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Research Article

Solvability for Discrete Fractional Boundary Value Problems with a p-Laplacian Operator

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This paper is concerned with the solvability for a discrete fractional p-Laplacian boundary value problem. Some existence and uniqueness results are obtained by means of the Banach contraction mapping principle. Additionally, two representative examples are presented to illustrate the effectiveness of the main results.

1. Introduction

For any number $a \in \mathbb{R}$ and each interval $\mathbb{I}$ of $\mathbb{R}$, we denote $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{I}_{\mathbb{N}_a} = \mathbb{I} \cap \mathbb{N}_a$ throughout this paper. It is also worth noting that, in what follows, we appeal to the convention that the empty sum is taken to be 0.

In this paper, we will consider the existence and uniqueness of solutions for the following discrete fractional boundary value problem involving a p-Laplacian operator

$$\begin{gathered}\Delta\left[\phi_p\left(\Delta_C^{\alpha} u\right)\right](t) = f\left(t+\alpha-1, u\left(t+\alpha-1\right)\right), \quad t \in [0,b]_{\mathbb{N}_0}, \\ u(\alpha-2) = \beta_1 u(\alpha+b+1), \\ \Delta u(\alpha-2) = \Delta u(\alpha-1) = \beta_2 \Delta u(\alpha+b),\end{gathered} \tag{1}$$

where $1 < \alpha \leq 2$, $b \in \mathbb{N}_1$, $\beta_1 \neq 1$, $\beta_2 \neq 1$, Δ is the forward difference operator with stepsize 1, Δ_C^{α} denotes the discrete Caputo fractional difference of order α, $f : [\alpha-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ϕ_p is the p-Laplacian operator; that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$.

The theory of fractional differential equations has become a new important branch of mathematics (see, e.g., [1–8]). At the same time, boundary value problems for fractional differential equations have received considerable attention [9–18]. It is well known that discrete analogues of differential equations can be very useful in applications [19, 20], especially for using computer to simulate the behavior of solutions for certain dynamic equations. Compared to continuous case, significantly less is known about the discrete fractional calculus. However, within the recent years, a lot of papers have appeared on discrete fractional calculus and discrete fractional boundary value problems; see [21–37]. For example, in [25], Atıcı and Eloe explored a discrete fractional conjugate boundary value problem with the Riemann-Liouville fractional difference. To the best of our knowledge, this is pioneering work on discussing boundary value problems in discrete fractional calculus. After that, Goodrich studied discrete fractional boundary value problems involving the Riemann-Liouville fractional difference intensively and obtained a series of excellent results; see [26–31]. In [33, 34], Bastos et al. considered the discrete fractional calculus of variations and established several necessary optimality conditions for fractional difference variational problems. Abdeljawad introduced the conception of Caputo fractional difference and developed some useful properties of it in [35]. Ferreira in [37] initially investigated the existence and uniqueness of solutions for some discrete fractional boundary value problems of order less than one by the Banach fixed point theorem.

Very recently, some authors have focused their attention on the existence of solutions for fractional boundary value problems with the p-Laplacian operator in continuous case [38–44]. However, as far as we know, few papers can be found

in the literature for the discrete fractional boundary value problems with the p-Laplacian operator [45].

Inspired by the aforementioned results, we will investigate the discrete fractional p-Laplacian boundary value problem (1) and establish some sufficient conditions for the existence and uniqueness of solutions to it by using the Banach contraction mapping principle.

The remainder of this paper is organized as follows. Section 2 preliminarily provides some necessary basic knowledge for the theory of discrete fractional calculus. In Section 3, the existence and uniqueness results for the solution to problem (1) will be established with the help of the contraction mapping principle. Finally, in Section 4, two concrete examples are provided to illustrate the possible applications of the established analytical results.

2. Preliminaries

For the convenience of the reader, we begin by presenting here some necessary basic definitions and lemmas on discrete fractional calculus theory.

Definition 1 (see [21]). For any t and ν, the falling factorial function is defined as

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \tag{2}$$

provided that the right-hand side is well defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2 (see [46]). The νth fractional sum of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad \text{for } t \in \mathbb{N}_{a+\nu}. \tag{3}$$

Definition 3 (see [35]). The νth Caputo fractional difference of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, $\nu \notin \mathbb{N}$, is defined by

$$\begin{aligned} \Delta_C^{\nu} f(t) &= \Delta^{-(n-\nu)} \Delta^n f(t) \\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-n+\nu} (t-s-1)^{\underline{n-\nu-1}} \Delta^n f(s), \\ &\qquad \text{for } t \in \mathbb{N}_{a+n-\nu}, \end{aligned} \tag{4}$$

where n is the smallest integer greater than or equal to ν and Δ^n is the nth forward difference operator. If $\nu = n \in \mathbb{N}$, then $\Delta_C^{\nu} f(t) = \Delta^n f(t)$.

Lemma 4 (see [45]). *Assume that $\nu > 0$ and f is defined on $\mathbb{N}_a$. Then*

$$\Delta^{-\nu} \Delta_C^{\nu} f(t) = f(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{\underline{n-1}}, \tag{5}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, and n is the smallest integer greater than or equal to ν.

Now, we state and prove the following lemma, which provides a representation for the solution to (1) if the solution exists.

Lemma 5. *Let $h : [\alpha-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$, and let $\beta_1, \beta_2 \neq 1$. Then the following problem*

$$\Delta\left[\phi_p\left(\Delta_C^{\alpha} u\right)\right](t) = h(t+\alpha-1), \quad t \in [0, b]_{\mathbb{N}_0}, \tag{6}$$

$$\begin{aligned} u(\alpha-2) &= \beta_1 u(\alpha+b+1), \\ \Delta u(\alpha-2) &= \Delta u(\alpha-1) = \beta_2 \Delta u(\alpha+b), \end{aligned} \tag{7}$$

has a unique solution

$$\begin{aligned} u(t) = {} & \frac{a(t)}{\Gamma(\alpha-1)} \sum_{s=0}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} \phi_q\left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1)\right) \\ & + \frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)} \\ & \times \sum_{s=0}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} \phi_q\left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1)\right) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} \phi_q\left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1)\right), \\ & \qquad t \in [\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}, \end{aligned} \tag{8}$$

where $a(t) = (\beta_2\{[\beta_1(\alpha+b+1)+2-\alpha] + (1-\beta_1)t\})/((1-\beta_1)(1-\beta_2))$.

Proof. The definition of the discrete Caputo fractional difference, together with condition $\Delta u(\alpha-2) = \Delta u(\alpha-1)$, implies that $\Delta_C^{\alpha} u(0) = 0$. So from (6), we have

$$\begin{aligned} \phi_p\left(\Delta_C^{\alpha} u(t)\right) &= \phi_p\left(\Delta_C^{\alpha} u(0)\right) + \sum_{s=0}^{t-1} h(s+\alpha-1) \\ &= \sum_{s=0}^{t-1} h(s+\alpha-1), \end{aligned} \tag{9}$$

and then

$$\Delta_C^{\alpha} u(t) = \phi_q\left(\sum_{s=0}^{t-1} h(s+\alpha-1)\right), \quad t \in [0, b+1]_{\mathbb{N}_0}. \tag{10}$$

Hence, in view of Lemma 4, we can get

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} \phi_q\left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1)\right) \\ & + c_0 + c_1 t, \end{aligned} \tag{11}$$

where $t \in [\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}$, $c_0, c_1 \in \mathbb{R}$.

Furthermore, we have

$$\Delta u(t) = \frac{1}{\Gamma(\alpha-1)} \sum_{s=0}^{t-(\alpha-1)} (t-s-1)^{\underline{\alpha-2}} \phi_q \times \left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1) \right) + c_1, \quad t \in [\alpha-2, \alpha+b]_{\mathbb{N}_{\alpha-2}}. \tag{12}$$

Then by conditions $u(\alpha-2) = \beta_1 u(\alpha+b+1)$, $\Delta u(\alpha-2) = \beta_2 \Delta u(\alpha+b)$, we can get

$$\begin{aligned} c_0 &= \frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)} \times \sum_{s=0}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} \phi_q \left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1) \right) \\ &\quad + \frac{\beta_2 [\beta_1(\alpha+b+1)+2-\alpha]}{(1-\beta_1)(1-\beta_2)\Gamma(\alpha-1)} \times \sum_{s=0}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} \phi_q \left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1) \right), \\ c_1 &= \frac{\beta_2}{(1-\beta_2)\Gamma(\alpha-1)} \times \sum_{s=0}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} \phi_q \left(\sum_{\tau=0}^{s-1} h(\tau+\alpha-1) \right). \end{aligned} \tag{13}$$

Substituting the values of c_0 and c_1 in (11), we get (8). This completes the proof. □

Finally, we list below the following basic properties of the p-Laplacian operator which will be used in the sequel.

(1) If $1 < p < 2$, $uv > 0$ and $|u|, |v| \geq m > 0$, then

$$\left| \phi_p(v) - \phi_p(u) \right| \leq (p-1) m^{p-2} |v-u|. \tag{14}$$

(2) If $p > 2$, $|u|, |v| \leq M$, then

$$\left| \phi_p(v) - \phi_p(u) \right| \leq (p-1) M^{p-2} |v-u|. \tag{15}$$

3. Main Results

In this section, we will use the Banach contraction mapping principle to prove the existence and uniqueness for the solution to problem (1).

Let $\mathbb{E}$ denote the Banach space of all functions from $[\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}$ into $\mathbb{R}$ endowed with the norm defined by $\|u\| = \max\{|u(t)|, t \in [\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}\}$.

For the sake of convenience to the following discussion, we set

$$\overline{a} = \max\left\{ |a(t)|, t \in [\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}} \right\}, \tag{16}$$

where $a(t)$ is as given in Lemma 5. Also, for any $u, v \in \mathbb{E}$, we denote

$$\begin{aligned} A(v,u)(t) &= \phi_q \left(\sum_{s=0}^{t-1} f(s+\alpha-1, v(s+\alpha-1)) \right) \\ &\quad - \phi_q \left(\sum_{s=0}^{t-1} f(s+\alpha-1, u(s+\alpha-1)) \right), \end{aligned} \tag{17}$$

for $t \in [0, b+1]_{\mathbb{N}_0}$. Obviously, $A(v,u)(0) = 0$.

In view of Lemma 5, we transform problem (1) as

$$u = \mathcal{F} u, \tag{18}$$

where $\mathcal{F} : \mathbb{E} \to \mathbb{E}$ is defined by

$$\begin{aligned} (\mathcal{F}u)(t) &= \frac{a(t)}{\Gamma(\alpha-1)} \sum_{s=0}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} \times \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau+\alpha-1, u(\tau+\alpha-1)) \right) \\ &\quad + \frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)} \sum_{s=0}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} \times \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau+\alpha-1, u(\tau+\alpha-1)) \right) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} \times \phi_q \left(\sum_{\tau=0}^{s-1} f(\tau+\alpha-1, u(\tau+\alpha-1)) \right), \end{aligned} \tag{19}$$

for $t \in [\alpha-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}$. It is clear to see that u is a solution of the problem (1) if and only if u is a fixed point of $\mathcal{F}$.

Now, we state the main results as follows.

Theorem 6. *Suppose $p > 2$, $\beta_1 \neq 1$, $\beta_2 \neq 1$, and the following condition holds:*

(H_1) *there exist positive numbers λ and k with*

$$k < (|1-\beta_1| b!) \times \left((q-1)(b+1) \times [|1-\beta_1| (\overline{a}\alpha + \alpha + b) + |\beta_1| (\alpha+b)] \times [\lambda \Gamma(\alpha+1)]^{q-2} \prod_{i=1}^{b-1} (\alpha+i) \right)^{-1}, \tag{20}$$

such that

$$\begin{aligned}&\lambda\alpha t^{\underline{\alpha-1}}\\&\quad\le f(t,u),\quad \text{for } (t,u)\in[\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}}\times\mathbb{R},\end{aligned}\tag{21}$$

$$\begin{aligned}&\left|f(t,v)-f(t,u)\right|\\&\quad\le k\left|v-u\right|,\quad \text{for } t\in[\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}},\ u,v\in\mathbb{R}.\end{aligned}\tag{22}$$

Then the problem (1) *has a unique solution.*

Proof. For any $u\in\mathbb{E}$, by (21), we can get that

$$\begin{aligned}&\sum_{s=0}^{t-1}f(s+\alpha-1,u(s+\alpha-1))\\&\quad\ge\lambda\sum_{s=0}^{t-1}\alpha(s+\alpha-1)^{\underline{\alpha-1}}\\&\quad=\lambda(t+\alpha-1)^{\underline{\alpha}}\ge\lambda\Gamma(\alpha+1),\quad t\in[1,b+1]_{\mathbb{N}_0}.\end{aligned}\tag{23}$$

Due to $p>2$ and $1/p+1/q=1$, we know that $1<q<2$. By (14) and (22), for any $u,v\in\mathbb{E}$, $t\in[1,b+1]_{\mathbb{N}_1}$, we have

$$\begin{aligned}\left|A(v,u)(t)\right|&\le(q-1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\\&\quad\times\sum_{s=0}^{t-1}\left|f(s+\alpha-1,v(s+\alpha-1))\right.\\&\qquad\left.-f(s+\alpha-1,u(s+\alpha-1))\right|\\&\le(q-1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\sum_{s=0}^{t-1}k\left\|v-u\right\|\\&\le k(q-1)(b+1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\left\|v-u\right\|.\end{aligned}\tag{24}$$

Next, for any $u,v\in\mathbb{E}$ and for each $t\in[\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}$, together with the fact that $A(v,u)(0)=0$, we obtain

$$\begin{aligned}&\left|(\mathcal{F}v)(t)-(\mathcal{F}u)(t)\right|\\&\quad=\left|\frac{a(t)}{\Gamma(\alpha-1)}\sum_{s=0}^{b+1}(\alpha+b-s-1)^{\underline{\alpha-2}}A(v,u)(s)\right.\\&\qquad+\frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)}\sum_{s=0}^{b+1}(\alpha+b-s)^{\underline{\alpha-1}}A(v,u)(s)\\&\qquad\left.+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}A(v,u)(s)\right|\\&\quad=\left|\frac{a(t)}{\Gamma(\alpha-1)}\sum_{s=1}^{b+1}(\alpha+b-s-1)^{\underline{\alpha-2}}A(v,u)(s)\right.\\&\qquad+\frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)}\sum_{s=1}^{b+1}(\alpha+b-s)^{\underline{\alpha-1}}A(v,u)(s)\\&\qquad\left.+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}A(v,u)(s)\right|\\&\quad\le\left\{\frac{\left|a(t)\right|}{\Gamma(\alpha-1)}\sum_{s=1}^{b+1}(\alpha+b-s-1)^{\underline{\alpha-2}}\right.\\&\qquad+\frac{\left|\beta_1\right|}{\left|1-\beta_1\right|\Gamma(\alpha)}\sum_{s=1}^{b+1}(\alpha+b-s)^{\underline{\alpha-1}}\\&\qquad\left.+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}\right\}\\&\qquad\times k(q-1)(b+1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\left\|v-u\right\|\\&\quad\le\left\{\frac{\overline{a}(\alpha+b-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\right.\\&\qquad\left.+\frac{\left|\beta_1\right|(\alpha+b)^{\underline{\alpha}}}{\left|1-\beta_1\right|\Gamma(\alpha+1)}+\frac{(\alpha+b)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right\}\\&\qquad\times k(q-1)(b+1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\left\|v-u\right\|\\&\quad=\frac{\left\{\left|1-\beta_1\right|\left[\overline{a}\alpha+\alpha+b\right]+\left|\beta_1\right|(\alpha+b)\right\}\Gamma(\alpha+b)}{\left|1-\beta_1\right|\Gamma(\alpha+1)\Gamma(b+1)}\\&\qquad\times k(q-1)(b+1)\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\left\|v-u\right\|\\&\quad=\left(\left(k(q-1)(b+1)\left[\left|1-\beta_1\right|(\overline{a}+\alpha+b)\right.\right.\right.\\&\qquad\left.+\left|\beta_1\right|(\alpha+b)\right]\left[\lambda\Gamma(\alpha+1)\right]^{q-2}\\&\qquad\left.\left.\times\prod_{i=1}^{b-1}(\alpha+i)\right)\times\left(\left|1-\beta_1\right|b!\right)^{-1}\right)\left\|v-u\right\|\\&\quad=L\left\|v-u\right\|,\end{aligned}\tag{25}$$

where $L=(k(q-1)(b+1)[|1-\beta_1|(\overline{a}\alpha+\alpha+b)+|\beta_1|(\alpha+b)][\lambda\Gamma(\alpha+1)]^{q-2}\prod_{i=1}^{b-1}(\alpha+i))/(|1-\beta_1|b!)$. From (20), we get that $0<L<1$, which implies that $\mathcal{F}$ is a contraction mapping. By means of the Banach contraction mapping principle, we get that $\mathcal{F}$ has a unique fixed point in $\mathbb{E}$; that is. the problem (1) has a unique solution. This completes the proof. □

With a similar proof to that of Theorem 6, we can get the following theorem.

Theorem 7. *Suppose $p>2$, $\beta_1\ne1$, $\beta_2\ne1$, and the following condition holds:*

(H_2) *there exist constants* $\lambda > 0$ *and* k *with*

$$\begin{aligned} 0 < k < & \left(\left|1-\beta_1\right| b!\right) \\ & \times \Bigg((q-1)(b+1) \\ & \quad \times \left[\left|1-\beta_1\right| (\overline{a}\alpha+\alpha+b) + \left|\beta_1\right| (\alpha+b)\right] \\ & \quad \times [\lambda\Gamma(\alpha+1)]^{q-2} \prod_{i=1}^{b-1} (\alpha+i) \Bigg)^{-1}, \end{aligned} \tag{26}$$

such that

$$\begin{aligned} & f(t,u) \\ & \quad \le -\lambda\alpha t^{\underline{\alpha-1}}, \quad \text{for } (t,u) \in [\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R}, \\ & \left|f(t,v) - f(t,u)\right| \\ & \quad \le k|v-u|, \quad \text{for } t \in [\alpha-1,b+\alpha-1]_{\mathbb{N}_{\alpha-1}},\ u,v \in \mathbb{R}. \end{aligned} \tag{27}$$

Then the problem (1) *has a unique solution.*

Theorem 8. *Suppose* $1 < p < 2$, $\beta_1 \neq 1$, $\beta_2 \neq 1$, *and the following condition holds:*

(H_3) *there exists a nonnegative function* $g : [\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$ *and* $\sum_{s=0}^{b} g(s+\alpha-1) = M > 0$ *such that*

$$\left|f(t,u)\right| \le g(t), \quad (t,u) \in [\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R}, \tag{28}$$

and there exists a positive constant k *such that*

$$\begin{aligned} & \left|f(t,v) - f(t,u)\right| \\ & \quad \le k|v-u|, \quad \text{for } t \in [\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}},\ u,v \in \mathbb{R}. \end{aligned} \tag{29}$$

Then the problem (1) *has a unique solution provided that*

$$\begin{aligned} k < & \left(\left|1-\beta_1\right| b!\right) \\ & \times \Bigg(\left[\left|1-\beta_1\right| (\overline{a}\alpha+\alpha+b) + \left|\beta_1\right| (\alpha+b)\right] \\ & \quad \times (q-1)(b+1) M^{q-2} \prod_{i=1}^{b-1} (\alpha+i) \Bigg)^{-1}. \end{aligned} \tag{30}$$

Proof. By (28), we can get that, for $t \in [1,b+1]_{\mathbb{N}_1}$,

$$\begin{aligned} & \left|\sum_{s=0}^{t-1} f(s+\alpha-1, u(s+\alpha-1))\right| \\ & \quad \le \sum_{s=0}^{t-1} \left|f(s+\alpha-1, u(s+\alpha-1))\right| \\ & \quad \le \sum_{s=0}^{b} g(s+\alpha-1) = M. \end{aligned} \tag{31}$$

In view of $1 < p < 2$ and $1/p + 1/q = 1$, we can get $q > 2$. From (15) and (29), for any $v, u \in \mathbb{E}$, we have

$$\begin{aligned} |A(v,u)(t)| & \le (q-1) M^{q-2} \\ & \quad \times \left|\sum_{s=0}^{t-1} f(s+\alpha-1, v(s+\alpha-1)) - \sum_{s=0}^{t-1} f(s+\alpha-1, u(s+\alpha-1))\right| \\ & \le (q-1) M^{q-2} \\ & \quad \times \sum_{s=0}^{t-1} \left|f(s+\alpha-1, v(s+\alpha-1)) - f(s+\alpha-1, u(s+\alpha-1))\right| \\ & \le (q-1) M^{q-2} \sum_{s=0}^{t-1} k\|v-u\| \\ & \le k(q-1) M^{q-2} t\|v-u\|, \quad t \in [1,b+1]_{\mathbb{N}_1}. \end{aligned} \tag{32}$$

Hence, for any $t \in [\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}$, by $A(u,v)(0) = 0$, we have

$$\begin{aligned} & |(\mathcal{F}v)(t) - (\mathcal{F}u)(t)| \\ & = \Bigg| \frac{a(t)}{\Gamma(\alpha-1)} \sum_{s=1}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} A(v,u)(s) \\ & \quad + \frac{\beta_1}{(1-\beta_1)\Gamma(\alpha)} \sum_{s=1}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} A(v,u)(s) \\ & \quad + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} A(v,u)(s) \Bigg| \\ & \le \Bigg\{ \frac{|a(t)|}{\Gamma(\alpha-1)} \sum_{s=1}^{b+1} (\alpha+b-s-1)^{\underline{\alpha-2}} \\ & \quad + \frac{|\beta_1|}{(|1-\beta_1|)\Gamma(\alpha)} \sum_{s=1}^{b+1} (\alpha+b-s)^{\underline{\alpha-1}} \\ & \quad + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} \Bigg\} \\ & \quad \times k(q-1)(b+1) M^{q-2} \|v-u\| \\ & \le \Bigg\{ \frac{\overline{a}(\alpha+b-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \\ & \quad + \frac{|\beta_1|(\alpha+b)^{\underline{\alpha}}}{(|1-\beta_1|)\Gamma(\alpha+1)} + \frac{(\alpha+b)^{\underline{\alpha}}}{\Gamma(\alpha+1)} \Bigg\} \\ & \quad \times k(q-1)(b+1) M^{q-2} \|v-u\| \end{aligned}$$

$$= \left(\left(k\left[\left|1-\beta_1\right|(\bar{a}\alpha+\alpha+b)+\left|\beta_1\right|(\alpha+b)\right]\right.\right.$$
$$\times (q-1)(b+1)M^{q-2}\prod_{i=1}^{b-1}(\alpha+i)\Big)$$
$$\left.\times\left(\left|1-\beta_1\right|b!\right)^{-1}\right)\|v-u\| = L\|v-u\|, \tag{33}$$

where $L = (k[|1-\beta_1|(\bar{a}\alpha+\alpha+b)+|\beta_1|(\alpha+b)](q-1)(b+1)M^{q-2}\prod_{i=1}^{b-1}(\alpha+i))/(|1-\beta_1|b!)$. In view of (30), $\mathscr{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

4. Examples

In this section, we will illustrate the possible application of the above established analytical results with the following two concrete examples.

Example 1. Consider the discrete fractional boundary value problem

$$\begin{aligned}
&\Delta\left[\phi_3\left(\Delta_C^{3/2}u\right)\right](t)\\
&\quad = 3\left(t+\frac{1}{2}\right)^{\underline{1/2}}\\
&\qquad\times\left[\frac{1}{2}+\sin^2\left(\frac{u(t+1/2)}{10\sqrt{3}}+\theta\right)\right.\\
&\qquad\left.+\frac{1}{390}\left|u\left(t+\frac{1}{2}\right)\right|\right], \quad t\in[0,2]_{\mathbb{N}_0},\\
&\qquad 10u\left(-\frac{1}{2}\right)=u\left(\frac{9}{2}\right),\\
&\quad 10\Delta u\left(-\frac{1}{2}\right)=10\Delta u\left(\frac{1}{2}\right)=\Delta u\left(\frac{7}{2}\right),
\end{aligned} \tag{34}$$

here θ is a real number.

Conclusion. Problem (34) has a unique nonnegative solution.

Proof. Corresponding to problem (1), $p=3>2$, $q=3/2$, $\alpha=3/2$, $\beta_1=1/10$, $\beta_2=1/10$, $b=2$, and $f(t,u)=3t^{\underline{1/2}}[1/2+\sin^2((u/10\sqrt{3})+\theta)+(1/390)|u|]$, $(t,u)\in[1/2,5/2]_{\mathbb{N}_{1/2}}\times\mathbb{R}$.

Choosing $\lambda=1$ and $k=3/100$, by direct calculation, we can verify that

$$\begin{aligned}
k &= \frac{3}{100} < \left(\left|1-\beta_1\right|b!\right)\\
&\quad\times\left((q-1)(b+1)\right.\\
&\qquad\times\left[\left|1-\beta_1\right|(\bar{a}\alpha+\alpha+b)+\left|\beta_1\right|(\alpha+b)\right]\\
&\qquad\left.\times[\lambda\Gamma(\alpha+1)]^{q-2}\prod_{i=1}^{b-1}(\alpha+i)\right)^{-1}\\
&= \frac{18\sqrt{3}\pi^{1/4}}{325}\approx 0.1277.
\end{aligned} \tag{35}$$

It is easy to verify that

$$\begin{aligned}
\lambda\alpha t^{\underline{\alpha-1}} &= \frac{3}{2t^{\underline{1/2}}}\\
&\le 3t^{\underline{1/2}}\left[\frac{1}{2}+\sin^2\left(\frac{u}{10\sqrt{3}}+\theta\right)+\frac{1}{390}|u|\right]\\
&= f(t,u), \quad (t,u)\in\left[\frac{1}{2},\frac{5}{2}\right]_{\mathbb{N}_{1/2}}\times\mathbb{R},\\
&\left|f(t,v)-f(t,u)\right|\\
&\quad = 3t^{\underline{1/2}}\left|\sin^2\left(\frac{v}{10\sqrt{3}}+\theta\right)\right.\\
&\qquad -\sin^2\left(\frac{u}{10\sqrt{3}}+\theta\right)\\
&\qquad\left.+\frac{1}{390}|v|-\frac{1}{390}|u|\right|\\
&\quad\le 3\left(\frac{5}{2}\right)^{\underline{1/2}}\left(\frac{1}{300}+\frac{1}{390}\right)|v-u|\\
&\quad = \frac{69}{4160}\sqrt{\pi}\,|v-u|\\
&\quad\approx 0.0294\,|v-u| < k\,|v-u|,
\end{aligned} \tag{36}$$

for $t\in[1/2,5/2]_{\mathbb{N}_{1/2}}$, $u,v\in\mathbb{R}$. Therefore, by Theorem 6, the boundary value problem (34) has a unique solution. Furthermore, from the nonnegativeness of f and the expression of $\mathscr{F}$, we also get that the unique solution of (34) is nonnegative. □

Example 2. Consider the nonlinear discrete fractional boundary value problem

$$\begin{aligned}
&\Delta\left[\phi_{3/2}\left(\Delta_C^{3/2}u\right)\right](t)\\
&\quad = \frac{3}{2}\left(t+\frac{1}{2}\right)^{\underline{1/2}}\\
&\qquad\times\sin^2\left(\frac{u(t+1/2)}{40}+\omega\right), \quad t\in[0,2]_{\mathbb{N}_0},\\
&\qquad 2u\left(-\frac{1}{2}\right)=-u\left(\frac{9}{2}\right),\\
&\quad 2\Delta u\left(-\frac{1}{2}\right)=2\Delta u\left(\frac{1}{2}\right)=\Delta u\left(\frac{7}{2}\right),
\end{aligned} \tag{37}$$

where ω is a real number.

Conclusion. Problem (37) has a unique solution.

Proof. The problem (37) can be regarded as problem (1), where $p=3/2<2$, $q=3>2$, $\alpha=3/2$, $\beta_1=-1/2$, $\beta_2=1/2$, $b=2$, and $f(t,u)=(3/2)t^{\underline{1/2}}\sin^2((u/40)+\omega)$, (t,u)

$\in [1/2, 5/2]_{\mathbb{N}_{1/2}} \times \mathbb{R}$. Taking $g(t) = (3/2)t^{1/2}$, $t \in [1/2, 5/2]_{\mathbb{N}_{1/2}}$, then $M = (105\Gamma(1/2))/32$. Let $k = 1/600 \approx 0.0017$; we have

$$\begin{aligned} k &< (|1-\beta_1| b!) \\ &\quad \times \Bigg([|1-\beta_1| (\overline{a}\alpha + \alpha + b) + |\beta_1| (\alpha + b)] \\ &\qquad \times (q-1)(b+1) M^{q-2} \prod_{i=1}^{b-1} (\alpha + i) \Bigg)^{-1} \\ &= \frac{64}{15225\sqrt{\pi}} \approx 0.0024. \end{aligned} \tag{38}$$

Moreover, we can verify that

$$\begin{aligned} |f(t,u)| &\le g(t), \quad (t,u) \in \left[\frac{1}{2}, \frac{5}{2}\right]_{\mathbb{N}_{1/2}} \times \mathbb{R}, \\ |f(t,v) - f(t,u)| &\le \frac{3}{2} t^{1/2} \frac{1}{40^2} |v-u| \\ &\le \frac{3}{2}\left(\frac{5}{2}\right)^{1/2} \frac{1}{40^2} |v-u| \\ &= \frac{9\sqrt{\pi}}{10240} |v-u| \\ &\approx 0.0016 |v-u| < k|v-u|, \end{aligned} \tag{39}$$

for $t \in [1/2, 5/2]_{\mathbb{N}_{1/2}}$, $v, u \in \mathbb{R}$.

Therefore, by Theorem 8, problem (37) has a unique solution. □

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