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Research Article

A Mixed Monotone Operator Method for the Existence and Uniqueness of Positive Solutions to Impulsive Caputo Fractional Differential Equations

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We establish some sufficient conditions for the existence and uniqueness of positive solutions to a class of initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative. Our analysis relies on a fixed point theorem for mixed monotone operators. Our result can not only guarantee the existence of a unique positive solution but also be applied to construct an iterative scheme for approximating it. An example is given to illustrate our main result.

1. Introduction

The purpose of this paper is to investigate the existence and uniqueness of positive solutions to the following impulsive initial value problem (IVP for short) for Caputo fractional-order differential equations:

$$\begin{gathered} {}^{c}D^{\alpha} y(t) = f(t, y(t), y(t)) + g(t, y(t)), \\ t \in [0, T], \ t \neq t_k, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \qquad y(0) = y_0, \end{gathered} \tag{1}$$

where $k = 1, 2, \ldots, m$, $0 < \alpha \leq 1$, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $g : [0, T] \times \mathbf{R} \to \mathbf{R}$ are given functions, $I_k : \mathbf{R} \to \mathbf{R}$, $y_0 \in \mathbf{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering, and biological sciences; see [1–6], for example. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monographs of Miller and Ross [3], Podlubny [5], Kilbas et al. [6], and the papers [7–21] and the references therein. In these papers, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors; see [10, 15, 19, 22, 23], for example.

In [7], Ahmad and Sivasundaram considered the following impulsive hybrid boundary value problem for nonlinear fractional differential equations:

$$\begin{gathered} {}^{c}D^{q} x(t) = f(t, x(t)), \qquad 1 < q \leq 2, \\ t \in [0, 1] \setminus \{t_1, t_2, \ldots, t_p\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), \\ \Delta x'(t_k) = J_k(x(t_k^-)), \quad t_k \in (0, 1), \ k = 1, 2, \ldots, p, \\ x(0) + x'(0) = 0, \qquad x(1) + x'(1) = 0, \end{gathered} \tag{2}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative, $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, $I_k, J_k : \mathbf{R} \to \mathbf{R}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$, and

$x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$, $k = 1, 2, \ldots, p$ for $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$. Based on contraction mapping principle and the Krasnoselskii fixed point theorem, they discussed some existence results for (2).

In [24], Benchohra and Slimani concerned the existence and uniqueness of solutions for the following initial value problem for Caputo fractional-order differential equations:

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0, T], \ t \neq t_k, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \qquad y(0) = y_0, \end{gathered} \tag{3}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, $f : [0, T] \times \mathbf{R} \to \mathbf{R}$ is a given function, $I_k : \mathbf{R} \to \mathbf{R}$, $y_0 \in \mathbf{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$. They gave an existence and uniqueness result for the IVP (3) which was based on the Banach fixed point theorem and also obtained two existence and uniqueness results; the first one was based on the Schaefer fixed point theorem and the second one was based on the nonlinear alternative of the Leray-Schauder type.

Different from the above works [7, 24], in this paper, we will use a fixed point theorem for mixed monotone operators to study the existence and uniqueness of positive solutions for the IVP (1). Our result can not only guarantee the existence of unique positive solution but also be applied to construct iterative scheme for approximating it.

With this context in mind, the outline of this paper is as follows. In Section 2, we will recall certain results from the theory of fractional calculus and some definitions, notations, and results of mixed monotone operators. In Section 3, we will provide some conditions under which the IVP (1) will have a unique positive solution. Finally, in Section 4, we will provide one example, which explicates the applicability of our main result.

2. Preliminaries

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proof of our main theorem.

Definition 1 (see [9]). Let $\nu > 0$ with $\nu \in \mathbf{R}$. Suppose that $y : [a, \infty) \to \mathbf{R}$. Then the νth Riemann-Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1} ds, \tag{4}$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbf{R}$, one defines the νth Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}} ds, \tag{5}$$

where $n \in N$ is the unique positive integer satisfying $n - 1 \leq \nu < n$ and $t > a$.

Definition 2 (see [11]). For a function y given on the interval $[a, b]$, the Caputo fractional-order derivative of order ν of y is defined by

$${}^cD_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{\nu+1-n}} ds, \tag{6}$$

where $n = [\nu] + 1$.

In the sequel, we present some basic concepts in the ordered Banach spaces for completeness and one fixed point theorem which will be used later. For convenience of readers, we suggest that one refers to [23, 25, 26] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$. That is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote that $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies that $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2.

Definition 3 (see [23, 25, 26]). Consider that $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y; that is, u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2$, and $v_1 \geq v_2$ imply that $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Lemma 4 (see [25]). *Let P be normal and let $A : P \times P \to P$ be a mixed monotone operator. Suppose that*

(a) there exist $v > \theta$ and $c > 0$, such that $\theta < A(v, \theta) \leq v$ and $A(\theta, v) \geq cA(v, \theta)$,

(b) for any $0 < a < b < 1$, there exists $\eta = \eta(a, b) > 0$, such that

$$\begin{gathered} A(tx, t^{-1}y) \geq t(1+\eta) A(x, y), \\ \forall a \leq t \leq b, \ \theta \leq y \leq x \leq v. \end{gathered} \tag{7}$$

Then A has a unique fixed point x^* in $[\theta, v]$ with $x^* > \theta$. Moreover, for any initial values x_0, $y_0 \in [\theta, v]$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \tag{8}$$

one has $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3. Main Result

In this section, we apply Lemma 4 to study the IVP (1) and then we obtain a new result on the existence and uniqueness of positive solutions. The existence and uniqueness result is relatively new to the fractional differential equations in this literature.

In our considerations we will work in the Banach space $C[0, T] = \{x : [0, T] \to \mathbf{R}$ is continuous$\}$ with the standard

norm $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,T], \quad x \le y \Longleftrightarrow x(t) \le y(t) \quad \text{for } t \in [0,T]. \tag{9}$$

Set $P = \{x \in C[0,T] \mid x(t) \ge 0, t \in [0,T]\}$, the standard cone. It is clear that P is a normal cone in $C[0,T]$ and the normality constant is 1. Consider the following set of functions:

$$\begin{aligned} PC[0,T] &= \{y : [0,T] \longrightarrow \mathbf{R} : y \in C(t_k, t_{k+1}], \\ &\quad k = 0,1,\ldots,m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), \\ &\quad k = 1,2,\ldots,m \text{ with } y(t_k^-) = y(t_k)\}. \end{aligned} \tag{10}$$

Then $PC[0,T]$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in[0,T]} |y(t)|. \tag{11}$$

Denote P_C by $P_C = \{x \in PC[0,T] \mid x(t) \ge 0, t \in [0,T]\}$; then P_C is a standard and normal cone in $PC[0,T]$.

Lemma 5 (see [24]). *Let $0 < \alpha \le 1$ and let $h : J \to \mathbf{R}$ be continuous. A function $y \in PC[0,T]$ is a solution of the fractional integral equation:*

$$y(t) = \begin{cases} y_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} h(s)\,ds, & \text{if } t \in [0,t_1], \\ y_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\,ds \\ \quad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1} h(s)\,ds \\ \quad + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)) & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{12}$$

where $k = 1,\ldots,m$, if and only if y is a solution of the fractional IVP

$$\begin{aligned} &{}^cD^\alpha y(t) = h(t), \quad t \in [0,T], \ t \ne t_k, \\ &\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\ldots,m, \\ &y(0) = y_0. \end{aligned} \tag{13}$$

Remark 6. The concept of solutions for fractional differential equations has been argued extensively; see [7, 18, 21, 24] and the references therein. In [18], the authors gave a counterexample to show that the result in Lemma 5 is not reasonable. However, the counterexample is inappropriate. From the very recent paper [21], the approach for finding the solution of impulsive fractional differential equations in [18] is inappropriate and some arguments like Lemma 5 are plausible.

For the convenience, we list the following conditions:

(H_1) $f : [0,T] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and $g : [0,T] \times [0,\infty) \to [0,\infty)$ is continuous;

(H_2) $f(t,u,v)$ is increasing in $u \in [0,\infty)$ for fixed $t \in [0,T]$ and $v \in [0,\infty)$ and decreasing in $v \in [0,\infty)$ for fixed $t \in [0,T]$ and $u \in [0,\infty)$, and $g(t,v)$ is decreasing in $v \in [0,\infty)$ for fixed $t \in [0,T]$;

(H_3) for any $0 < p < q < 1$, there exist β_1, β_2, and $\beta_3 \in (0,\infty)$ which depend on q, such that

$$\begin{aligned} &f(t, \lambda u, \lambda^{-1} v) \ge \frac{\lambda}{1-\lambda^{\beta_1}} f(t,u,v), \\ &\forall t \in [0,T], \ p \le \lambda \le q, u, v \in [0,\infty), \\ &g(t, \lambda^{-1} v) \ge \frac{\lambda}{1-\lambda^{\beta_2}} g(t,v), \\ &\forall t \in [0,T], \ p \le \lambda \le q, v \in [0,\infty), \\ &iI_k(\lambda u) \ge \frac{\lambda}{1-\lambda^{\beta_3}} I_k(u), \\ &\forall k = 1,2,\ldots m, \ p \le \lambda \le q, u \in [0,\infty); \end{aligned} \tag{14}$$

(H_4) the functions $I_k : [0,\infty) \to \mathbf{R}$ are continuous and nondecreasing and there exists a constant $M^* > 0$ such that $|I_k(u)| \le M^*$ for each $u \in [0,\infty)$, $k = 1,\ldots,m$;

(H_5) $y_0 > mM^* > 0$.

Theorem 7. *Assume that (H_1)–(H_5) are satisfied and there exists $R > 0$ such that*

$$y_0 + \frac{(m+1)(M_1 + M_2)}{\Gamma(\alpha+1)} T^\alpha + mM^* \le R, \tag{15}$$

where $M_1 = \max_{t\in[0,T]} f(t,R,0)$, $M_2 = \max_{t\in[0,T]} g(t,0)$.

Then the IVP (1) has a unique positive solution u^* in $P_{[\theta,R]} = \{y \in P_C \mid 0 \le y(t) \le R, t \in [0,T]\}$ with $u^*(t) \not\equiv 0$, $t \in [0,T]$. Moreover, for any initial values x_0, $y_0 \in P_C$, constructing successively the sequences

$$\begin{aligned} x_{n+1}(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, x_n(s), y_n(s)) \\ &\qquad + g(s, y_n(s))]\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, x_n(s), y_n(s)) \\ &\qquad + g(s, y_n(s))]\,ds \\ &\quad + \sum_{0<t_k<t} I_k(x_n(t_k^-)), \end{aligned}$$

$$\begin{aligned} y_{n+1}(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, y_n(s), x_n(s)) \\ &\qquad + g(s, x_n(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, y_n(s), x_n(s)) \\ &\qquad + g(s, x_n(s))] \, ds \\ &+ \sum_{0<t_k<t} I_k(y_n(t_k^-)), \end{aligned} \tag{16}$$

where $n = 0, 1, 2, \ldots$, then one has $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Proof. To begin with, from Lemma 5, the IVP (1) has an integral formulation given by

$$\begin{aligned} u(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, u(s), u(s)) \\ &\qquad + g(s, u(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, u(s), u(s)) \\ &\qquad + g(s, u(s))] \, ds \\ &+ \sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{17}$$

Define an operator $A : P_C \times P_C \to PC[0, T]$ by

$$\begin{aligned} A(u, v)(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, u(s), v(s)) \\ &\qquad + g(s, v(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, u(s), v(s)) \\ &\qquad + g(s, v(s))] \, ds \\ &+ \sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{18}$$

It is easy to prove that u is the solution of the IVP (1) if and only if u is the fixed point of A. From (H_1), (H_4), and (H_5), we know that

$$\begin{aligned} A(u, v)(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, u(s), v(s)) \\ &\qquad + g(s, v(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, u(s), v(s)) \\ &\qquad + g(s, v(s))] \, ds \\ &+ \sum_{0<t_k<t} I_k(u(t_k^-)), \\ &\geq y_0 - mM^* > 0, \end{aligned} \tag{19}$$

for any $u, v \in P_C$, $t \in [0, T]$. So, $A : P_C \times P_C \to P_C$. In the sequel we check that A satisfies all assumptions of Lemma 4.

Firstly, we prove that A is a mixed monotone operator. In fact, for $u_i, v_i \in P_C$, $i = 1, 2$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, and $t \in [0, T]$, and by (H_2) and Definition 3, we have that

$$\begin{aligned} A(u_1, v_1)(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, u_1(s), v_1(s)) \\ &\qquad + g(s, v_1(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, u_1(s), v_1(s)) \\ &\qquad + g(s, v_1(s))] \, ds \\ &+ \sum_{0<t_k<t} I_k(u_1(t_k^-)) \\ \geq y_0 &+ \frac{1}{\Gamma(\alpha)} \\ &\times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [f(s, u_2(s), v_2(s)) \\ &\qquad + g(s, v_2(s))] \, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [f(s, u_2(s), v_2(s)) \\ &\qquad + g(s, v_2(s))] \, ds \end{aligned}$$

$$+\sum_{0<t_k<t} I_k\left(u_2\left(t_k^-\right)\right)$$
$$= A\left(u_2, v_2\right)(t). \tag{20}$$

That is, $A(u_1, v_1) \geq A(u_2, v_2)$.

Next, we show that A satisfies the condition (a) of Lemma 4. Now, we set $v(t) \equiv R(t \in [0, T])$. Then, $v > \theta$.

On the one hand, from (H_1) and (H_5), we have

$$\begin{aligned} A(v,\theta)(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f(s,R,0) + g(s,0)\right] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[f(s,R,0) + g(s,0)\right] ds \\ &\quad + \sum_{0<t_k<t} I_k(R), \\ &\geq y_0 - mM^* > 0. \end{aligned} \tag{21}$$

That is to say, $A(v, \theta) > \theta$. From (15), for any $t \in [0, T]$, we obtain

$$\begin{aligned} A(v,\theta)(t) &\leq y_0 + \frac{m(M_1+M_2)}{\Gamma(\alpha+1)} T^{\alpha} + \frac{(M_1+M_2)}{\Gamma(\alpha+1)} T^{\alpha} + mM^* \\ &\leq R = v(t), \end{aligned} \tag{22}$$

That is, $A(v, \theta) \leq v$.

On the other hand, we take $c \in (0, 1)$ such that

$$c \leq \min\left\{\frac{m_1+m_2}{M_1+M_2}, \frac{m_3}{M^*}\right\}, \tag{23}$$

where $m_1 = \min_{t\in[0,T]} f(t, 0, R)$, $m_2 = \min_{t\in[0,T]} g(t, R)$, and $m_3 = \min_{k\in\{1,\ldots,m\}} I_k(0)$. For any $t \in [0, T]$,

$$\begin{aligned} A(\theta,v)(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f(s,0,R) + g(s,R)\right] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[f(s,0,R) + g(s,R)\right] ds \\ &\quad + \sum_{0<t_k<t} I_k(0) \\ &\geq y_0 + \frac{1}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} (m_1+m_2)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} (m_1+m_2)\, ds \\ &\quad + \sum_{0<t_k<t} m_3 \\ &\geq y_0 + \frac{1}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} c(M_1+M_2)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} c(M_1+M_2)\, ds \\ &\quad + \sum_{0<t_k<t} cM^* \\ &\geq cy_0 + \frac{c}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f(s,R,0) + g(s,0)\right] ds \\ &\quad + \frac{c}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[f(s,R,0) + g(s,0)\right] ds \\ &\quad + c\sum_{0<t_k<t} I_k(R) \\ &= cA(v,\theta)(t). \end{aligned} \tag{24}$$

That is to say, $A(\theta, v) \geq cA(v, \theta)$. Therefore, condition (a) of Lemma 4 holds.

Finally, we show that A satisfies condition (b) of Lemma 4. Let $0 < a < b < 1$ and

$$\eta = \min\left\{\frac{1}{b} - 1, \frac{1}{1-a^{\beta_1}} - 1, \frac{1}{1-a^{\beta_2}} - 1, \frac{1}{1-a^{\beta_3}} - 1\right\} > 0. \tag{25}$$

For any $a \leq \lambda \leq b$, $x, y \in P_C$, and $t \in [0, T]$, by (H_3) we have

$$\begin{aligned} A\left(\lambda x, \lambda^{-1} y\right)(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \times \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f\left(s, \lambda x(s), \lambda^{-1} y(s)\right) + g\left(s, \lambda^{-1} y(s)\right)\right] ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[f\left(s,\lambda x(s),\lambda^{-1}y(s)\right)\right.\\
&\qquad\left.+g\left(s,\lambda^{-1}y(s)\right)\right]ds\\
&+\sum_{0<t_k<t}I_k\left(\lambda x\left(t_k^-\right)\right)\\
&\geq y_0+\frac{1}{\Gamma(\alpha)}\\
&\times\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left(t_k-s\right)^{\alpha-1}\left[\frac{\lambda}{1-\lambda^{\beta_1}}f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\frac{\lambda}{1-\lambda^{\beta_2}}g\left(s,y(s)\right)\right]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[\frac{\lambda}{1-\lambda^{\beta_1}}f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\frac{\lambda}{1-\lambda^{\beta_2}}g\left(s,y(s)\right)\right]ds\\
&+\sum_{0<t_k<t}\frac{\lambda}{1-\lambda^{\beta_3}}I_k\left(x\left(t_k^-\right)\right)\\
&\geq y_0+\frac{1}{\Gamma(\alpha)}\\
&\times\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left(t_k-s\right)^{\alpha-1}\left[\frac{\lambda}{1-a^{\beta_1}}f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\frac{\lambda}{1-a^{\beta_2}}g\left(s,y(s)\right)\right]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[\frac{\lambda}{1-a^{\beta_1}}f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\frac{\lambda}{1-a^{\beta_2}}g\left(s,y(s)\right)\right]ds\\
&+\sum_{0<t_k<t}\frac{\lambda}{1-a^{\beta_3}}I_k\left(x\left(t_k^-\right)\right)\\
&\geq b(1+\eta)y_0+\frac{1}{\Gamma(\alpha)}\\
&\times\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left(t_k-s\right)^{\alpha-1}\left[\lambda(1+\eta)f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\lambda(1+\eta)g\left(s,y(s)\right)\right]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[\lambda(1+\eta)f\left(s,x(s),y(s)\right)\right.\\
&\qquad\left.+\lambda(1+\eta)g\left(s,y(s)\right)\right]ds\\
&+\sum_{0<t_k<t}\lambda(1+\eta)I_k\left(x\left(t_k^-\right)\right)\\
&\geq\lambda(1+\eta)A(x,y)(t).
\end{aligned}
\tag{26}
$$

That is to say, $A(\lambda x,\lambda^{-1}y)\geq\lambda(1+\eta)A(x,y)$, for all $a\leq\lambda\leq b$, $x,y\in P_C$. Hence, condition (b) of Lemma 4 is satisfied. So, the conclusion of Theorem 7 follows from Lemma 4. □

4. An Example

We present one example to illustrate Theorem 7.

Example 1. Consider the following fractional IVP:

$$
\begin{aligned}
{}^cD^{1/5}y(t)&=\tau(t)+\delta(t)+\frac{y(t)}{1+y(t)}\\
&\quad+\frac{2}{1+y(t)}+2,\quad t\in[0,1],\ t\neq\frac{1}{3},\\
\Delta y\big|_{t=(1/3)}&=\frac{y\left(1/3^-\right)}{4+y\left(1/3^-\right)}+1,\qquad y(0)=2,
\end{aligned}
\tag{27}
$$

where $\tau,\delta\in C([0,1,0,\infty))$.

In this example,

$$
\begin{gathered}
\alpha=\frac{1}{5},\qquad T=1,\qquad t_1=\frac{1}{3},\\
m=1,\qquad y_0=2,\\
f(t,x,y)=\tau(t)+1+\frac{x}{1+x}+\frac{1}{1+y},\\
g(t,y)=\delta(t)+1+\frac{1}{1+y},\\
I_1(x)=1+\frac{x}{4+x}.
\end{gathered}
\tag{28}
$$

Then, $f(t,x,y)\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$ is continuous and increasing in $x\in[0,\infty)$ for fixed $t\in[0,1]$ and $y\in[0,\infty)$ and decreasing in $y\in[0,\infty)$ for fixed $t\in[0,1]$ and $x\in[0,\infty)$. And $g(t,y)\in C([0,1]\times[0,\infty),[0,\infty))$ is continuous and decreasing in $y\in[0,\infty)$ for fixed $t\in[0,1]$. Moreover, for any $0<p<q<1$, we take

$$
\begin{gathered}
\beta_1\geq\log_q\left(1-\frac{3q}{1+2q}\right),\qquad\beta_2\geq\log_q\left(1-\frac{2q}{1+q}\right),\\
\beta_3\geq\log_q\left(1-\frac{2q}{1+q}\right).
\end{gathered}
\tag{29}
$$

For any $p\leq\lambda\leq q$, $t\in[0,1]$, and $x,y\in[0,\infty)$, we have

$$
\begin{aligned}
f\left(t,\lambda x,\lambda^{-1}y\right)&=\tau(t)+1+\frac{\lambda x}{1+\lambda x}+\frac{\lambda}{\lambda+y}\\
&\geq\frac{\lambda}{q}(\tau(t)+1)+\lambda\left(\frac{x}{1+x}+\frac{1}{1+y}\right).
\end{aligned}
\tag{30}
$$

We show that

$$
f\left(t,\lambda x,\lambda^{-1}y\right)\geq\frac{\lambda}{1-\lambda^{\beta_1}}f(t,x,y),
\tag{31}
$$

if and only if we can prove that

$$\frac{1}{q}(\tau(t)+1)+\frac{x}{1+x}+\frac{1}{1+y} \geq \frac{1}{1-\lambda^{\beta_1}}\left(\tau(t)+1+\frac{x}{1+x}+\frac{1}{1+y}\right). \tag{32}$$

That is to say,

$$(\tau(t)+1)\left(\frac{1}{q}-\frac{1}{1-\lambda^{\beta_1}}\right) \geq \left(\frac{1}{1-\lambda^{\beta_1}}-1\right)\left(\frac{x}{1+x}+\frac{1}{1+y}\right). \tag{33}$$

From (29), we get

$$\lambda^{\beta_1} \leq q^{\beta_1} \leq 1-\frac{3q}{1+2q}, \tag{34}$$

and then

$$\frac{1}{q}-\frac{1}{1-\lambda^{\beta_1}} \geq 2\left(\frac{1}{1-\lambda^{\beta_1}}-1\right). \tag{35}$$

Hence, (33) is satisfied. So, (31) holds.

Similarly, we obtain that

$$\begin{aligned} g\left(t,\lambda^{-1}y\right) &= \delta(t)+1+\frac{\lambda}{\lambda+y} \\ &\geq \frac{\lambda}{q}(\delta(t)+1)+\frac{\lambda}{1+y} \\ &\geq \frac{\lambda}{1-\lambda^{\beta_2}}\left(\delta(t)+1+\frac{1}{1+y}\right) \\ &= \frac{\lambda}{1-\lambda^{\beta_2}}g(t,y), \\ I_1(\lambda x) &= 1+\frac{\lambda x}{4+\lambda x} \geq \frac{\lambda}{q}+\frac{\lambda x}{4+x} \\ &\geq \frac{\lambda}{1-\lambda^{\beta_3}}\left(1+\frac{x}{4+x}\right) = \frac{\lambda}{1-\lambda^{\beta_3}}I_1(x). \end{aligned} \tag{36}$$

Further, $I_1(x)$ is continuous, increasing in $x \in [0,\infty)$, and

$$\left|I_1(x)\right| = \left|\frac{x}{4+x}\right| \leq 1 \equiv M^*, \quad \forall x \in [0,\infty). \tag{37}$$

And, we also have $y_0 = 2 > 1 = mM^* > 0$.

Finally, owing to $\tau, \delta \in C([0,1],[0,\infty))$, we have that

$$\tau^* = \max_{t\in[0,1]}\tau(t), \qquad \delta^* = \max_{t\in[0,1]}\delta(t). \tag{38}$$

We take $R > 0$ such that

$$3+\frac{2\left(5+\tau^*+\delta^*\right)}{\Gamma((1/5)+1)} \leq R. \tag{39}$$

And thus,

$$\begin{aligned} M_1 &= \max_{t\in[0,1]} f(t,R,0) \\ &= \max_{t\in[0,1]}\left(\tau(t)+1+\frac{R}{1+R}+1\right) \leq 3+\tau^*, \\ M_2 &= \max_{t\in[0,1]} g(t,0) = \max_{t\in[0,1]}(\delta(t)+1+1) = \delta^*+2. \end{aligned} \tag{40}$$

Then,

$$\begin{aligned} y_0 &+ \frac{(m+1)(M_1+M_2)}{\Gamma(\alpha+1)}T^\alpha + mM^* \\ &\leq 2+\frac{2}{\Gamma((1/5)+1)}\left(3+\tau^*+2+\delta^*\right)+1 \\ &= 3+\frac{2\left(5+\tau^*+\delta^*\right)}{\Gamma((1/5)+1)} \leq R. \end{aligned} \tag{41}$$

Hence, (15) holds. Therefore, all the conditions of Theorem 7 are satisfied. An application of Theorem 7 implies that problem (27) has a unique positive solution.

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