Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 869621, 6 pages
http://dx.doi.org/10.1155/2013/869621

Research Article

Exponential Attractor for Lattice System of Nonlinear Boussinesq Equation

Min Zhao and Shengfan Zhou

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Correspondence should be addressed to Shengfan Zhou; zhoushengfan@yahoo.com

Received 14 July 2013; Accepted 13 August 2013

Academic Editor: Zhan Zhou

We study the lattice dynamical system of a nonlinear Boussinesq equation. We first verify the Lipschitz continuity of the continuous semigroup associated with the system. Then, we provide an estimation of the tail of the difference between two solutions of the system. Finally, we obtain the existence of an exponential attractor of the system.

1. Introduction

Lattice dynamical systems (LDSs) have a wide range of applications in many areas such as electrical engineering, chemical reaction theory, laser systems, material science, and biology [1, 2]. In recent years, many works about the asymptotic behavior of LDSs have been done, which include the global attractor, see [3–11] and the references therein. However, the global attractor sometimes attracts orbits at a relatively slow speed and it might take an unexpected long time to be reached. For this reason, the exponential attractor having finite fractal dimension and attracting all bounded sets exponentially was introduced, and it has been studied for a large class of LDSs, see [12–15] and the references therein. Han presented in [13] some sufficient conditions for the existence of exponential attractor for LDSs in the weighted space of infinite sequences and applied the result to obtain the existence of exponential attractors for some LDSs. Zhou and Han in [15] presented some sufficient conditions for the existence of uniform exponential attractor for LDSs, which is easier to verify the existence of exponential attractor for some LDSs. Abdallah in [3] considered the following initial problem of lattice system of nonlinear Boussinesq equation:

$$\ddot{u}_i + \delta\dot{u}_i + \alpha(Au)_i + \beta(Bu)_i + \lambda u_i - \frac{1}{3}k\left(D\left(D^*u\right)^3\right)_i = f_i, \quad i \in \mathbb{Z}, \tag{1}$$

$$u_i(0) = u_{i,0}, \quad \dot{u}_i(0) = u_{1i,0}, \quad i \in \mathbb{Z}, \tag{2}$$

where δ, α, λ, and k are positive constants, β is a real constant; for $i \in \mathbb{Z}$, $u_i, f_i \in \mathbb{R}$; $u = (u_i)_{i\in\mathbb{Z}}$ and A, B, D, and D^* are linear operators (see Section 3 for details). Equation (1) can be regarded as a spatial discretization of the following nonlinear damped Boussinesq equation on $\mathbb{R}$:

$$u_{tt} + \delta u_t + \alpha u_{xxxx} + \beta u_{xx} + \lambda u - k u_x^2 u_{xx} = f(x), \tag{3}$$

which appears in many fields of physics and mechanics, for example, long waves in shallow water, nonlinear elastic beam systems, thermomechanical phase transitions, and some Hamiltonian mechanics. Abdallah has in [3] investigated the existence and finite-dimensional approximation of the global attractor for (1) under the following conditions:

$$f = (f_i)_{i\in\mathbb{Z}} \in l^2, \quad \lambda > 4|\beta|. \tag{4}$$

In this paper, motivated by the ideas of [13, 15], we will further prove the existence of an exponential attractor for the system (1) under the condition (4).

The paper is organized as follows. In Section 2, we present some preliminaries. Section 3 is devoted to the existence of an exponential attractor for (1).

2. Preliminaries

In this section, we present the definition of an exponential attractor and some sufficient conditions for the existence of an exponential attractor for a semigroup in a separable Hilbert space from [13, 15].

Let E_s be a separable Hilbert space, let $\mathcal{O}_s$ be a bounded subset of E_s, and let $\{S(t)\}_{t\geq 0}$ be a semigroup acting on $\mathcal{O}_s$ which satisfy: $S(t)S(s) = S(t+s)$, $S(0) = I_{E_s}$, for all $t, s \geq 0$, and $S(t)\mathcal{O}_s \subseteq \mathcal{O}_s$ for $t \geq 0$, where I_{E_s} is the identity operator on E_s.

Definition 1. A set $\mathcal{M}_s$ is called an exponential attractor for the semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{O}_s$, if

(i) $\mathcal{M}_s$ is compact;

(ii) $\mathcal{B}_s \subseteq \mathcal{M}_s \subseteq \mathcal{O}_s$, where $\mathcal{B}_s$ is the global attractor;

(iii) $S(t)\mathcal{M}_s \subseteq \mathcal{M}_s$, $t > 0$;

(iv) $\mathcal{M}_s$ has a finite fractal dimension;

(v) there exist two positive constants a_1 and a_2 such that $\operatorname{dist}(S(t)u, \mathcal{M}_s) \leq a_1 e^{-a_2 t}$ for all $u \in \mathcal{O}_s$, $t \geq 0$.

Let E_s^N be a N-dimensional subspace of E_s, $N \in \mathbb{N}$. We define the bounded N-dimensional orthogonal projection $P_N : E_s \to E_s^N$ from E_s into E_s^N and $Q_N = I_{E_s} - P_N$.

As a direct consequence of [13, Theorem 2.5] and [15, Theorem 2.1], we have the following theorem.

Theorem 2. *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on E_s and let $\mathcal{O}_s$ be a closed bounded subset of E_s such that $S(t)\mathcal{O}_s \subseteq \mathcal{O}_s$, for $t \geq 0$. If there exist $t^* > 0$, a constant $L = L(t^*) > 0$ and a N-dimensional subspace E_s^N of E_s ($N = N(t^*) \in \mathbb{N}$) such that for any $u_1, u_2 \in \mathcal{O}_s$,*

$$\begin{gathered} \|S(t)u_1 - S(t)u_2\|_{E_s} \leq L\|u_1 - u_2\|_{E_s}, \quad \forall t \in [0, t^*], \\ \|Q_N(S(t)u_1 - S(t)u_2)\|_{E_s}^2 \leq \frac{1}{128}\|u_1 - u_2\|_{E_s}^2, \end{gathered} \tag{5}$$

Then,

(i) *$S(t^*) = S^*$ has an exponential attractor $\mathcal{M}_s^*$ on $\mathcal{O}_s$ with $\dim_f(\mathcal{M}_s^*) \leq K_0 N \ln\sqrt{L^2+1}$, where K_0 is a constant;*

(ii) *$\mathcal{M}_s = \bigcup_{0\leq t\leq t^*} S(t)\mathcal{M}_s^*$ is an exponential attractor for $\{S(t)\}_{t\geq 0}$ on $\mathcal{O}_s$ that $\dim_f(\mathcal{M}_s) \leq \dim_f(\mathcal{M}_s^*) + 1$, and there exist two positive constants a_1 and a_2 such that $\operatorname{dist}(S(t)u, \mathcal{M}_s) \leq a_1 e^{-a_2 t}$ for all $u \in \mathcal{O}_s$, $t \geq 0$.*

3. Exponential Attractor for System (1)

Let $l^2 = \{u = (u_i)_{i\in\mathbb{Z}} : u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}} u_i^2 < +\infty\}$ and equip it with the inner product and norm as

$$\begin{gathered} (u, v) = \sum_{i\in\mathbb{Z}} u_i v_i, \qquad \|u\|^2 = (u, u), \\ u = (u_i)_{i\in\mathbb{Z}}, \qquad v = (v_i)_{i\in\mathbb{Z}} \in l^2. \end{gathered} \tag{6}$$

Then, $(l^2, \|\cdot\|, (\cdot,\cdot))$ is a separable Hilbert space. The linear operators A, B, D, and D^* are defined from l^2 into l^2 as follows: for any $u = (u_i)_{i\in\mathbb{Z}} \in l^2$,

$$\begin{gathered} (Au)_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}, \\ (Bu)_i = u_{i+1} - 2u_i + u_{i-1}, \\ (Du)_i = u_{i+1} - u_i, \\ (D^*u)_i = u_i - u_{i-1}, \quad \forall i \in \mathbb{Z}, \end{gathered} \tag{7}$$

then, $A = B^2$, $B = D^*D = DD^*$.

The system (1) with initial data (2) is equivalent to the following vector form:

$$\begin{gathered} \ddot{u} + \delta\dot{u} + \alpha(Au) + \beta(Bu) + \lambda u \\ - \frac{1}{3}kD(D^*u)^3 = f, \quad \forall t > 0, \end{gathered} \tag{8}$$

$$u(0) = (u_{i,0})_{i\in\mathbb{Z}}, \qquad \dot{u}(0) = (u_{1i,0})_{i\in\mathbb{Z}},$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $Au = ((Au)_i)_{i\in\mathbb{Z}}$, $Bu = ((Bu)_i)_{i\in\mathbb{Z}}$, $D(D^*u)^3 = ((D(D^*u)^3)_i)_{i\in\mathbb{Z}}$, $f = (f_i)_{i\in\mathbb{Z}}$.

Letting

$$v = \dot{u} + \varepsilon u, \qquad \varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where } \varepsilon > 0, \tag{9}$$

then, the system (8) can be written as the following initial value problem:

$$\begin{gathered} \dot{\varphi} + C(\varphi) = F(\varphi), \\ \varphi(0) = (u(0), v(0))^T = (u(0), \dot{u}(0) + \varepsilon u(0))^T, \end{gathered} \tag{10}$$

where

$$\begin{gathered} C(\varphi) = \begin{pmatrix} \varepsilon u - v \\ \alpha Au + \lambda u + (\delta - \varepsilon)(v - \varepsilon u) \end{pmatrix}, \\ F(\varphi) = \begin{pmatrix} 0 \\ -\beta Bu + \frac{1}{3}kD(D^*u)^3 + f \end{pmatrix}. \end{gathered} \tag{11}$$

We define

$$\begin{gathered} (u, v)_\lambda = (Bu, Bv) + \lambda(u, v), \\ \|u\|_\lambda = \left(\|Bu\|^2 + \lambda\|u\|^2\right)^{1/2}, \quad \forall u = (u_i)_{i\in\mathbb{Z}}, \\ v = (v_i)_{i\in\mathbb{Z}} \in l^2. \end{gathered} \tag{12}$$

Then, the bilinear form $(\cdot,\cdot)_\lambda$ is an inner product on l^2 and the induced norm $\|\cdot\|_\lambda$ is equivalent to $\|\cdot\|$. Let $l_\lambda^2 = (l^2, (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ and let $H = l_\lambda^2 \times l^2$, then, H is a separable Hilbert space with the following norm:

$$\begin{gathered} \|\varphi\|_H = \left(\sum_{i\in\mathbb{Z}} \left((Bu)_i^2 + \lambda u_i^2 + v_i^2\right)\right)^{1/2}, \\ \forall \varphi = (u_i, v_i)_{i\in\mathbb{Z}} \in H. \end{gathered} \tag{13}$$

In this section, we will study the existence of an exponential attractor of (10) in the space H.

Lemma 3 (see [3]). *Assume* (4) *holds. Then, there exist small* $\varepsilon > 0$ *and* $M_1 > 0$, *such that*

$$\varepsilon^2 + 3\varepsilon \le 2\delta, \qquad \delta \ge 4\varepsilon, \qquad \varepsilon(1+\delta) + 4|\beta| \le \lambda,$$
$$\lambda\left(1 - \frac{M_1}{4\lambda} - 2M_1\right) \ge 4|\beta|. \tag{14}$$

Moreover,

(1) *for any initial data* $\varphi(0) = (u(0), v(0))^T \in H$, *there exists a unique solution* $\varphi(t) = (u(t), v(t))^T$ *of* (10), *such that* $\varphi(\cdot) \in \mathscr{C}([0,\infty), H) \cap \mathscr{C}^1((0,\infty), H)$, *and the solution map*

$$S_\varepsilon(t) : \varphi(0) = (u(0), v(0))^T \in H \longmapsto \varphi(t) = (u(t), v(t))^T \in H, \quad t > 0, \tag{15}$$

generates a continuous semigroup $\{S_\varepsilon(t)\}_{t\ge0}$ *on* H.

(2) *The semigroup* $\{S_\varepsilon(t)\}_{t\ge0}$ *possesses a closed bounded absorbing ball* $\mathcal{O} = B(0, r_0) = \{\varphi \in H : \|\varphi\|_H \le r_0\} \subset H$, *where* $r_0 = 2\|f\|/\sqrt{M_3}$, $M_3 = M_1M_2$, $M_2 = \min\{1/8, \delta/8\varepsilon, \alpha/2, M_1\}$. *Therefore, there exists a constant* $T_0 = T_0(\mathcal{O})$ *such that* $S_\varepsilon(t)\mathcal{O} \subseteq \mathcal{O}$, *for* $t \ge T_0$.

(3) *For any* $\eta > 0$, *there exist* $K(\eta) \in \mathbb{N}$ *and* $T(\eta) \ge 0$ *such that the solution* $\varphi(t) = ((u_i(t), v_i(t)))_{i\in\mathbb{Z}}$ *of* (10) *with* $\varphi(0) \in \mathcal{O}$ *satisfies*

$$\sum_{|i|\ge K(\eta)} \|\varphi_i(t)\|_H^2 = \sum_{|i|\ge K(\eta)} \left((Bu(t))_i^2 + \lambda(u_i(t))^2 + (v_i(t))^2\right) \le \eta, \quad \forall t \ge T(\eta). \tag{16}$$

(4) *The semigroup* $\{S_\varepsilon(t)\}_{t\ge0}$ *of* (10) *possesses a global attractor* $\mathscr{B} \subset \mathcal{O} \subset H$.

In the following, we first verify the Lipschitz continuity of $\{S_\varepsilon(t)\}_{t\ge0}$ and provide an estimation of the tail of the difference between two solutions of (10). Then, we obtain the existence of an exponential attractor of (10) by Theorem 2.

For $j = 1, 2$, $\varphi^{(j0)} \in \mathcal{O}$, $t \ge 0$, let $\varphi^{(j)}(t) = S_\varepsilon(t)\varphi^{(j0)} = (u^{(j)}(t), v^{(j)}(t))$ be the solutions of (10). Set $\Phi(t) = \varphi^{(1)}(t) - \varphi^{(2)}(t) = S_\varepsilon(t)\varphi^{(10)} - S_\varepsilon(t)\varphi^{(20)} = (\omega(t), \zeta(t)) = ((\omega_i(t))_{i\in\mathbb{Z}}, (\zeta_i(t))_{i\in\mathbb{Z}})$, we have by (10) that

$$\dot{\Phi} + C(\Phi) = F\left(\varphi^{(1)}\right) - F\left(\varphi^{(2)}\right), \qquad \Phi(0) = \varphi^{(10)} - \varphi^{(20)}. \tag{17}$$

Lemma 4. *Assume that* (4) *and* (14) *hold. Let*

$$M_4 = \max\left\{\frac{2+4\varepsilon^2}{\lambda - 4|\beta| + \varepsilon\delta - 2\varepsilon^2}, 8\right\},$$
$$M_5 = \min\left\{\frac{1}{8}, \frac{\alpha}{2}, \frac{\lambda - 4|\beta|}{2\lambda}, \frac{\delta - 2\varepsilon}{4\varepsilon}\right\},$$
$$M_6 = \max\left\{\frac{\alpha+1}{2}, \frac{2\lambda + 4|\beta| + \varepsilon\delta - \varepsilon^2}{2\lambda}\right\},$$
$$L(t) = \frac{M_6}{M_5}e^{|-\varepsilon + 8kr_0^2M_4|t}, \tag{18}$$

Then,

(1)

$$\left\|S_\varepsilon(t)\varphi^{(10)} - S_\varepsilon(t)\varphi^{(20)}\right\|_H^2 \le L(t)\left\|\varphi^{(10)} - \varphi^{(20)}\right\|_H^2, \quad \forall t \ge 0. \tag{19}$$

(2) *There exist* $T^* > 0$ *and* $M^* \in \mathbb{N}$, *such that*

$$\sum_{|i|>M^*} \left\|\left(S_\varepsilon(T^*)\varphi^{(10)} - S_\varepsilon(T^*)\varphi^{(20)}\right)_i\right\|_H^2 \le \frac{1}{128}\left\|\varphi^{(10)} - \varphi^{(20)}\right\|_H^2, \tag{20}$$

where

$$\|\Phi_i(t)\|_H^2 = \|(\omega_i(t), \zeta_i(t))\|_H^2 = (B\omega(t))_i^2 + \lambda(\omega_i(t))^2 + (\zeta_i(t))^2, \quad \forall i \in \mathbb{Z}. \tag{21}$$

Proof. (1) Taking the inner product $(\cdot,\cdot)_H$ of (17) with $\Phi(t)$, we obtain

$$\left(\ddot{\omega} + \delta\dot{\omega} + \alpha(A\omega) + \beta(B\omega) + \lambda\omega - \left(\frac{1}{3}kD\left(D^*u^{(1)}\right)^3 - \frac{1}{3}kD\left(D^*u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right) = 0. \tag{22}$$

We can write (22) into the following form:

$$\frac{d}{dt}P(t) + N(t) = 0, \tag{23}$$

where

$$P(t) = \frac{1}{2}\|\dot{\omega}\|^2 + \frac{\alpha}{2}\|B\omega\|^2 - \frac{\beta}{2}\|D\omega\|^2 + \frac{\lambda}{2}\|\omega\|^2 + \varepsilon(\dot{\omega}, \omega) + \frac{\varepsilon\delta}{2}\|\omega\|^2,$$
$$N(t) = (\delta - \varepsilon)\|\dot{\omega}\|^2 + \varepsilon\alpha\|B\omega\|^2 - \varepsilon\beta\|D\omega\|^2 + \varepsilon\lambda\|\omega\|^2 - \frac{1}{3}k\left(D\left(\left(D^*u^{(1)}\right)^3 - \left(D^*u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right). \tag{24}$$

Then,

$$\begin{aligned}
&\varepsilon P(t) - N(t) \\
&= \frac{3\varepsilon - 2\delta}{2}\|\dot{\omega}\|^2 - \frac{\varepsilon\alpha}{2}\|B\omega\|^2 + \frac{\varepsilon\beta}{2}\|D\omega\|^2 \\
&\quad + \frac{\varepsilon(\varepsilon\delta - \lambda)}{2}\|\omega\|^2 + \varepsilon^2(\dot{\omega}, \omega) \\
&\quad + \frac{1}{3}k\left(D\left(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right) \\
&\leq \frac{\varepsilon^2 + 3\varepsilon - 2\delta}{2}\|\dot{\omega}\|^2 \\
&\quad + \frac{\varepsilon(\varepsilon(1+\delta) + 4|\beta| - \lambda)}{2}\|\omega\|^2 \\
&\quad + \frac{1}{3}k\left(D\left(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right) \\
&\leq \frac{1}{3}k\left(D\left(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right).
\end{aligned} \tag{25}$$

Since

$$\begin{aligned}
P(t) &\geq \frac{1}{4}\|\dot{\omega}\|^2 + \frac{\alpha}{2}\|B\omega\|^2 \\
&\quad + \left(\frac{\lambda - 4|\beta| + \varepsilon\delta - 2\varepsilon^2}{2}\right)\|\omega\|^2,
\end{aligned} \tag{26}$$

thus,

$$\begin{aligned}
&\frac{1}{3}k\left(D\left(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3\right), \dot{\omega} + \varepsilon\omega\right) \\
&= \frac{1}{3}k\left(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3, D^*(\dot{\omega} + \varepsilon\omega)\right) \\
&\leq 8kr_0^2\left(2\|\dot{\omega}\|^2 + \left(1 + 2\varepsilon^2\right)\|\omega\|^2\right) \\
&\leq 8kr_0^2 M_4 P(t).
\end{aligned} \tag{27}$$

From (23), (25), and (27), it follows that for $t > 0$,

$$\frac{d}{dt}P(t) \leq \left(-\varepsilon + 8kr_0^2 M_4\right)P(t). \tag{28}$$

Applying Gronwall's inequality to (28), we obtain

$$P(t) \leq e^{(-\varepsilon + 8kr_0^2 M_4)t}P(0). \tag{29}$$

Since

$$\begin{aligned}
P(t) &\geq \frac{1}{4}\|\dot{\omega}\|^2 + \frac{\alpha}{2}\|B\omega\|^2 + \frac{\lambda - 4\beta}{2}\|\omega\|^2 + \frac{\varepsilon\delta - 2\varepsilon^2}{2}\|\omega\|^2 \\
&\geq M_5\|\Phi(t)\|_H^2,
\end{aligned} \tag{30}$$

$$\begin{aligned}
P(t) &\leq \frac{1}{2}\|\dot{\omega}\|^2 + \frac{\alpha}{2}\|B\omega\|^2 + 2|\beta|\|\omega\|^2 + \frac{\lambda}{2}\|\omega\|^2 \\
&\quad + \varepsilon(\dot{\omega}, \omega) + \frac{\varepsilon\delta}{2}\|\omega\|^2 \leq M_6\|\Phi(t)\|_H^2.
\end{aligned} \tag{31}$$

From (29) to (31), it follows that for $t > 0$,

$$\|\Phi(t)\|_H^2 \leq \frac{M_6}{M_5}e^{(-\varepsilon + 8kr_0^2 M_4)t}\|\Phi(0)\|_H^2. \tag{32}$$

(2) Choosing a smooth increasing function $\xi \in \mathscr{C}^1(\mathbb{R}_+, \mathbb{R})$ satisfies

$$\begin{aligned}
&\xi(s) = 0, \quad 0 \leqslant s < 1, \\
&0 \leqslant \xi(s) \leqslant 1, \quad 1 \leqslant s < 2, \\
&\xi(s) = 1, \quad s \geqslant 2, \\
&\left|\xi'(s)\right| \leq C_0, \quad s \in \mathbb{R}_+,
\end{aligned} \tag{33}$$

where C_0 is a positive constant. For $t \geq 0$, let $\Psi_i = \xi(|i|/M)\Phi_i$, $y = (\xi(|i|/M)\omega_i(t))_{i\in\mathbb{Z}}$, $z = (\xi(|i|/M)\zeta_i(t))_{i\in\mathbb{Z}}$, where $M \in \mathbb{N}$. Taking the inner product $(\cdot,\cdot)_H$ of (17) with $\Psi = \{\Psi_i(t)\}_{i\in\mathbb{Z}}$, we obtain

$$\begin{aligned}
&\Big(\ddot{\omega} + \delta\dot{\omega} + \alpha(A\omega) + \beta(B\omega) + \lambda\omega \\
&\quad - \left(\frac{1}{3}kD\left(D^* u^{(1)}\right)^3 - \frac{1}{3}kD\left(D^* u^{(2)}\right)^3\right), z\Big) = 0.
\end{aligned} \tag{34}$$

Similar to (4.3)–(4.5) in [3], we can get

$$\frac{d}{dt}P_1(t) + N_1(t) = 0, \tag{35}$$

where

$$\begin{aligned}
P_1(t) &= \sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\left(\frac{1}{2}\dot{\omega}_i^2 + \frac{\alpha}{2}(B\omega)_i^2 - \frac{\beta}{2}(D\omega)_i^2\right. \\
&\qquad\left. + \frac{\lambda}{2}\omega_i^2 + \varepsilon\dot{\omega}_i\omega_i + \frac{\varepsilon\delta}{2}\omega_i^2\right), \\
N_1(t) &= \sum_{i\in\mathbb{Z}}\Bigg(\xi\left(\frac{|i|}{M}\right)\Big((\delta - \varepsilon)\dot{\omega}_i^2 + \varepsilon\alpha(B\omega)_i^2 \\
&\qquad - \varepsilon\beta(D\omega)_i^2 + \varepsilon\lambda\omega_i^2\Big)\Bigg) \\
&\quad + \frac{1}{3}k\Bigg(\left(D^* u^{(1)}\right)^3 - \left(D^* u^{(2)}\right)^3, \\
&\qquad \left(\xi\left(\frac{|i|}{M}\right)(D^*\zeta)_i\right)_{i\in\mathbb{Z}}\Bigg) \\
&\quad + \sum_{i\in\mathbb{Z}}\Bigg(\alpha(B\omega)_i\left((Bz)_i - \xi\left(\frac{|i|}{M}\right)(B\zeta)_i\right) \\
&\qquad + \beta(D\omega)_i\left((Dz)_i - \xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right) \\
&\qquad + \frac{1}{3}k\left(\left(D^* u^{(1)}\right)_i^3 - \left(D^* u^{(2)}\right)_i^3\right) \\
&\qquad \times \left((Dz)_i - \xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right)\Bigg).
\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}\varepsilon P_1(t)-N_1(t) &\le -\frac{1}{3}k\left(\left(D^*u^{(1)}\right)^3-\left(D^*u^{(2)}\right)^3,\left(\xi\left(\frac{|i|}{M}\right)(D^*\zeta)_i\right)_{i\in\mathbb{Z}}\right)\\ &\quad-\sum_{i\in\mathbb{Z}}\left(\alpha(B\omega)_i\left((Bz)_i-\xi\left(\frac{|i|}{M}\right)(B\zeta)_i\right)\right.\\ &\qquad+\beta(D\omega)_i\left((Dz)_i-\xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right)\\ &\qquad+\frac{1}{3}k\left(\left(D^*u^{(1)}\right)_i^3-\left(D^*u^{(2)}\right)_i^3\right)\\ &\qquad\left.\times\left((Dz)_i-\xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right)\right).\end{aligned} \tag{37}$$

By (3) of Lemma 3, there exist $K_1=K(\lambda\varepsilon/16kM_4)$, $T_1=T(\lambda\varepsilon/16kM_4)$, such that

$$\begin{aligned}&\sum_{|i|\ge K_1}\left(\left(Bu^{(j)}(t)\right)_i^2+\lambda\left(u_i^{(j)}(t)\right)^2+\left(v_i^{(j)}(t)\right)^2\right)\\ &\quad\le\frac{\lambda\varepsilon}{16kM_4},\qquad j=1,2,\ \forall t\ge T_1.\end{aligned} \tag{38}$$

This implies that

$$\left(u_i^{(1)}(t)\right)^2+\left(u_i^{(2)}(t)\right)^2\le\frac{\varepsilon}{8kM_4},\qquad\forall|i|>K_1,\ t\ge T_1. \tag{39}$$

Then, for $M>K_1+1$, $t\ge T_1$,

$$\begin{aligned}&-\frac{1}{3}k\left(\left(D^*u^{(1)}\right)^3-\left(D^*u^{(2)}\right)^3,\left(\xi\left(\frac{|i|}{M}\right)(D^*\zeta)_i\right)_{i\in\mathbb{Z}}\right)\\ &\quad\le\frac{1}{3}k\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\left|(D^*\omega)_i(D^*\zeta)_i\right|\\ &\qquad\times\left(\left(D^*u^{(1)}\right)_i^2+\left(D^*u^{(1)}\right)_i\left(D^*u^{(2)}\right)_i+\left(D^*u^{(2)}\right)_i^2\right)\\ &\quad\le k\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\left|(D^*\omega)_i(D^*\zeta)_i\right|\\ &\qquad\times\left(\left(u^{(1)}\right)_i^2+\left(u^{(2)}\right)_i^2+\left(u^{(1)}\right)_{i-1}^2+\left(u^{(2)}\right)_{i-1}^2\right)\\ &\quad\le\frac{\varepsilon}{2}P_1(t)+\frac{C_0\varepsilon M_6}{4M}\|\Phi\|_H^2.\end{aligned} \tag{40}$$

Since

$$\begin{aligned}&-\sum_{i\in\mathbb{Z}}\left(\alpha(B\omega)_i\left((Bz)_i-\xi\left(\frac{|i|}{M}\right)(B\zeta)_i\right)\right.\\ &\qquad+\beta(D\omega)_i\left((Dz)_i-\xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right)\\ &\qquad+\frac{1}{3}k\left(\left(D^*u^{(1)}\right)_i^3-\left(D^*u^{(2)}\right)_i^3\right)\\ &\qquad\left.\times\left((Dz)_i-\xi\left(\frac{|i|}{M}\right)(D\zeta)_i\right)\right)\\ &\quad\le\frac{C_0}{M}\left(\alpha\|B\omega\|^2+\left(2|\beta|+384kr_0^4\right)\|\omega\|^2\right.\\ &\qquad\left.+\left(\frac{6\alpha+3|\beta|+k}{6}\right)\|\zeta\|^2\right)\\ &\quad\le\frac{C_0M_7}{M}\|\Phi\|_H^2,\end{aligned} \tag{41}$$

where $M_7=\max\{\alpha,(2|\beta|+384kr_0^4)/\lambda,(6\alpha+3|\beta|+k)/6\}$. From (32), (35), (37), and (40)-(41), it follows that for $M>K_1+1$, $t\ge T_1$,

$$\frac{d}{dt}P_1(t)\le-\frac{\varepsilon}{2}P_1(t)+\frac{C_0M_6M_8}{M_5M}e^{(-\varepsilon+8kr_0^2M_4)t}\|\Phi(0)\|_H^2, \tag{42}$$

where $M_8=\varepsilon M_6/4+M_7$. Applying Gronwall's inequality to (42) from T_2 to t, where $T_2=\max\{T_0,T_1\}$, we obtain that for $M>K_1+1$,

$$\begin{aligned}P_1(t)&\le e^{-(\varepsilon/2)(t-T_2)}P_1(T_2)\\ &\quad+\frac{2C_0M_6M_8}{\left|-\varepsilon+16kr_0^2M_4\right|M_5M}e^{(-\varepsilon+8kr_0^2M_4)t}\\ &\quad\times\|\Phi(0)\|_H^2.\end{aligned} \tag{43}$$

Similar to (30), we can get

$$M_5\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\|\Phi_i(t)\|_H^2\le P_1(t). \tag{44}$$

Since

$$P_1(T_2)\le P(T_2). \tag{45}$$

By (31)-(32), we obtain

$$P(T_2)\le\frac{M_6^2}{M_5}e^{(-\varepsilon+8kr_0^2M_4)T_2}\|\Phi(0)\|_H^2. \tag{46}$$

From (43) to (46), it follows that for $t \geq T_2$, $M > K_1 + 1$,

$$\sum_{i\in\mathbb{Z}} \xi\left(\frac{|i|}{M}\right)\left\|\Phi_i(t)\right\|_H^2 \leq \left(\frac{M_6^2}{M_5^2} e^{-(\varepsilon t+\varepsilon T_2-16kr_0^2 M_4 T_2)/2} + \frac{2C_0 M_6 M_8}{\left|-\varepsilon + 16kr_0^2 M_4\right| M_5^2 M} e^{(-\varepsilon+8kr_0^2 M_4)t}\right)\|\Phi(0)\|_H^2. \tag{47}$$

Letting

$$T^* = \max\left\{\frac{1}{\varepsilon}\left(2\left(\ln\left(256M_6^2\right) - \ln M_5^2\right) + 16kr_0^2 M_4 T_2 - \varepsilon T_2\right), T_2\right\};$$
$$M^* = \max\left\{\frac{1024 C_0 M_6 M_8}{\left|-\varepsilon + 16kr_0^2 M_4\right| M_5^2} e^{(-\varepsilon+8kr_0^2 M_4)T^*}, 2K_1 + 3\right\}, \tag{48}$$

we then have

$$\frac{M_6^2}{M_5^2} e^{-(\varepsilon T^*+\varepsilon T_2-16kr_0^2 M_4 T_2)/2} + \frac{2C_0 M_6 M_8}{\left|-\varepsilon + 16kr_0^2 M_4\right| M_5^2 M^*} e^{(-\varepsilon+8kr_0^2 M_4)T^*} \leq \frac{1}{128},$$
$$\sum_{|i|>M^*}\left\|\Phi_i(T^*)\right\|^2 \leq \frac{1}{128}\|\Phi(0)\|^2. \tag{49}$$

□

As a direct consequence of (1)-(2), (4) of Lemma 3, (1)-(2) of Lemma 4 and Theorem 2, we have our main result.

Theorem 5. *Assume that* (4) *and* (14) *hold. Then, the semigroup* $\{S_\varepsilon(t)\}_{t\geq 0}$ *of* (10) *possesses an exponential attractor* $\mathscr{M}$ *on* $\mathscr{A} = \overline{\bigcup_{t\geq T_0} S_\varepsilon(t)\mathscr{O}}$ *with* (*i*) $\mathscr{M}$ *is compact;* (*ii*) $\mathscr{B} \subset \mathscr{M} \subset \mathscr{O}$, *where* $\mathscr{B}$ *is the global attractor;* (*iii*) $\mathscr{M}$ *has a finite fractal dimension* $\dim_f(\mathscr{M}) \leq 2K_0(2M^* + 1)\ln\sqrt{L(T^*) + 1} + 1$, *where* K_0 *is a constant and* T^* *and* M^* *are as in* (48)*; and* (*iv*) *there exist two positive constants* k_1 *and* k_2 *such that* $\operatorname{dist}(S_\varepsilon(t)u, \mathscr{M}) \leq k_1 e^{-k_2 t}$ *for all* $u \in \mathscr{O}$, $t \geq 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 11071165 and Zhejiang Normal University (ZC304011068).

References

[1] H. Chate and M. Courbage, "Lattice systems," *Physica D*, vol. 103, no. 1–4, pp. 1–612, 1997.

[2] S. N. Chow, *Lattice Dynamical Systems, in Dynamical System*, vol. 1822 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2003.

[3] A. Y. Abdallah, "Global attractor for the lattice dynamical system of a nonlinear Boussinesq equation," *Abstract and Applied Analysis*, vol. 2005, no. 6, pp. 655–671, 2005.

[4] A. Y. Abdallah, "Long-time behavior for second order lattice dynamical systems," *Acta Applicandae Mathematicae*, vol. 106, no. 1, pp. 47–59, 2009.

[5] P. W. Bates, K. Lu, and B. Wang, "Attractors for lattice dynamical systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 11, no. 1, pp. 143–153, 2001.

[6] H. Li and S. Zhou, "Structure of the global attractor for a second order strongly damped lattice system," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 1426–1446, 2007.

[7] J. C. Oliveira, J. M. Pereira, and G. Perla Menzala, "Attractors for second order periodic lattices with nonlinear damping," *Journal of Difference Equations and Applications*, vol. 14, no. 9, pp. 899–921, 2008.

[8] B. Wang, "Dynamics of systems on infinite lattices," *Journal of Differential Equations*, vol. 221, no. 1, pp. 224–245, 2006.

[9] C. Zhao and S. Zhou, "Upper semicontinuity of attractors for lattice systems under singular perturbations," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 72, no. 5, pp. 2149–2158, 2010.

[10] S. Zhou, "Attractors for second order lattice dynamical systems," *Journal of Differential Equations*, vol. 179, no. 2, pp. 605–624, 2002.

[11] S. Zhou, "Attractors and approximations for lattice dynamical systems," *Journal of Differential Equations*, vol. 200, no. 2, pp. 342–368, 2004.

[12] X. Fan and H. Yang, "Exponential attractor and its fractal dimension for a second order lattice dynamical system," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 2, pp. 350–359, 2010.

[13] X. Han, "Exponential attractors for lattice dynamical systems in weighted spaces," *Discrete and Continuous Dynamical Systems*, vol. 31, no. 2, pp. 445–467, 2011.

[14] A. Y. Abdallah, "Exponential attractors for second order lattice dynamical systems," *Communications on Pure and Applied Analysis*, vol. 8, no. 3, pp. 803–813, 2009.

[15] S. Zhou and X. Han, "Uniform exponential attractors for non-autonomous KGS and Zakharov lattice systems with quasiperiodic external forces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 78, pp. 141–155, 2013.