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Research Article

Monotonicity of Eventually Positive Solutions for a Second Order Nonlinear Difference Equation

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We derive several sufficient conditions for monotonicity of eventually positive solutions on a class of second order perturbed nonlinear difference equation. Furthermore, we obtain a few nonexistence criteria for eventually positive monotone solutions of this equation. Examples are provided to illustrate our main results.

1. Introduction

The theory of difference equations and their applications have received intensive attention. In the last few years, new research achievements kept emerging (see [1–7]). Among them, in [3], Saker considered the second order nonlinear delay difference equation

$$\Delta\left(p_n \Delta x_n\right) + q_n f\left(x_{n-\sigma}\right) = 0, \quad n \geqslant 0. \tag{1}$$

Saker used the Riccati transformation technique to obtain several sufficient conditions which guarantee that every solution of (1) oscillates or converges to zero. In [4], Rath et al. considered the more general second order equations

$$\begin{gathered} \Delta\left(r_n \Delta\left(y_n - p_n y_{n-m}\right)\right) + q_n G\left(y_{n-k}\right) = 0, \quad n \geqslant 0, \\ \Delta\left(r_n \Delta\left(y_n - p_n y_{n-m}\right)\right) + q_n G\left(y_{n-k}\right) = f_n, \quad n \geqslant 0. \end{gathered} \tag{2}$$

They found necessary conditions for the solutions of the above equations to be oscillatory or tend to zero. Following this trend, this paper is concerned with the second order perturbed nonlinear difference equation

$$\Delta\left(a_n \Delta x_n\right) + P\left(n, x_n, x_{n+1}\right) = Q\left(n, x_n, \Delta x_n\right), \quad n \geqslant 0, \tag{3}$$

where $\{a_n\}$ is a positive sequence, $P, Q : N \times R^2 \rightarrow R$ are two continuous functions, and Δ is the forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$.

In [8], Li and Cheng considered the special case of (3)

$$\Delta\left(p_{n-1} \Delta x_{n-1}\right) + q_n f\left(x_n\right) = 0, \quad n \geqslant 0. \tag{4}$$

They got the sufficient conditions for asymptotically monotone solutions of (4). Enlightened by [8, 9], in this paper, we derive several sufficient conditions for monotonicity of eventually positive solutions on (3) and obtain a few nonexistence criteria for eventually positive monotone solutions of (3). Our results improve and generalize results in [8]. We also provide examples to illustrate our main results.

For convenience, these essential conditions used in main results are listed as follows:

(H_1) there exists a continuous function $f : R \rightarrow R$ such that $xf(x) > 0$ for all $x \neq 0$;

(H_2) f is a derivable function and $f'(x) \geqslant 0$ for $x \neq 0$;

(H_3) there exist two sequences $\{p_n\}$ and $\{q_n\}$, such that $P(n, x_n, x_{n+1})/f(x_{n+1}) \geqslant p_n$ and $Q(n, x_n, \Delta x_n)/f(x_{n+1}) \leqslant q_n$ for $x_n \neq 0$;

(H_4) $\sum_{n=n_0}^{\infty} 1/a_n = +\infty$, n_0 is a positive integral number,

where a_n, $P(n, x_n, x_{n+1})$ and $Q(n, x_n, \Delta x_{n+1})$ are all in (3).

2. Main Results

We first state a result which relates a positive sequence and a positive nondecreasing function. Its proof can be found in [8].

Lemma 1 (see [8]). *Let $f(x)$ be a positive nondecreasing function defined for $x > 0$. Let $\{x_k\}$ be a real sequence such that $x_k > 0$ for $i \leqslant k \leqslant j+1$. Then*

$$\sum_{k=i}^{j} \frac{\Delta x_k}{f(x_{k+1})} \leqslant \int_{x_i}^{x_{j+1}} \frac{du}{f(u)} \leqslant \sum_{k=i}^{j} \frac{\Delta x_k}{f(x_k)}. \quad (5)$$

Theorem 2. *Suppose that conditions (H_1)–(H_4) hold, p_n and q_n satisfy the following conditions:*

(H_5) $\sum_{s=n_0}^{\infty}(p_s - q_s) < +\infty$;

(H_6) $\liminf_{n\to\infty} \sum_{s=n_0}^{n}(p_s - q_s) \geqslant 0$

for all n_0. Then eventually positive solutions of (3) are eventually monotone increasing.

Proof. Suppose that $\{x_n\}$ is a positive solution of (3), say $x_n > 0$ for $n > N > n_0$. If conclusion cannot hold, without any loss of generality, assume $\Delta x_N \leqslant 0$, in view of (3) and conditions, we have

$$\Delta\left(\frac{a_n \Delta x_n}{f(x_n)}\right) = \frac{\Delta(a_n \Delta x_n)}{f(x_{n+1})} - \frac{a_n (\Delta x_n)^2 f'(x_n + \theta \Delta x_n)}{f(x_n) f(x_{n+1})} \leqslant q_n - p_n \quad (0 < \theta < 1), \quad (6)$$

by summing (6) from N to $n-1$, then

$$\frac{a_n \Delta x_n}{f(x_n)} \leqslant \frac{a_N \Delta x_N}{f(x_N)} - \sum_{s=N}^{n-1}(p_s - q_s). \quad (7)$$

Making use of condition (H_6), we know $\Delta x_n < 0$ for $n \geqslant N$. Summing (3) and using (H_3), we have

$$\begin{aligned} a_n \Delta x_n &\leqslant a_N \Delta x_N - \sum_{s=N}^{n-1} f(x_{s+1})(p_s - q_s) \\ &= a_N \Delta x_N - f(x_{n+1}) \sum_{s=N}^{n-1}(p_s - q_s) \\ &\quad + \sum_{s=N}^{n-1} \Delta f(x_s)\left(\sum_{t=N}^{s-1}(p_t - q_t)\right) \\ &\leqslant a_N \Delta x_N. \end{aligned} \quad (8)$$

By summing (8), we then see that

$$x_{n+1} \leqslant x_N + a_N \Delta x_N \sum_{s=N}^{n} \frac{1}{a_s} \longrightarrow -\infty \quad (\text{as } n \longrightarrow \infty), \quad (9)$$

which contradicts the fact $x_n > 0$. The proof is complete. □

Example 3. Consider the difference equation

$$\Delta\left(\frac{\Delta x_n}{n^2}\right) + x_{n+1}\left(r(n, x_n) + \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^3}\right) = x_{n+1} r(n, x_n), \quad n \geqslant 0, \quad (10)$$

where $r(n, x_n)$ is any function of n and x_n. By taking $f(x) = x$, we have

$$\begin{gathered} \frac{P(n, x_n, x_{n+1})}{f(x_{n+1})} = r(n, x_n) + \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^3} = p_n, \\ \frac{Q(n, x_n, \Delta x_n)}{f(x_{n+1})} = r(n, x_n) = q_n. \end{gathered} \quad (11)$$

So conditions of Theorem 2 hold. By Theorem 2, (10) has a positive monotone increasing solution $\{x_n\} = \{n\}$.

Theorem 4. *If conditions (H_1)–(H_4) hold, there exist $M > 0$ and $j > n_0$ for $n_0 \geqslant M$ such that*

$$\limsup_{n\to\infty} \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=n_0}^{k-1}(p_s - q_s) > 0. \quad (12)$$

Then eventually positive solutions $\{x_n\}$ of (3) are eventually monotone increasing or $\liminf_{n\to\infty} x_n = 0$.

Proof. Suppose $\{x_n\}$ is a positive solution of (3), there exists $N > n_0$ such that $x_n > 0$ for $n > N$. Let $\Delta x_N \leqslant 0$, and

$$\limsup_{n\to\infty} \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=N}^{k-1}(p_s - q_s) > 0, \quad j > N. \quad (13)$$

If $\liminf_{n\to\infty} x_n \neq 0$, then there exist $T \geqslant N$ and a number $\alpha > 0$ such that $x_n > \alpha > 0$ for $n \geqslant T$; in view of (7), we get

$$\frac{\Delta x_n}{f(x_n)} \leqslant \frac{a_N \Delta x_N}{f(x_N)} \cdot \frac{1}{a_n} - \frac{1}{a_n} \sum_{s=N}^{n-1}(p_s - q_s). \quad (14)$$

Summing (14) and making use of Lemma 1, we know

$$\begin{aligned} \int_{x_j}^{\alpha} \frac{du}{f(u)} &\leqslant \int_{x_j}^{x_{n+1}} \frac{du}{f(u)} \\ &\leqslant \frac{a_N \Delta x_N}{f(x_N)} \sum_{k=j}^{n} \frac{1}{a_k} - \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=N}^{k-1}(p_s - q_s). \end{aligned} \quad (15)$$

By (H_4), the right side of (15) tends to $-\infty$ as $n \to \infty$, whereas the left side is finite. This contradiction completes our proof. □

Example 5. Consider the difference equation

$$\Delta\left(\frac{\Delta x_n}{\sqrt{n}}\right) + \frac{x_{n+1}}{\sqrt{n}} = \frac{\sqrt{n+2}}{n+1} x_{n+1}, \quad n \geqslant 0. \quad (16)$$

By taking $f(x) = x$, we have $P(n, x_n, x_{n+1})/f(x_{n+1}) = 1/\sqrt{n} = p_n$, $Q(n, x_n, \Delta x_n)/f(x_{n+1}) = \sqrt{n+2}/n+1 = q_n$. So conditions of Theorem 4 hold. By Theorem 4, (16) has a positive monotone increasing solution $\{x_n\} = \{\sqrt{n}\}$.

Theorem 6. *If conditions* (H_1)–(H_3) *hold, and*

$$\liminf_{n\to\infty}\frac{1}{a_n}\sum_{s=n_0}^{n-1}(p_s-q_s)>0 \tag{17}$$

holds for all n_0*. Then eventually positive solutions* $\{x_n\}$ *of* (3) *are eventually monotone increasing or eventually monotone decreasing and* $\lim_{n\to\infty}x_n=0$.

Proof. Suppose $\{x_n\}$ is a positive solution of (3), there exists $N>n_0$ such that $x_n>0$ for $n>N$. Let $\Delta x_N\leqslant 0$ and $\liminf_{n\to\infty}1/a_n\sum_{s=N}^{n-1}(p_s-q_s)>0$, then there exists $\beta>0$ such that

$$\frac{1}{a_n}\sum_{s=N}^{n-1}(p_s-q_s)\geqslant\beta>0,\quad n\geqslant N. \tag{18}$$

From (7), we have

$$\Delta x_n\leqslant -f(x_n)\cdot\frac{1}{a_n}\sum_{s=N}^{n-1}(p_s-q_s)\leqslant -\beta f(x_n)<0,\quad n>N. \tag{19}$$

If $\lim_{n\to\infty}x_n\neq 0$, then there exists $c>0$ such that $x_n\geqslant c>0$. There is no harm in assumption $x_n\geqslant c$ for $n\geqslant N$. Summing (19), we obtain

$$c\leqslant x_{n+1}\leqslant x_N-(n+1-N)\beta f(c)\longrightarrow-\infty\quad(n\longrightarrow\infty), \tag{20}$$

which is a contrary. The proof is complete. □

Example 7. Consider the difference equation

$$\Delta\left(n^2\Delta x_n\right)+x_{n+1}\left(r(n,x_n)+\frac{1}{n+2}\right)=x_{n+1}r(n,x_n), \tag{21}$$

where $r(n,x_n)$ is any function of n and x_n. By taking $f(x)=x$, we have

$$\begin{aligned}\frac{P(n,x_n,x_{n+1})}{f(x_{n+1})}&=r(n,x_n)+\frac{1}{n+2}=p_n,\\ \frac{Q(n,x_n,\Delta x_n)}{f(x_{n+1})}&=r(n,x_n)=q_n.\end{aligned} \tag{22}$$

So conditions of Theorem 6 hold. By Theorem 6, (21) has a monotone decreasing positive solution $\{x_n\}=\{1/n\}$.

Theorem 8. *If conditions* (H_1)–(H_3) *hold and*

(H_7) $\limsup_{n\to\infty}\sum_{s=k}^{n}1/a_s\sum_{t=n_0}^{s-1}(p_t-q_t)=+\infty$;

(H_8) *for all* $\varepsilon>0$, $\int_0^{\varepsilon}du/f(u)<+\infty$.

Then eventually positive solutions of (3) *are eventually monotone increasing.*

Proof. Suppose $x_n>0$ for $n>N>n_0$, $\{x_n\}$ is a solution of (3), and $\limsup_{n\to\infty}\sum_{s=k}^{n}1/a_s\sum_{t=N}^{s-1}(p_t-q_t)=\infty$. If the result does not hold, without any loss of generality, assume $\Delta x_N\leqslant 0$. In view of (7), we see that

$$\begin{aligned}\frac{\Delta x_n}{f(x_n)}&\leqslant\frac{a_N\Delta x_N}{f(x_N)}\cdot\frac{1}{a_n}-\frac{1}{a_n}\sum_{t=N}^{n-1}(p_t-q_t)\\&\leqslant-\frac{1}{a_n}\sum_{t=N}^{n-1}(p_t-q_t).\end{aligned} \tag{23}$$

Summing (23) and using Lemma 1, we know

$$\int_{x_k}^{x_{n+1}}\frac{du}{f(u)}\leqslant\sum_{s=k}^{n}\frac{\Delta x_s}{f(x_s)}\leqslant-\sum_{s=k}^{n}\frac{1}{a_s}\sum_{t=N}^{s-1}(p_t-q_t). \tag{24}$$

This is a contradiction. The proof is complete. □

Remark 9. In Theorems 2 and 4, condition (H_4) is essential; that is, the series with positive terms $\sum_{n=n_0}^{\infty}1/a_n$ is divergent, but it is not required in Theorems 6 and 8.

Remark 10. The eventually positive solutions in Theorems 4 and 8 are increasing it is not necessarily so in Theorem 6.

Next, we will derive several nonexistence criteria for eventually positive monotone solutions of (3).

Theorem 11. *If conditions* (H_1)–(H_3) *hold and*

$$\limsup_{n\to\infty}\sum_{k=1}^{n}(p_k-q_k)=+\infty. \tag{25}$$

Then, (3) *cannot have any eventually positive monotone increasing solutions.*

Proof of Theorem 11 is obvious. If $x_n>0$ is an eventually positive increasing solution, by means of conditions, (7) is a contrary.

Theorem 12. *If conditions* (H_1)–(H_3) *hold, and there is a nonnegative and nondegenerate sequence* $\{\varphi_n\}$ *such that*

$$\limsup_{n\to\infty}\frac{\sum_{k=n_0}^{n}\varphi_{k+1}/a_k\sum_{s=n_0}^{k-1}(p_s-q_s)}{\sum_{k=n_0}^{n}\varphi_{k+1}/a_k}=\infty \tag{26}$$

holds for all n_0*. Then,* (3) *cannot have any eventually positive nondecreasing solutions.*

Proof. Suppose that $\{x_n\}$ is a positive solution of (3), there exists $N>n_0$ such that $x_n>0$ and $\Delta x_n\geqslant 0$ for $n>N$. Multiplying (7) by φ_{n+1}/a_n, we have

$$\frac{\varphi_{n+1}\Delta x_n}{f(x_n)}+\frac{\varphi_{n+1}}{a_n}\sum_{s=N}^{n-1}(p_s-q_s)\leqslant\frac{\varphi_{n+1}}{a_n}\cdot\frac{a_N\Delta x_N}{f(x_N)}. \tag{27}$$

So we obtain

$$\sum_{k=N}^{n}\frac{\varphi_{k+1}}{a_k}\sum_{s=N}^{k-1}(p_s-q_s)\leqslant\frac{a_N\Delta x_N}{f(x_N)}\sum_{k=N}^{n}\frac{\varphi_{k+1}}{a_k}. \tag{28}$$

This is contrary to our condition. The proof is complete. □

Theorem 13. *If* (H_1) *and* (H_3) *hold,* $\{a_n\}$ *is a nondecreasing sequence,* $f(x)$ *is a nondecreasing function, and there is a nonnegative sequence* $\{\varphi_n\}$*, where* $\{\Delta\varphi_n\}$ *is bounded, and*

(H_9) $\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\varphi_{s+1}(p_s-q_s)/a_{s+1}=+\infty$ *for all* n_0*;*

(H_{10}) $0<\int_{\varepsilon}^{+\infty}du/f(u)<+\infty$, $\varepsilon>0$.

Then, (3) *cannot have any eventually positive monotone increasing solutions.*

Proof. Suppose that $\{x_n\}$ is a solution of (3), and there exists $N>n_0$ such that $x_n>0$ and $\Delta x_n>0$ for $n>N$. Multiplying (3) by $\varphi_{n+1}/a_{n+1}f(x_{n+1})$ and summing from N to $n-1$ again, we have

$$\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}f(x_{s+1})}\Delta(a_s\Delta x_s)\leqslant\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}(q_s-p_s). \tag{29}$$

Namely,

$$\begin{aligned}&\frac{\varphi_n\Delta x_n}{f(x_n)}-\frac{\varphi_N\Delta x_N}{f(x_N)}-\sum_{s=N}^{n-1}a_s\Delta x_s\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)\\&\quad\leqslant\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}(q_s-p_s).\end{aligned} \tag{30}$$

As $\{a_n\}$ is a nondecreasing sequence, we get

$$\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)=\frac{\varphi_{s+1}}{a_{s+1}f(x_{s+1})}-\frac{\varphi_s}{a_sf(x_s)}\leqslant\frac{\Delta\varphi_s}{a_{s+1}f(x_{s+1})}. \tag{31}$$

Thus

$$a_s\Delta x_s\cdot\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)\leqslant\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}. \tag{32}$$

From (30), we obtain

$$\frac{\varphi_n\Delta x_n}{f(x_n)}-\frac{\varphi_N\Delta x_N}{f(x_N)}+\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}(p_s-q_s)\leqslant\sum_{s=N}^{n-1}\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}, \tag{33}$$

using Lemma 1 and conditions, we have

$$\sum_{s=N}^{n-1}\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}\leqslant M\sum_{s=N}^{n-1}\frac{\Delta x_s}{f(x_{s+1})}\leqslant M\int_{x_N}^{+\infty}\frac{du}{f(u)},\quad M>0. \tag{34}$$

By letting $n\to\infty$, we see that the left-hand side of (33) is bounded, this is contrary to our condition (H_9). The proof is complete. □

By means of proof of Theorem 13, we get

Corollary 14. *If* (H_1)*,* (H_3)*, and* (H_9) *hold,* $\{a_n\}$ *is a nondecreasing sequence,* $f(x)$ *is a nondecreasing function, and there is a nonnegative sequence* $\{\varphi_n\}$*,* $\{\Delta\varphi_n\}$ *is bounded, and* $0<\int_0^{\varepsilon}du/f(u)<+\infty$ *for* $\varepsilon>0$*. Then,* (3) *cannot have any eventually positive nondecreasing bounded solutions.*

Corollary 15. *Suppose* (H_1)*,* (H_3)*, and* (H_9) *hold,* $\{a_n\}$ *is a nondecreasing sequence,* $f(x)$ *is a nondecreasing function, and there is a nonnegative nonincreasing sequence* $\{\varphi_n\}$*. Then,* (3) *cannot have any eventually positive monotone increasing solutions.*

Theorem 16. *Suppose* (H_1)*,* (H_3)*, and* (H_5) *hold,* $f(x)$ *is a nondecreasing function, and*

(H_{11}) $\limsup_{n\to\infty}\sum_{k=n_0}^{n}1/a_k\sum_{s=k}^{\infty}(p_s-q_s)=+\infty$ *for all* n_0*;*

(H_{12}) $0<\int_{\varepsilon}^{+\infty}du/f(u)<+\infty$, $\varepsilon>0$.

Then, (3) *cannot have any eventually positive nondecreasing solutions.*

Proof. Assume to the contrary that there exists $N>n_0$ such that $x_n>0$ and $\Delta x_n>0$ for $n>N$. $\{x_n\}$ is a solution of (3). By means of (3) and (H_3), we get

$$\frac{\Delta(a_n\Delta x_n)}{f(x_{n+1})}\leqslant(q_n-p_n), \tag{35}$$

by summing (35) from N to $n-1$, thus

$$\begin{aligned}&\frac{a_n\Delta x_n}{f(x_{n+1})}-\frac{a_N\Delta x_N}{f(x_{N+1})}-\sum_{s=N}^{n-1}a_{s+1}\Delta x_{s+1}\Delta\left(\frac{1}{f(x_{s+1})}\right)\\&\quad\leqslant\sum_{s=N}^{n-1}(q_s-p_s).\end{aligned} \tag{36}$$

As $f(x)$ is a nondecreasing function, we know $\Delta x_{s+1}\Delta(1/f(x_{s+1}))\leqslant0$, so

$$\sum_{s=N}^{\infty}(p_s-q_s)\leqslant\frac{a_N\Delta x_N}{f(x_{N+1})}. \tag{37}$$

In view of Lemma 1, we see that

$$\sum_{N=T}^{n}\frac{1}{a_N}\sum_{s=N}^{\infty}(p_s-q_s)\leqslant\sum_{N=T}^{n}\frac{\Delta x_N}{f(x_{N+1})}\leqslant\int_{x_T}^{x_{n+1}}\frac{du}{f(u)}. \tag{38}$$

This contradiction establishes our assertion. □

By means of proof of Theorem 16, we obtain the following.

Corollary 17. *Suppose* (H_1)*,* (H_3)*,* (H_5)*, and* (H_{11}) *hold,* $f(x)$ *is a nondecreasing function. Then* (3) *cannot have any eventually positive nondecreasing bounded solutions.*

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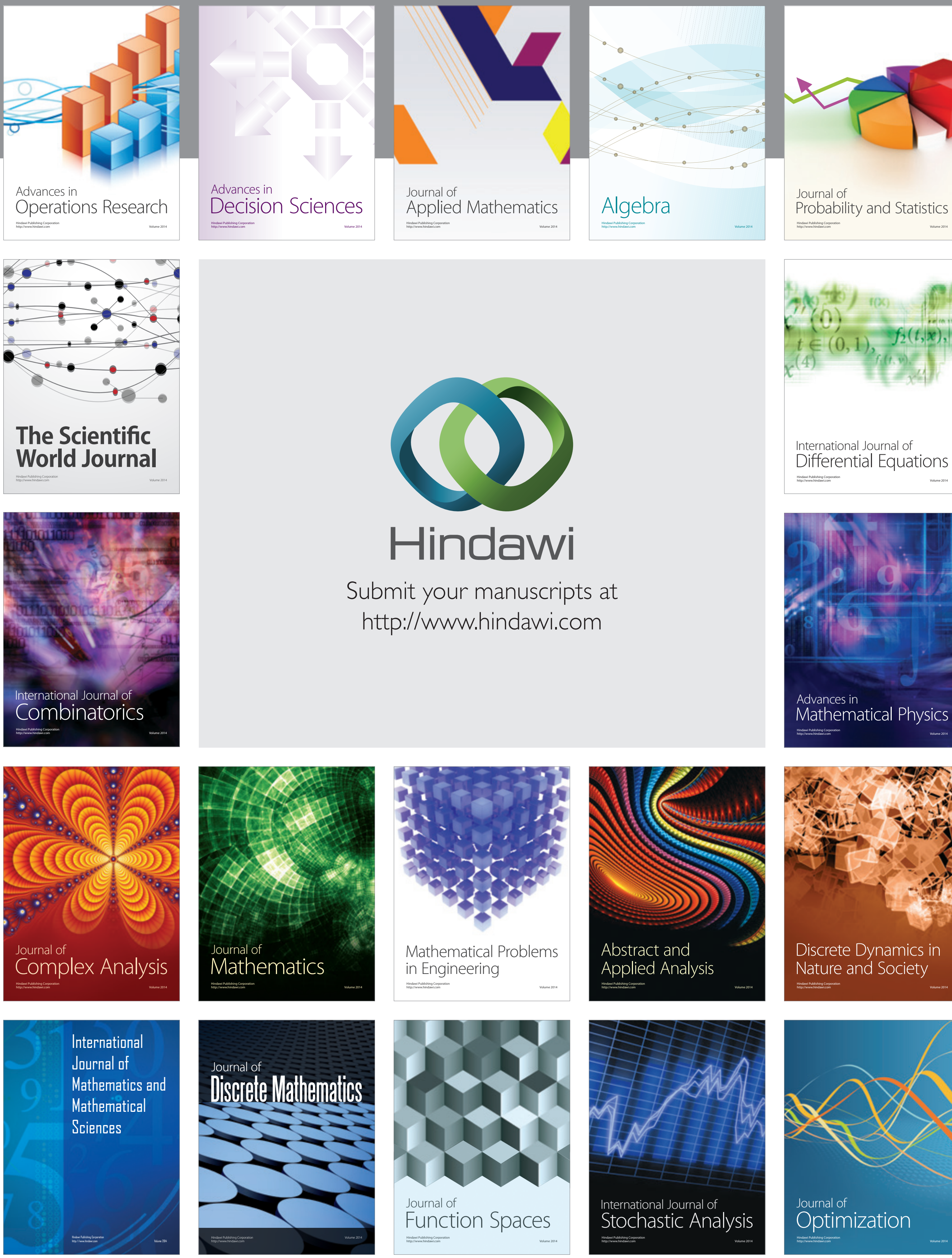
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