}{\sigma_S}\left(1 + \frac{k}{\nu}\bar{a}_{\overline{T-t}|r}\right),$$
$$\pi_B^* = \frac{\sigma_r(\lambda_2\sigma_S - \lambda_1\eta_1)}{\sigma_B\sigma_S}\left(1 + \frac{k}{\nu}\bar{a}_{\overline{T-t}|r}\right), \tag{44}$$

where $\pi_S^*$ is the same as the result of Gao [2], but $\pi_B^*$ is different from that result because Gao [2] made mistakes in calculation.

In this section, to make it easier for us to discuss the parameters' effect on the optimal investment strategies, we suppose that $\beta_0 > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$, where the assumption is generally in line with reality.

**Lemma 4.** *Consider*

$$I(t,r) > 0, \qquad \frac{dI(t,r)}{dt} < 0, \qquad \frac{d\pi_S^*}{dt} < 0. \tag{45}$$

*Proof.* Since $p < 1$, $k > 0$, $\beta_0 > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, by differentiating $I(t,r)$ with the respect to $t$, we have

$$\begin{aligned} \bar{a}_{\overline{T-t|}} &= \frac{1-e^{-\beta_0(T-t)}}{\beta_0} > 0, \quad I(t,r) > 0, \\ \frac{dI(t,r)}{dt} &= \frac{kl}{v}\frac{d\bar{a}_{\overline{T-t|}}}{dt} = -\frac{kl}{v}e^{-\beta_0(T-t)} < 0, \\ \frac{d\pi_S^*}{dt} &= \frac{\lambda_1+\eta_3\sigma_S^2}{(1-p)\sigma_S}\frac{dI(t,r)}{dt} < 0. \end{aligned} \tag{46}$$

□

**Lemma 5.** *Consider*

$$\begin{aligned} \frac{dJ(t)}{dt} &= \begin{cases} > 0, & (p<0), \\ < 0, & (0<p<1), \end{cases} \\ J(t) &= \begin{cases} \le 1, & (p<0), \\ \ge 1, & (0<p<1). \end{cases} \end{aligned} \tag{47}$$

*Proof.* Since $p < 1$, we have

$$m_1 \times m_2 = \frac{(\lambda_2-\eta_2)^2 p}{(1-p)^2} = \begin{cases} < 0, & (p<0), \\ > 0, & (0<p<1). \end{cases} \tag{48}$$

Here, we just consider the condition of $\alpha_4 > 0$. Differentiating $B(t)$ with the respect to $t$, we have

$$\begin{aligned} \frac{dB(t)}{dt} &= \frac{-(m_1-m_2)^2(k_1m_1/2m_2)e^{(1/2)k_1(m_1-m_2)(T-t)}}{\left(1-(m_1/m_2)\ e^{(1/2)k_1(m_1-m_2)(T-t)}\right)^2} \\ &= \begin{cases} > 0, & (p<0) \\ < 0, & (0<p<1), \end{cases} \\ \frac{dJ(t)}{dt} &= \frac{1-p}{\alpha_4}\frac{dB(t)}{dt} = \begin{cases} > 0, & (p<0), \\ < 0, & (0<p<1). \end{cases} \end{aligned} \tag{49}$$

In addition, noting that $B(T) = 0$ and $J(T) = 1$, we get

$$J(t) = \begin{cases} \le 1, & (p<0), \\ \ge 1, & (0<p<1). \end{cases} \tag{50}$$

□

**Lemma 6.** *Consider*

$$f(T-t) > 0, \qquad \frac{df(T-t)}{dt} < 0. \tag{51}$$

*Proof.* Since $T - t > 0$, and $k_1 > 0$, we have,

$$\begin{aligned} m &= \sqrt{(b-k_1\lambda_2)^2 + 2k_1} > |b-k_1\lambda_2| > 0, \\ e^{m(T-t)} &> 1, \\ f(T-t) &= \frac{2\left(e^{m(T-t)}-1\right)}{m-(b-k_1\lambda_2)+e^{m(T-t)}(m+b-k_1\lambda_2)} > 0, \\ \frac{df(T-t)}{dt} &= -\frac{4m^2e^{m(T-t)}}{\left(m-(b-k_1\lambda_2)+e^{m(T-t)}(m+b-k_1\lambda_2)\right)^2} < 0. \end{aligned} \tag{52}$$

□

**Lemma 7.** *Whether $d\pi_B^*/dt$ is positive or negative or neither is not established, and it is affected by the coefficient of relative risk aversion $p$ and the other parameters.*

*Proof.* By differentiating $\pi_B^*$ with the respect to $t$, we have

$$\begin{aligned} \frac{d\pi_B^*}{dt} &= \frac{-1}{f^2(T-t)}\frac{df(T-t)}{dt} \\ &\quad \times \left\{(\eta_2-\eta_1\eta_3) - \frac{\alpha_4}{1-p}I(t,r)J(t)\right\} \\ &\quad - \frac{\alpha_4}{(1-p)f(T-t)}\left\{I(t,r)\frac{dJ(t)}{dt} + \frac{dI(t,r)}{dt}J(t)\right\}. \end{aligned} \tag{53}$$

On the bases of Lemmas 4 and 6, we get

$$\begin{aligned} I(t,r) &> 0, \qquad \frac{dI(t,r)}{dt} < 0, \\ f(T-t) &> 0, \qquad \frac{df(T-t)}{dt} < 0. \end{aligned} \tag{54}$$

Meanwhile, based on Lemma 5, we get

$$\begin{aligned} \frac{dJ(t)}{dt} &= \begin{cases} > 0, & (p<0), \\ < 0, & (0<p<1), \end{cases} \\ J(t) &= \begin{cases} \le 1, & (p<0), \\ \ge 1, & (0<p<1). \end{cases} \end{aligned} \tag{55}$$

Therefore, whether $d\pi_B^*/dt$ is positive or negative or neither is very complicated. □

**Lemma 8.** *Consider*

$$\frac{d\pi_S^*}{dl} > 0, \qquad \frac{d\pi_B^*}{dl} = \begin{cases} -, & (p<0), \\ < 0, & (0<p<1). \end{cases} \tag{56}$$

*Proof.* Since $p < 1$, $k > 0$, $\beta_0 > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, therefore

$$\bar{a}_{\overline{T-t|}} > 0, \qquad \frac{dI(t,r)}{dl} = \frac{k}{v}\bar{a}_{\overline{T-t|}} > 0$$
$$\frac{d\pi_S^*}{dl} = \frac{\lambda_1 + \eta_3\sigma_S^2}{(1-p)\sigma_S}\frac{dI(t,r)}{dl} > 0. \tag{57}$$

According to Lemmas 5 and 6, we get

$$J(t) = \begin{cases} \leq 1, & (p < 0), \\ \geq 1, & (0 < p < 1), \end{cases} \qquad f(T-t) > 0. \tag{58}$$

Similarly, we just consider the condition of $\alpha_4 > 0$. So,

$$\frac{d\pi_B^*}{dl} = -\frac{\alpha_4 J(t)}{(1-p) f(T-t)}\frac{dI(t,r)}{dl} = \begin{cases} -, & (p < 0), \\ < 0, & (0 < p < 1). \end{cases} \tag{59}$$

□

*Remark 9.* The parameter $p$ is the coefficient of the relative risk aversion. Hence, the plan member would like to avoid risk strongly if they get high $p$.

Lemma 4 shows that the optimal proportion invested in stock $\pi_S^*$ depends on the time $t$ and is a monotone decreasing function with respect to time $t$, but the trend is not affected by $p$. The stock is regarded as high risk, whose purpose is to satisfy the risk appetite of the plan member and hedge the risk. So as the retirement date approaches, the risk appetite begins to decrease so that the optimal proportion invested in stock is monotonically decreasing. It is concluded that as the retirement date approaches, there is a gradual switch from high-risk investment (i.e., stock) into low-risk investment (i.e., cash and bonds).

Thus it can be seen that, as the retirement date approaches, the plan member will think more about how to invest between cash and bonds. However, Lemma 7 indicates that the effect of the time $t$ on $\pi_B^*$ depends on the risk aversion coefficient $p$ and the other parameters under the power utility. Consequently, as the retirement date approaches, how to invest between cash and bonds mainly depends on the risk aversion coefficient $p$ and the other parameters.

In agreement with Cairns et al. [1], instead of switching from high-risk assets into low-risk assets, in the stochastic interest rate framework, the optimal investment strategies involve a switch between different types of low-risk assets (i.e., cash and bonds).

Lemma 8 reveals that the optimal proportion invested in stock $\pi_S^*$ is a monotone increasing function with respect to the salary numeraire $l$, which means that the plan member will be more reluctant to invest in stock when the salary numeraire $l$ becomes larger, but the trend is not affected by $p$. However, the effect of $l$ on the optimal proportion invested in bonds $\pi_B^*$ depends on the risk aversion coefficient $p$ under the power utility. When $0 < p < 1$, $\pi_B^*$ is a monotone decreasing function with respect to $l$. Because the plan members would like to avoid risk strongly if they get high $p$, they invest in cash more as $l$ increases. But when the risk aversion coefficient $p < 0$, $\pi_B^*$ depends on the risk aversion coefficient $p$ and the other parameters.

*5.2. The Explicit Solution for The CARA Utility Function.* Assume that the plan member takes an exponential utility function:

$$U(x) = -\frac{1}{q}e^{-qx}, \quad (\text{with } q > 0). \tag{60}$$

The absolute risk aversion of a decision maker with the utility described in (60) is constant, and (60) is a CARA utility.

According to $g(T,r,z) = (U')^{-1}(z)$ and the CARA utility function, we obtain

$$g(T,r,z) = -\frac{1}{q}\ln z. \tag{61}$$

So, we conjecture a solution to (25) with the following form:

$$g(t,r,z) = -\frac{1}{q}\left[b(t)(\ln z + m(t,r))\right] + a(t), \tag{62}$$

with the boundary conditions given by $b(T) = 1$, $a(T) = 0$, $m(T,s) = 0$.

Therefore,

$$\begin{aligned} g_t &= -\frac{1}{q}\left[b'(t)(\ln z + m(t,r)) + b(t)m_t\right] + a'(t), \\ g_r &= -\frac{1}{q}b(t)m_r, \qquad g_z = -\frac{b(t)}{qz}, \\ g_{zz} &= \frac{b(t)}{qz^2}, \qquad g_{rr} = -\frac{1}{q}b(t)m_{rr}, \qquad g_{rz} = 0. \end{aligned} \tag{63}$$

Putting these derivatives into (25), we derive

$$\begin{aligned} &\left(\beta_0 b(t) - b'(t)\right)\ln z + \left(a'(t) - \beta_0 a(t) - k\right)q \\ &\quad - \Big(m_t + \frac{1}{2}\sigma_r^2 m_{rr} - \beta_0 m + (a - br)m_r \\ &\quad + (\lambda_2 - \eta_2)\sigma_r^2 m_r - (\beta_0 + \beta_1) \\ &\quad + \frac{1}{2}(\lambda_2 - \eta_2)^2\sigma_r^2 + \frac{b'(t)}{b(t)}m\Big)b(t) = 0, \\ &\beta_0 = \lambda_1\eta_3\sigma_S + \lambda_2\eta_2\sigma_r^2 + 2\eta_3^2\sigma_S^2 - m_L, \\ &\beta_1 = \eta_3\sigma_S\left(\frac{1}{2}\eta_3\sigma_S^3 - \eta_3\sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{64}$$

Again we can split this equation into three equations:

$$\beta_0 b(t) - b'(t) = 0, \tag{65}$$

$$a'(t) - \beta_0 a(t) - k = 0, \tag{66}$$

$$\begin{aligned} m_t + \frac{1}{2}\sigma_r^2 m_{rr} - \beta_0 m + (a - br) m_r + (\lambda_2 - \eta_2)\sigma_r^2 m_r \\ - (\beta_0 + \beta_1) + \frac{1}{2}(\lambda_2 - \eta_2)^2 \sigma_r^2 + \frac{b'(t)}{b(t)} m = 0. \end{aligned} \tag{67}$$

Combining with the account boundary conditions: $b(T) = 1$ and $a(T) = 0$, the solutions to (65) and (66) are

$$\begin{aligned} b(t) &= e^{\beta_0 (t-T)}, \\ a(t) &= -k\left(\frac{1 - e^{-\beta_0 (T-t)}}{\beta_0}\right). \end{aligned} \tag{68}$$

We conjecture a solution of (67) with the following structure:

$$m(t, r) = A(t) + B(t) r, \tag{69}$$

with the boundary conditions: $A(T) = 0$ and $B(T) = 0$.

Putting this into (67), we obtain

$$\begin{aligned} A_t + aB + (\lambda_2 - \eta_2) k_2 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_2 - (\beta_0 + \beta_1) \\ + r\left\{B_t - bB + (\lambda_2 - \eta_2) k_1 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1\right\} = 0. \end{aligned} \tag{70}$$

By matching coefficients, we can decompose (70) into two conditions:

$$\begin{aligned} B_t - bB + (\lambda_2 - \eta_2) k_1 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1 = 0, \\ A_t + aB + (\lambda_2 - \eta_2) k_2 B + \frac{1}{2}(\lambda_2 - \eta_2)^2 k_2 - (\beta_0 + \beta_1) = 0. \end{aligned} \tag{71}$$

Taking into account the boundary conditions, the solutions to (71) are

$$\begin{aligned} B(t) &= \frac{\theta_3}{\theta_1}\left(1 - e^{\theta_1 (t-T)}\right), \\ A(t) &= \left(\frac{\theta_2 \theta_3}{\theta_1} + \theta_4 - \beta_0 - \beta_1\right)(T - t) + \frac{\theta_2 \theta_3}{\theta_1^2}\left(e^{\theta_1 (t-T)} - 1\right), \end{aligned} \tag{72}$$

where

$$\begin{aligned} &\theta_1 = b - (\lambda_2 - \eta_2) k_1, \qquad \theta_2 = a + (\lambda_2 - \eta_2) k_2 \\ &\theta_3 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1, \qquad \theta_4 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_4, \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\beta_1 = \eta_3 \sigma_S \left(\frac{1}{2}\eta_3 \sigma_S^3 - \eta_3 \sigma_S^2 - \lambda_1\right) - \frac{1}{2}\lambda_1^2. \end{aligned} \tag{73}$$

From the above calculation, we finally obtain the optimal investment strategies under the CARA utility.

**Proposition 10.** *The optimal investment strategies are given by*

$$\begin{aligned} \pi_0(t) &= 1 - \pi_B(t) - \pi_S(t), \\ \pi_S^* &= \eta_3 + \frac{\left(\lambda_1 + \eta_3 \sigma_S^2\right) l}{q \sigma_S v} b(t), \\ \pi_B^* &= \frac{1}{f(T - t)}\left\{(\eta_2 - \eta_1 \eta_3) + \frac{l}{qv}(\alpha_4 + B(t)) b(t)\right\}, \end{aligned} \tag{74}$$

*where*

$$\begin{aligned} &b(t) = e^{\beta_0 (t-T)}, \qquad B(t) = \frac{\theta_3}{\theta_1}\left(1 - e^{\theta_1 (t-T)}\right), \\ &\beta_0 = \lambda_1 \eta_3 \sigma_S + \lambda_2 \eta_2 \sigma_r^2 + 2\eta_3^2 \sigma_S^2 - m_L, \\ &\theta_1 = b - (\lambda_2 - \eta_2) k_1, \qquad \theta_3 = \frac{1}{2}(\lambda_2 - \eta_2)^2 k_1 \\ &\alpha_4 = \frac{\left(\eta_2 \sigma_S + \lambda_1 \eta_1 + \eta_1 \eta_3 \sigma_S^2 - \lambda_2 \sigma_S\right)}{\sigma_S}, \\ &f(t) = \frac{2\left(e^{mt} - 1\right)}{m - (b - k_1 \lambda_2) + e^{mt}(m + b - k_1 \lambda_2)}, \\ &m = \sqrt{(b - k_1 \lambda_2)^2 + 2k_1}. \end{aligned} \tag{75}$$

**Lemma 11.** *Consider*

$$\frac{db(t)}{dt} > 0, \qquad \frac{d\pi_S^*}{dt} > 0. \tag{76}$$

*Proof.* Since $\beta_0 > 0$, $q > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, by differentiating $b(t)$ with the respect to $t$, we have

$$\begin{aligned} \frac{db(t)}{dt} &= \beta_0 e^{\beta_0 (t-T)} > 0, \\ \frac{d\pi_S^*}{dt} &= \frac{\left(\lambda_1 + \eta_3 \sigma_S^2\right) l}{q \sigma_S v} \frac{db(t)}{dt} > 0. \end{aligned} \tag{77}$$

□

**Lemma 12.** *Consider*

$$\frac{d\pi_S^*}{dl} > 0. \tag{78}$$

*Proof.* Since $r > 0$, $\beta_0 > 0$, $q > 0$, $\lambda_1 > 0$, $\eta_3 > 0$, and $\sigma_S > 0$, we obtain

$$\begin{aligned} b(t) &= e^{\beta_0 (t-T)} > 0, \\ \frac{d\pi_S^*}{dl} &= \frac{\left(\lambda_1 + \eta_3 \sigma_S^2\right) b(t)}{q \sigma_S v} > 0. \end{aligned} \tag{79}$$

□

*Remark 13.* Lemma 11 shows that the effect of $t$ on $\pi_S^*$ is then different from the situation of the power utility. The optimal proportion invested in stock $\pi_S^*$ depends on the time $t$ and is a monotone increasing function with respect to time $t$. Meanwhile, we cannot find the monotone increasing or decreasing effect of $t$ on $\pi_B^*$. So under the exponential utility,

as the retirement date approaches, the plan member will distribute more assets to invest in stock or less asset to invest in low-risk assets (i.e., cash and bonds).

This can be explained by the risky tolerance, namely, $-U'(x)/U''(x) = q^{-1}$, which is only a constant. This indicates that for an exponential utility, due to the independence of a risk tolerance coefficient on wealth, the optimal proportion invested in stock $\pi_S^*$ is independent of the profitability of risky assets and the wealth. As the wealth gives an insight into the accumulated profit gained from risky assets, the plan member will buy more risky assets as the wealth increases.

Lemma 12 reveals that the optimal proportion invested in stock $\pi_S^*$ is a monotone increasing function with respect to the salary numeraire $l$, which is the same as the situation of the power utility. However, the regular change in the effect of $l$ on $\pi_B^*$ is not found.

Nevertheless, the change trend of $t$ or $l$ on $\pi_S^*$ is not affected by the absolute risk aversion coefficient $p$, which is the same as the power utility.

## 6. Conclusions

We have analyzed an investment problem for a defined contribution pension plan with stochastic salary under the affine interest rate model. In view of the related literatures, we have adopted the CRRA and CARA utility functions. And then, the problem of the maximization of the terminal relative wealth's utility has been solved analytically by the Legendre transform and dual theory. As above mentioned, we have analyzed the effect of different parameters on the optimal investment strategies under the CRRA and CARA utility functions, respectively, and compared their differences. So, this paper extends the research of Gao [2] and Cairns et al. [1].

The further research on the stochastic optimal control of DC mainly spread our work under the more generalized situation: (i) assuming the salary to be affected by non-hedgeable risk source under the research framework; (ii) assuming the risky asset to follow a constant elasticity of variance (CEV) model, and so forth. It is noteworthy that the optimal solution with the extended framework is very difficult. Nevertheless, the above methodology cannot be applied to the extended framework, which will result in a more sophisticated nonlinear partial differential equation and cannot tackle it at present.

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