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Research Article

Oscillation for a Class of Fractional Differential Equation

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We consider the oscillation for a class of fractional differential equation $[r(t)g((D_-^\alpha y)(t))]' - p(t)f(\int_t^\infty (s-t)^{-\alpha} y(s)ds) = 0$, for $t > 0$, where $0 < \alpha < 1$ is a real number and $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order α of y. By generalized Riccati transformation technique, oscillation criteria for a class of nonlinear fractional differential equation are obtained.

1. Introduction

Fractional differential equations have been of great interest recently. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self-similar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. There have appeared lots of works in which fractional derivatives are used for a better description of considered material properties; mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications; see [1–6].

It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of linear fractional differential equations. Recently, there are many papers dealing with the qualitative theory, especially the existence of solutions (or positive solutions) of nonlinear initial (or boundary) value problems for fractional differential equation (or system) by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, Adomian decomposition method, etc.); see [7–11].

The oscillation theory as a part of the qualitative theory of differential equations has been developed rapidly in the last decades, and there has been a great deal of work on the oscillatory behavior of integer order differential equations. However, there are only very few papers dealing with the oscillation of fractional differential equation; see [12–15].

Grace et al. [12] initiated the oscillatory theory of fractional differential equations

$$\begin{gathered} D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \\ \lim_{t\to a^+} J_a^{1-q} x(t) = b_1, \end{gathered} \tag{1}$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0 < q < 1$ and the functions f_1, f_2, and v are continuous. By the expression of solution and some inequalities, oscillation criteria are obtained for a class of nonlinear fractional differential equations. The results are also stated when the Riemann-Liouville differential operator is replaced by Caputo's differential operator.

Chen [13] considered the oscillation of the fractional differential equation

$$\begin{aligned} &\left[r(t)\left(D_-^\alpha y\right)^\eta(t)\right]' \\ &\quad - q(t) f\left(\int_t^\infty (s-t)^{-\alpha} y(s)\,ds\right) = 0, \quad \text{for } t > 0, \end{aligned} \tag{2}$$

where $D_-^\alpha y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y, $\eta > 0$ is a quotient of odd positive integers, r and q are positive continuous functions on $[t_0, \infty)$ for a certain $t_0 > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(u)/(u^\eta) > K$ for a certain constant $K > 0$ and for all $u \neq 0$. They established some oscillation criteria for the equation by using a generalized Riccati transformation technique and an inequality.

In 2013, Chen [15] studied oscillatory behavior of the fractional differential equation with the form

$$\left(D_-^{1+\alpha} y\right)(t) - p(t)\left(D_-^{\alpha} y\right)(t) + q(t) f\left(\int_t^{\infty} (v-t)^{-\alpha} y(v)\, dv\right) = 0 \quad \text{for } t > 0, \tag{3}$$

where $D_-^{\alpha} y$ is the Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y.

To the best of our knowledge, nothing is known regarding the oscillatory behavior for the following fractional differential equation:

$$\left[r(t)\, g\left(\left(D_-^{\alpha} y\right)(t)\right)\right]' - p(t) f\left(\int_t^{\infty} (s-t)^{-\alpha} y(s)\, ds\right) = 0, \quad \text{for } t > 0, \tag{4}$$

where $0 < \alpha < 1$ is a real number, $D_-^{\alpha} y$ is the Liouville right-sided fractional derivative of order α of y defined by $(D_-^{\alpha} y)(t) := -(1/\Gamma(1-\alpha)) d/dt \int_t^{\infty} (s-t)^{-\alpha} y(s) ds$ for $t \in \mathbb{R}^+ := (0,\infty)$, here $\Gamma(\cdot)$ is the gamma function defined by $\Gamma(t) := \int_t^{\infty} s^{t-1} e^{-s} ds$ for $t \in \mathbb{R}^+$, and the following conditions are assumed to hold:

(H_1) r and p are two positive continuous functions on $t \in [t_0, \infty)$ for a certain $t_0 > 0$;

(H_2) $g^{-1} \in C(\mathbb{R}, \mathbb{R})$ are continuous functions with $ug^{-1}(u) > 0$ for $u \neq 0$, and there exists some positive constant γ_1 such that $g^{-1}(uv) \geq \gamma_1 g^{-1}(u) g^{-1}(v)$ for $uv \neq 0$;

(H_3) $f, g : \mathbb{R} \to \mathbb{R}$ are continuous function with $xf(x) > 0$, $xg(x) > 0$ for $x \neq 0$, and there exist positive constants k_1, k_2 such that $f(x)/x \geq k_1$, $x/g(x) \geq k_2$ for all $x \neq 0$.

By a solution of (4), we mean a nontrivial function $y \in C(\mathbb{R}^+, \mathbb{R})$ with $\int_t^{\infty} (s-t)^{-\alpha} y(s) ds \in C^1(\mathbb{R}^+, \mathbb{R})$, and $r(t) g((D_-^{\alpha} y)(t)) \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies (4) for $t > 0$. Our attention is restricted to those solutions of (4) which exist on $\mathbb{R}^+$ and satisfy $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \geq 0$. A solution y of (4) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (4) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory. These materials can be found in the recent literature; see [12, 13, 16, 17].

Definition 1 (see [16]). The Liouville right-sided fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ on the half-axis $\mathbb{R}^+$ is given by

$$\left(I_-^{\alpha} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s)\, ds, \quad \text{for } t > 0, \tag{5}$$

provided that the right side is pointwise defined on $\mathbb{R}^+$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [16]). The Liouville right-sided fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ on the half-axis $\mathbb{R}^+$ is given by

$$\begin{aligned}\left(D_-^{\alpha} f\right)(t) &:= (-1)^{\lceil\alpha\rceil} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \left(I_-^{\lceil\alpha\rceil-\alpha} f\right)(t) \\ &= (-1)^{\lceil\alpha\rceil} \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_t^{\infty} (s-t)^{\lceil\alpha\rceil-\alpha-1} f(s)\, ds, \\ &\qquad \text{for } t > 0,\end{aligned} \tag{6}$$

where $\lceil\alpha\rceil := \min\{z \in \mathbb{Z} : z \geq \alpha\}$, provided that the right side is pointwise defined on $\mathbb{R}^+$.

The following lemma is fundamental in the proofs of our main results.

Lemma 3 (see [13]). *Let y be a solution of* (4) *and*

$$G(t) := \int_t^{\infty} (s-t)^{-\alpha} y(s)\, ds, \quad \text{for } \alpha \in (0,1),\ t > 0. \tag{7}$$

Then

$$G'(t) = -\Gamma(1-\alpha)\left(D_-^{\alpha} y\right)(t), \quad \text{for } \alpha \in (0,1),\ t > 0. \tag{8}$$

Lemma 4 (see [17]). *If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \leq (m-1) Y^m. \tag{9}$$

3. Main Results

Theorem 5. *Suppose that (H_1)–(H_3) and*

$$\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right) ds = \infty \tag{10}$$

hold. Furthermore, assume that there exists a positive function $\delta \in C^1[t_0, \infty)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k_1 \delta(s) p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2 \Gamma(1-\alpha)\delta(s)}\right] ds = \infty, \tag{11}$$

where k_1, k_2 are defined as in (H_3). Then every solution of (4) *is oscillatory.*

Proof. Suppose that y is a nonoscillatory solution of (4). Without loss of generality, we may assume that y is an eventually positive solution of (4). Then there exists $t_1 \in [t_0, \infty)$ such that

$$y(t) > 0, \quad G(t) > 0 \quad \text{for } t \in [t_1, \infty), \tag{12}$$

where G is defined as in (7). Therefore, it follows from (4) that

$$\left[r(t)\, g\left(\left(D_{-}^{\alpha} y\right)(t)\right)\right]' = p(t)\, f(G(t)) > 0 \quad \text{for } t \in [t_1, \infty). \tag{13}$$

Thus, $r(t)g((D_{-}^{\alpha}y)(t))$ is strictly increasing on $[t_1, \infty)$ and is eventually of one sign. Since $r(t) > 0$ for $t \in [t_0, \infty)$ and (H_2), we see that $(D_{-}^{\alpha}y)(t)$ is eventually of one sign. We now claim that

$$\left(D_{-}^{\alpha} y\right)(t) < 0, \quad \text{for } t \in [t_1, \infty). \tag{14}$$

If not, then $(D_{-}^{\alpha}y)(t)$ is eventually positive, and there exists $t_2 \in [t_1, \infty)$ such that $(D_{-}^{\alpha}y)(t) > 0$. Since $r(t)g((D_{-}^{\alpha}y)(t))$ is strictly increasing on $[t_1, \infty)$, it is clear that $r(t)g((D_{-}^{\alpha}y)(t)) \geq r(t_2)g((D_{-}^{\alpha}y)(t_2)) := c > 0$ for $t \in [t_2, \infty)$. Therefore, from (8), we have

$$\begin{aligned} -\frac{G'(t)}{\Gamma(1-\alpha)} &= \left(D_{-}^{\alpha} y\right)(t) \\ &\geq g^{-1}\left(\frac{c}{r(t)}\right) \geq \gamma_1 g^{-1}(c)\, g^{-1}\left(\frac{1}{r(t)}\right), \\ &\qquad \text{for } t \in [t_2, \infty). \end{aligned} \tag{15}$$

Then, we get

$$g^{-1}\left(\frac{1}{r(t)}\right) \leq -\frac{G'(t)}{\gamma_1 g^{-1}(c)\,\Gamma(1-\alpha)}, \quad \text{for } t \in [t_2, \infty). \tag{16}$$

Integrating the above inequality from t_2 to t, we have

$$\begin{aligned} \int_{t_2}^{t} g^{-1}\left(\frac{1}{r(s)}\right) ds &\leq -\frac{G(t) - G(t_2)}{\gamma_1 g^{-1}(c)\,\Gamma(1-\alpha)} \\ &< \frac{G(t_2)}{\gamma_1 g^{-1}(c)\,\Gamma(1-\alpha)}, \quad \text{for } t \in [t_2, \infty). \end{aligned} \tag{17}$$

Letting $t \to \infty$, we see

$$\int_{t_2}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right) ds \leq \frac{G(t_2)}{\gamma_1 g^{-1}(c)\,\Gamma(1-\alpha)} < \infty. \tag{18}$$

This contradicts (10). Hence, (14) holds.

Define the function w by the generalized Riccati substitution

$$w(t) = \delta(t)\, \frac{-r(t)\, g\left(\left(D_{-}^{\alpha} y\right)(t)\right)}{G(t)}, \quad \text{for } t \in [t_1, \infty). \tag{19}$$

Then we have $w(t) > 0$ for $t \in [t_1, \infty)$. From (19), (4), (8), and (H_1)–(H_3), it follows that

$$\begin{aligned} w'(t) &= \frac{\delta(t)}{G(t)}\left[-r(t)\, g\left(\left(D_{-}^{\alpha} y\right)(t)\right)\right]' \\ &\quad + \left(\frac{\delta(t)}{G(t)}\right)' \left[-r(t)\, g\left(\left(D_{-}^{\alpha} y\right)(t)\right)\right] \\ &= -\delta(t)\, p(t)\, \frac{f(G(t))}{G(t)} \\ &\quad + \frac{\delta'(t)\, G(t) - \delta(t)\, G'(t)}{G^2(t)} \left[-r(t)\, g\left(\left(D_{-}^{\alpha} y\right)(t)\right)\right] \\ &= -\delta(t)\, p(t)\, \frac{f(G(t))}{G(t)} + \frac{\delta'(t)}{\delta(t)} w(t) - \frac{G'(t)}{G(t)} w(t) \\ &= -\delta(t)\, p(t)\, \frac{f(G(t))}{G(t)} + \frac{\delta'(t)}{\delta(t)} w(t) \\ &\quad - \frac{\Gamma(1-\alpha)\, w^2(t)}{\delta(t)\, r(t)} \frac{D_{-}^{\alpha} y(t)}{g\left(D_{-}^{\alpha} y(t)\right)} \\ &\leq -k_1 \delta(t)\, p(t) + \frac{\delta'(t)}{\delta(t)} w(t) - \frac{k_2 \Gamma(1-\alpha)}{\delta(t)\, r(t)} w^2(t). \end{aligned} \tag{20}$$

Taking

$$\begin{aligned} m &= 2, \\ X &= \sqrt{\frac{k_2 \Gamma(1-\alpha)}{\delta(t)\, r(t)}}\, w(t), \\ Y(t) &= \frac{1}{2} \sqrt{\frac{\delta(t)\, r(t)}{k_2 \Gamma(1-\alpha)}}\, \frac{\delta'(t)}{\delta(t)}, \end{aligned} \tag{21}$$

from Lemma 4 and (20) we get

$$w'(t) \leq -k_1 \delta(t)\, p(t) + \frac{r(t)\left(\delta'(t)\right)^2}{4 k_2 \Gamma(1-\alpha)\, \delta(t)}. \tag{22}$$

Integrating both sides of the inequality (22) from t_0 to t, we obtain

$$\begin{aligned} \infty > w(t_0) &> w(t_0) - w(t) \\ &\geq \int_{t_0}^{t} \left[k_1 \delta(s)\, p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4 k_2 \Gamma(1-\alpha)\, \delta(s)}\right] ds. \end{aligned} \tag{23}$$

Taking the limit supremum of both sides of the above inequality as $t \to \infty$, we get

$$\begin{aligned} &\limsup_{t \to \infty} \int_{t_0}^{t} \left[k_1 \delta(s)\, p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4 k_2 \Gamma(1-\alpha)\, \delta(s)}\right] ds \\ &\quad < w(t_0) < \infty, \end{aligned} \tag{24}$$

which contradicts (11). The proof is complete. □

Theorem 6. *Suppose that (H_1)–(H_3) and* (10) *hold. Furthermore, suppose that there exist a positive function $\delta \in C^1[t_0,\infty)$ and a function $H \in C(\mathbb{D},\mathbb{R})$, where $\mathbb{D} := \{(t,s) : t \geq s \geq t_0\}$, such that*

$$H(t,t) = 0 \quad \text{for } t \geq t_0, \qquad H(t,s) > 0 \quad \text{for } (t,s) \in \mathbb{D}_0, \tag{25}$$

where $\mathbb{D}_0 := \{(t,s) : t > s \geq t_0\}$, and H has a nonpositive continuous partial derivative $H_s'(t,s) := \partial H(t,s)/\partial s$ on D_0 with respect to the second variable and satisfies

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t-1} H(t,s) \times \left[k_1\delta(s)\, p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2\Gamma(1-\alpha)\,\delta(s)} \right] ds = \infty, \tag{26}$$

where k_1, k_2, and δ are defined as in Theorem 5. Then all solutions of (4) *are oscillatory.*

Proof. Suppose that y is a nonoscillatory solution of (4). Without loss of generality, we may assume that y is an eventually positive solution of (4). We proceed as in the proof of Theorem 5 to get (22), that is,

$$w'(t) \leq -k_1\delta(t)\, p(t) + \frac{r(t)\left(\delta'(t)\right)^2}{4k_2\Gamma(1-\alpha)\,\delta(t)}. \tag{27}$$

Multiplying the previous inequality by $H(t,s)$ and integrating from t_0 to $t-1$, for $t \in [t_1+1,\infty)$, we obtain

$$\begin{aligned} &\int_{t_0}^{t-1} H(t,s) \left[k_1\delta(s)\, p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2\Gamma(1-\alpha)\,\delta(s)} \right] ds \\ &\quad \leq -\left[H(t,s)\, w(s)\right]_{t_0}^{t-1} + \int_{t_0}^{t-1} H_s'(t,s)\, w(s)\, ds \\ &\quad \leq H(t,t_0)\, w(t_0) + \int_{t_0}^{t-1} H_s'(t,s)\, w(s)\, ds \\ &\quad \leq H(t,t_0)\, w(t_0). \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} &\frac{1}{H(t,t_0)} \int_{t_0}^{t-1} H(t,s) \left[k_1\delta(s)\, p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2\Gamma(1-\alpha)\,\delta(s)} \right] ds \\ &\quad \leq w(t_0) < \infty, \end{aligned} \tag{29}$$

which is a contradiction to (26). The proof is complete. □

Next, we consider the case

$$\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right) ds < \infty, \tag{30}$$

which yields that (10) does not hold. In this case, we have the following results.

Theorem 7. *Suppose that (H_1)–(H_3) and* (30) *hold, g is an increasing function, and that there exists a positive function $\delta \in C^1[t_0,\infty)$ such that* (11) *holds. Furthermore, assume that for every constant $T \geq t_0$,*

$$\int_{T}^{\infty} g^{-1}\left(\frac{1}{r(t)} \int_{T}^{t} p(s)\, ds\right) dt = \infty. \tag{31}$$

Then every solution y of (4) *is oscillatory or satisfies $\lim_{t\to\infty} \int_{t_0}^{\infty} (s-t)^{-\alpha} y(s) ds = 0$.*

Proof. Assume that y is a nonoscillatory solution of (4). Without loss of generality, assume that y is an eventually positive solution of (4). Proceeding as in the proof of Theorem 5, there are two cases for the sign of $(D_-^\alpha y)(t)$. The proof when $(D_-^\alpha y)(t)$ is eventually negative is similar to that of Theorem 5 and hence is omitted.

Next, assume that $(D_-^\alpha y)(t)$ is eventually positive. Then there exists $t_2 \geq t_1$ such that $(D_-^\alpha y)(t) > 0$ for $t \geq t_2$. From (8), we get $G'(t) < 0$ for $t \geq t_2$. Thus, we get $\lim_{t\to\infty} G(t) := M \geq 0$ and $G(t) \geq M$. We now claim that $M = 0$. Assume not, that is, $M > 0$, then from (H_3) we get

$$\begin{aligned} &\left[r(t)\, g\left((D_-^\alpha y)(t)\right)\right]' \\ &\quad = p(t)\, f(G(t)) \geq k_1 M p(t), \quad \text{for } t \in [t_2,\infty). \end{aligned} \tag{32}$$

Integrating both sides of the last inequality from t_2 to t, we have

$$\begin{aligned} r(t)\, g\left((D_-^\alpha y)(t)\right) &\geq r(t_2)\, g\left((D_-^\alpha y)(t_2)\right) + k_1 M \int_{t_2}^{t} p(s)\, ds \\ &> k_1 M \int_{t_2}^{t} p(s)\, ds, \quad \text{for } t \in [t_2,\infty). \end{aligned} \tag{33}$$

Hence, from (8), we get

$$\begin{aligned} -\frac{G'(t)}{\Gamma(1-\alpha)} = \left(D_-^\alpha y\right)(t) &\geq g^{-1}\left(\frac{k_1 M \int_{t_2}^{t} p(s)\, ds}{r(t)}\right) \\ &\geq \gamma_1 g^{-1}(k_1 M)\, g^{-1}\left(\frac{\int_{t_2}^{t} p(s)\, ds}{r(t)}\right), \\ &\qquad \text{for } t \in [t_2,\infty). \end{aligned} \tag{34}$$

Integrating both sides of the last inequality from t_2 to t, we obtain

$$G(t) \le G(t_2) - \Gamma(1-\alpha)\gamma_1 g^{-1}(k_1 M) \times \int_{t_2}^{t} g^{-1}\left(\frac{\int_{t_2}^{u} p(s)\,ds}{r(u)}\right) du, \quad \text{for } t \in [t_2, \infty). \tag{35}$$

Letting $t \to \infty$, from (31), we get $\lim_{t\to\infty} G(t) = -\infty$. This contradicts $G(t) > 0$. Therefore, we have $M = 0$, that is, $\lim_{t\to\infty} G(t) = 0$. In view of (7), we see that the proof is complete. □

Theorem 8. *Suppose that* (H_1)–(H_3) *and* (30) *hold and* g *is an increasing function. Let* $\delta(t)$ *and* $H(t,s)$ *be defined as in Theorem 6 such that* (26) *holds. Furthermore, assume that for every constant* $T \ge t_0$, (31) *holds. Then every solution* y *of* (4) *is oscillatory or satisfies* $\lim_{t\to\infty} \int_{t_0}^{\infty} (s-t)^{-\alpha} y(s)ds = 0$.

Proof. Assume that y is a nonoscillatory solution of (4). Without loss of generality, assume that y is an eventually positive solution of (4). Proceeding as in the proof of Theorem 5, there are two cases for the sign of $(D_-^{\alpha} y)(t)$. The proof when $(D_-^{\alpha} y)(t)$ is eventually negative is similar to that of Theorem 6 and hence is omitted. The proof when $(D_-^{\alpha} y)(t)$ is eventually positive is similar to that of the proof of Theorem 7 and thus is omitted. The proof is complete. □

4. Example

Example 1. Consider the fractional differential equation

$$\left[t^{1/3}\left(D_-^{1/2} y\right)(t)\right]' - t\left(\int_t^{\infty} (s-t)^{-\alpha} y(s)\,ds\right) = 0, \quad \text{for } t > 0. \tag{36}$$

In (36), $\alpha = 1/2$, $r(t) = t^{1/3}$, $p(t) = t$, and $f(x) = g(x) = x$. Take $t_0 = 1$, $k_1 = k_2 = 1$. It is clear that conditions (H_1)–(H_3) and (10) hold. Furthermore, taking $\delta(t) = t$, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k_1 \delta(s) p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2\Gamma(1-\alpha)\delta(s)}\right] ds = \limsup_{t\to\infty} \int_1^t \left[s^2 - \frac{s^{1/3}}{4\sqrt{\pi}s}\right] ds = \infty, \tag{37}$$

which shows that (11) holds. Therefore, by Theorem 5 every solution of (36) is oscillatory.

Example 2. Consider the fractional differential equation

$$\left[t^{3/2}\left(D_-^{1/2} y\right)(t)\right]' - t\left(\int_t^{\infty} (s-t)^{-\alpha} y(s)\,ds\right) = 0, \quad \text{for } t > 0. \tag{38}$$

In (38), $\alpha = 1/2$, $r(t) = t^{3/2}$, $p(t) = t$, and $f(x) = g(x) = x$. Take $t_0 = 1$, $k_1 = k_2 = 1$. It is clear that conditions (H_1)–(H_3) and (30) hold. Taking $\delta(t) = t$, we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k_1 \delta(s) p(s) - \frac{r(s)\left(\delta'(s)\right)^2}{4k_2\Gamma(1-\alpha)\delta(s)}\right] ds = \limsup_{t\to\infty} \int_1^t \left[s^2 - \frac{s^{3/2}}{4\sqrt{\pi}s}\right] ds = \infty, \tag{39}$$

which shows that (11) holds.

Furthermore, for every constant $T \ge 1$, we have

$$\int_T^{\infty} g^{-1}\left(\frac{1}{r(t)} \int_T^t p(s)\,ds\right) dt = \int_T^{\infty} \left(\frac{1}{t^{3/2}} \int_T^t s\,ds\right) dt = \infty, \tag{40}$$

which shows that (31) holds. Therefore, by Theorem 7 every solution of (38) is oscillatory or satisfies $\lim_{t\to\infty} \int_{t_0}^{\infty} (s-t)^{-\alpha} y(s)ds = 0$.

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