Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2014, Article ID 568129, 6 pages
http://dx.doi.org/10.1155/2014/568129

Research Article

Analytic Continuation of Euler Polynomials and the Euler Zeta Function

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to C. S. Ryoo; ryoocs@hnu.kr

Received 10 January 2014; Accepted 10 March 2014; Published 3 April 2014

Academic Editor: Binggen Zhang

We study that the Euler numbers E_n and Euler polynomials $E_n(z)$ are analytically continued to $E(s)$ and $E(s,w)$. We investigate the new concept of dynamics of the zeros of analytica continued polynomials. Finally, we observe an interesting phenomenon of "scattering" of the zeros of $E(s,w)$.

1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the ring of integers, the field of real numbers, and the complex numbers, respectively. Recently, many mathematicians have studied different kinds of the Euler, Bernoulli, and Genocchi numbers and polynomials (see [1–19]). The computing environment would make more and more rapid progress and there has been increasing interest in solving mathematical problems with the aid of computers. By using software, mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Numerical experiments of Bernoulli polynomials, Euler polynomials, and Genocchi polynomials have been the subject of extensive study in recent years and much progress has been made both mathematically and computationally. Using computer, a realistic study for Euler polynomials $E_n(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the Euler polynomials $E_n(x)$ in complex plane. First, we introduce the Euler numbers and Euler polynomials. As a well-known definition, the Euler numbers E_n are defined by

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1}$$

Here is the list of the first Euler numbers:

$$E_0 = 1, \quad E_1 = -\frac{1}{2}, \quad E_2 = 0, \quad E_3 = \frac{1}{4},$$
$$E_4 = 0, \quad E_5 = -\frac{1}{2}, \quad E_6 = 0, \quad E_7 = \frac{17}{8},$$
$$E_8 = 0, \quad E_9 = -\frac{31}{2}, \quad E_{10} = 0, \quad E_{11} = \frac{691}{4},$$
$$E_{12} = 0, \quad E_{13} = -\frac{5461}{2}, \quad E_{14} = 0,$$
$$E_{15} = \frac{929569}{16}, \quad E_{16} = 0,$$
$$E_{17} = -\frac{3202291}{2}, \quad E_{18} = 0,$$
$$E_{19} = \frac{221930581}{4}, \quad E_{20} = 0,$$
$$E_{21} = -\frac{4722116521}{2}, \quad E_{22} = 0,$$
$$E_{23} = \frac{968383680827}{8},$$
$$E_{24} = 0,$$
$$E_{25} = -\frac{14717667114151}{2},$$

$$E_{26} = 0,$$
$$E_{27} = \frac{2093660879252671}{4},$$
$$E_{28} = 0. \tag{2}$$

The Euler polynomials $E_n(x)$ are defined by the generating function:

$$F(x,t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right) e^{xt}, \tag{3}$$

where we use the technique method notation by replacing $E(x)^n$ by $E_n(x)$ symbolically.

Because

$$\frac{\partial F}{\partial x}(x,t) = tF(x,t) = \sum_{n=0}^{\infty} \frac{dE_n}{dx}(x) \frac{t^n}{n!}, \tag{4}$$

an important relation follows:

$$\frac{dE_k}{dx}(x) = kE_{k-1}(x). \tag{5}$$

Then, it is easy to deduce that $E_k(x)$ are polynomials of degree k. Here is the list of the first Euler polynomials:

$$\begin{gathered}
E_0(x) = 1, \qquad E_1(x) = x - \frac{1}{2}, \\
E_2(x) = x^2 - x, \qquad E_3(x) = x^3 - \frac{3x^2}{2} + \frac{1}{4}, \\
E_4(x) = x^4 - 2x^3 + x, \qquad E_5(x) = x^5 - \frac{5x^2}{2} - \frac{1}{2}, \\
E_6(x) = x^6 - 3x^5 + 5x^3 - 3x, \\
E_7(x) = x^7 - \frac{7x^6}{2} + \frac{35x^4}{4} - \frac{21x^2}{2} + \frac{17}{8}, \\
E_8(x) = x^8 - 4x^7 + 14x^5 - 28x^3 + 17x, \\
E_9(x) = x^9 - \frac{9x^8}{2} + 21x^6 - 63x^4 + \frac{153x^2}{2} - \frac{31}{2}, \\
E_{10}(x) = x^{10} - 5x^9 + 30x^7 - 126x^5 + 255x^3 - 155x.
\end{gathered} \tag{6}$$

2. Generating Euler Polynomials and Numbers

Since

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} &= \frac{2}{e^t+1} e^{xt} = e^{Et} e^{xt} = e^{(E+x)t} \\
&= \sum_{n=0}^{\infty} (E+x)^n \frac{t^n}{n!},
\end{aligned} \tag{7}$$

we have the following theorem.

Theorem 1. *For n, one has*

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}. \tag{8}$$

Definition 2. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, define the Euler zeta function by

$$\zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}; \tag{9}$$

see [7–10].

Notice that the Euler zeta function can be analytically continued to the whole complex plane, and these zeta functions have the values of the Euler numbers at negative integers. That is, Euler numbers are related to the Euler zeta function as

$$\zeta_E(-n) = E_n. \tag{10}$$

Definition 3. We define the Hurwitz zeta function $\zeta_E(s,x)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \in \mathbb{R}$ with $0 \le x < 1$ by

$$\zeta_E(s,x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}; \tag{11}$$

see [3, 7–10, 12].

Euler polynomials are related to the Hurwitz zeta function as

$$\zeta_E(-n,x) = E_n(x). \tag{12}$$

We now consider the function $E(s)$ as the analytic continuation of Euler numbers. From the above analytic continuation of Euler numbers, we consider

$$E_n \longmapsto E(s), \qquad \zeta_E(-n) = E_n \longmapsto \zeta_E(-s) = E(s). \tag{13}$$

All the Euler numbers E_n agree with $E(n)$, the analytic continuation of Euler numbers evaluated at n (see Figure 1),

$$E_n = E(n) \quad \text{for } n \ge 1, \tag{14}$$

$$\text{except } E(0) = -1, \quad \text{but } E_0 = 1. \tag{15}$$

In fact, we can express $E'(s)$ in terms of $\zeta_E'(s)$, the derivative of $\zeta_E(s)$,

$$\begin{gathered}
E(s) = \zeta_E(-s), \qquad E'(s) = -\zeta_E'(-s), \\
E'(2n+1) = -\zeta_E'(-2n-1) \quad \text{for } n \in \mathbb{N} \cup \{0\}.
\end{gathered} \tag{16}$$

From relation (16), we can define the other analytic continued half of Euler numbers as

$$\begin{gathered}
E(s) = \zeta_E(-s), \\
E(-s) = \zeta_E(s) \Longrightarrow E(-n) = \zeta_E(n), \quad n \in \mathbb{N}.
\end{gathered} \tag{17}$$

By (17), we have

$$\lim_{n\to\infty} E_{-n} = \zeta_E(n) = -2. \tag{18}$$

The curve $E(s)$ runs through the points $E_{-n} = E(-n)$ and grows ~ -2 asymptotically as $-n \to \infty$ (see Figure 2).

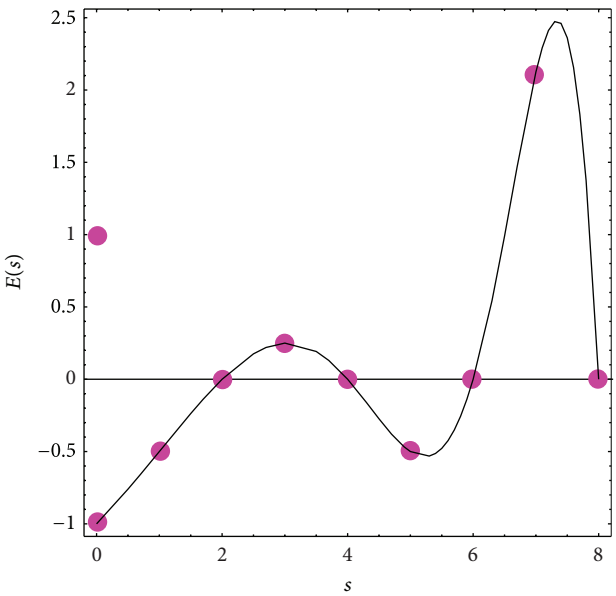

FIGURE 1: The curve $E(s)$ runs through the points of all E_n except E_0.

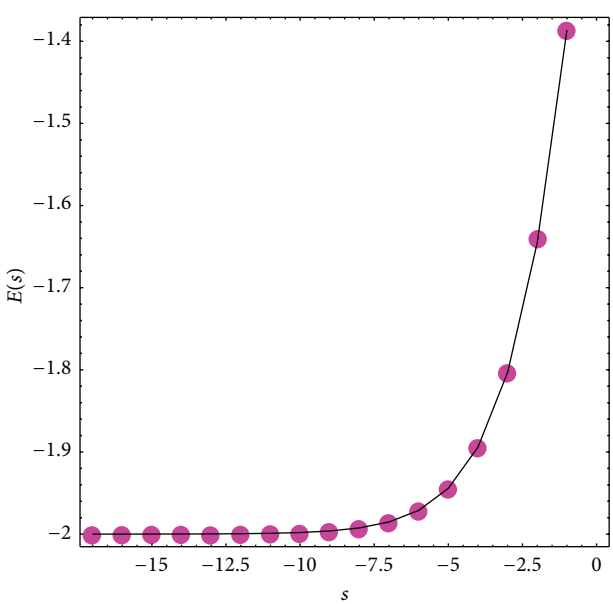

FIGURE 2: The curve $E(s)$ runs through the points E_{-n}.

3. Analytic Continuation of Euler Polynomials

Looking back at (1) and (3), we can see that the sign convention of E_0 was actually arbitrary. Equation (15) suggests that consistent definition of Euler numbers should really have been

$$\sum_{n=0}^{\infty}(-1)^{n+1}E_n\frac{t^n}{n!}=\frac{2}{e^t+1},\quad |t|<\pi,\ n=0,1,2,\ldots\in\mathbb{N}\cup\{0\} \tag{19}$$

which only changes the sign in the conventional definition of the only nonzero even Euler numbers, E_0, from $E_0 = 1$ to $E_0 = E(0) = -1$.

By using Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^{n+1}E_n(x)\frac{t^n}{n!}&=\left(\frac{2}{e^t+1}\right)e^{xt}\\&=\left(\sum_{k=0}^{\infty}(-1)^{k+1}E_n\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}E_kx^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{20}$$

For consistency with the redefinition of $E_n = E(n)$ in (19), Euler polynomials should be analogously redefined as

$$E_n(x)=\sum_{n=0}^{n}(-1)^{n+k}\binom{n}{k}E_kx^{n-k}. \tag{21}$$

The analytic continuation can be then obtained as

$$\begin{aligned}&n\longmapsto s\in\mathbb{R},\qquad x\longmapsto w\in\mathbb{C},\\&E_k\longmapsto E(k+s-[s])=\zeta_E(-(k+(s-[s]))),\\&\binom{n}{k}\longmapsto\frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\&\Longrightarrow E_n(x)\longmapsto E(s,w)\\&=\sum_{k=-1}^{[s]}\frac{(-1)^{k+[s]}\Gamma(1+s)E(k+s-[s])w^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\&=\sum_{k=0}^{[s]+1}\frac{(-1)^{k-1+[s]}\Gamma(1+s)E((k-1)+s-[s])w^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)},\end{aligned} \tag{22}$$

where $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part.

By (22), we obtain analytic continuation of Euler polynomials:

$$E_0(w)=-1,\qquad E_1(w)=E(1,w)=-0.5+w,$$
$$E_2(w)=E(2,w)=w-w^2,$$

$$\begin{aligned}
E(2.2, w) &\approx 0.0799175 + 0.866594w \\
&\quad - 1.19765154w^2 + 0.116134w^3, \\
E(2.4, w) &\approx 0.147412 + 0.693349w \\
&\quad - 1.36116w^2 + 0.279503w^3, \\
E(2.6, w) &\approx 0.199873 + 0.485705w \\
&\quad - 1.47547w^2 + 0.486915w^3, \\
E(2.8, w) &\approx 0.234806 + 0.251271w \\
&\quad - 1.52609w^2 + 0.731015w^3, \\
E_3(w) &= E(3, w) = 0.25 - 1.5w^2 + w^3.
\end{aligned} \tag{23}$$

By using (23), we plot the deformation of the curve $E(2, w)$ into the curve of $E(3, w)$ via the real analytic continuation $E(s, w)$, $2 \le s \le 3$, $w \in \mathbb{R}$ (see Figure 3).

Next, we investigate the beautiful zeros of the $E(s, w)$ by using a computer. We plot the zeros of $E(s, w)$ for $s = 9, 9.6, 9.8, 10$ and $w \in \mathbb{C}$ (Figure 4).

In Figure 4(a), we choose $s = 9$. In Figure 4(b), we choose $s = 9.6$. In Figure 4(c), we choose $s = 9.8$. In Figure 4(d), we choose $s = 10$.

Since

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(1-x) \frac{(-1)^n t^n}{n!} &= F(1-x, -t) \\
&= \frac{2}{e^{-t}+1} e^{(1-x)(-t)} = \frac{2}{e^t+1} e^{xt} \\
&= F(x, t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},
\end{aligned} \tag{24}$$

we obtain

$$E_n(x) = (-1)^n E_n(1-x). \tag{25}$$

Hence, we have the following theorem.

Theorem 4. *If $n \equiv 1 \pmod 2$, then $E_n(1/2) = 0$, for $n \in \mathbb{N}$.*

The question is, what happens with the reflexive symmetry (25) when one considers Euler polynomials? Prove that $E_n(x)$, $x \in \mathbb{C}$, has $\mathrm{Re}(x) = 1/2$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions. However, we observe that $E(s, w)$, $w \in \mathbb{C}$, does not have $\mathrm{Re}(w) = 1/2$ reflection symmetry analytic complex functions (Figure 4).

Our numerical results for approximate solutions of real zeros of $E(s, w)$ are displayed. We observe a remarkably regular structure of the complex roots of Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of Euler polynomials (Table 1). Next, we calculated an approximate solution satisfying $E(s, w)$, $w \in \mathbb{R}$. The results are given in Table 2.

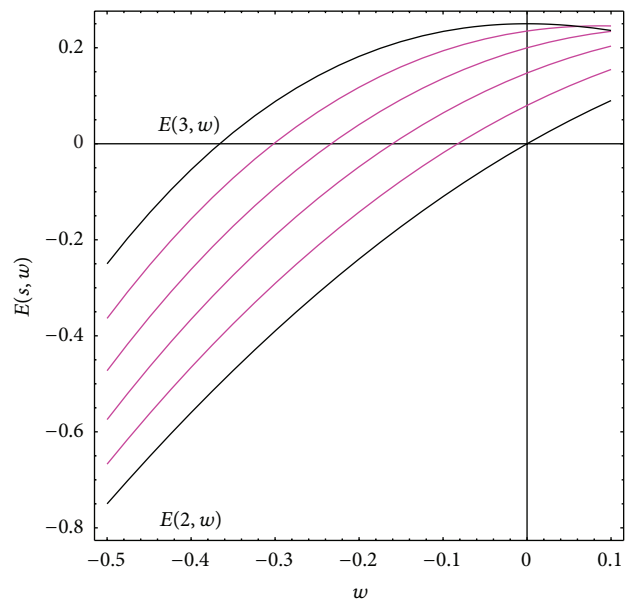

Figure 3: The curve of $E(s, w)$, $2 \le s \le 3$, $-0.5 \le w \le 0.2$.

Table 1: Numbers of real and complex zeros of $E(s, w)$.

s	Real zeros	Complex zeros
1.5	2	0
2.5	3	0
3.5	4	0
4.5	5	0
5.5	2	4
6.5	3	4
7.5	4	4
8.5	5	4
9	5	4
9.6	6	4
9.8	6	4
10	6	4

Table 2: Approximate solutions of $E(s, w) = 0$, $w \in \mathbb{R}$.

s	w
6	0.0000, 1.000
6.5	−0.24986, 0.749547, 6.94675
7	−0.49773, 0.49999, 1.49773
7.5	−0.731622, 0.249968, 1.23584, 7.86641
8	−0.93231, −0.0000242346, 1.00002, 1.93232
8.5	−1.09269, −0.250159, 0.750199, 1.64301, 8.78539
9	−1.21928, −0.501115, 0.500956, 1.49919, 2.22008
9.6	−1.32375, −0.810502, 0.206935, 1.19507, 1.90363, 7.44777
9.8	−1.34314, −0.922069, 0.112954, 1.08884, 1.90735, 4.56809
10	−1.34708, −1.0482, 0.0238982, 0.97898, 2.03661, 2.35464

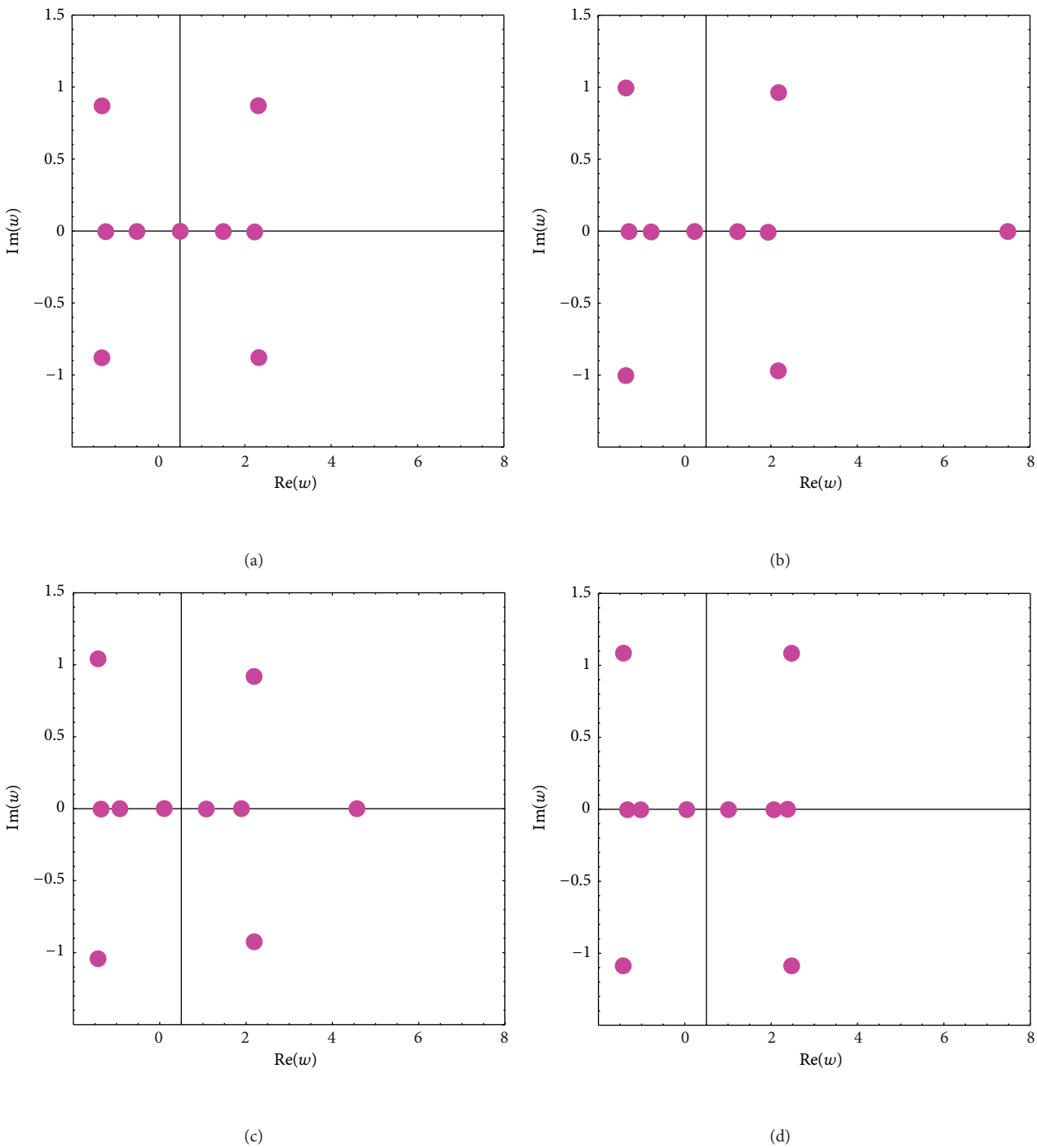

Figure 4: Zeros of $E(s, w)$ for $s = 9, 9.6, 9.8, 10$.

Euler polynomials $E_n(w)$ are polynomials of degree n. Thus, $E_n(w)$ has n zeros and $E_{n+1}(w)$ has $n + 1$ zeros. When discrete n is analytically continued to continuous parameter s, it naturally leads to the following question: how does $E(s, w)$, the analytic continuation of $E_n(w)$, pick up an additional zero as s increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials, the idea of looking at how the zeros move about in the w complex plane as we vary the parameter s. For more studies and results in this subject you may see [11, 14–16].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] S. Araci and M. Acikgoz, "A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 22, pp. 399–406, 2012.

[2] R. Ayoub, "Euler zeta function," *The American Mathematical Monthly*, vol. 81, pp. 1067–1086, 1974.

[3] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.

[4] I. N. Cangul, H. Ozden, and Y. Simsek, "A new approach to q-Genocchi numbers and their interpolation functions," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 12, pp. e793–e799, 2009.

[5] L.-C. Jang, "On multiple generalized ω-Genocchi polynomials and their applications," *Mathematical Problems in Engineering*, vol. 2010, Article ID 316870, 8 pages, 2010.

[6] J. Y. Kang, H. Y. Lee, and N. S. Jung, "Some relations of the twisted q-Genocchi numbers and polynomials with weight α and weak Weight β," *Abstract and Applied Analysis*, vol. 2012, Article ID 860921, 9 pages, 2012.

[7] M. S. Kim and S. Hu, "On p-adic Hurwitz-type Euler Zeta functions," *Journal of Number Theory*, no. 132, pp. 2977–3015, 2012.

[8] T. Kim, "Barnes-type multiple q-zeta functions and q-Euler polynomials," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, Article ID 255201, 11 pages, 2010.

[9] T. Kim, "Note on the Euler q-zeta functions," *Journal of Number Theory*, vol. 129, no. 7, pp. 1798–1804, 2009.

[10] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.

[11] T. Kim, C. S. Ryoo, L. C. Jang, and S. H. Rim, "Exploring the q-riemann zeta function and q-bernoulli polynomials," *Discrete Dynamics in Nature and Society*, vol. 2005, no. 2, pp. 171–181, 2005.

[12] H. Ozden and Y. Simsek, "A new extension of q-Euler numbers and polynomials related to their interpolation functions," *Applied Mathematics Letters*, vol. 21, no. 9, pp. 934–939, 2008.

[13] K. H. Park, S.-H. Rim, and E. J. Moon, "On Genocchi numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 898471, 7 pages, 2008.

[14] C. S. Ryoo, "A numerical computation on the structure of the roots of q-extension of Genocchi polynomials," *Applied Mathematics Letters*, vol. 21, no. 4, pp. 348–354, 2008.

[15] C. S. Ryoo, "Calculating zeros of the second kind Euler polynomials," *Journal of Computational Analysis and Applications*, vol. 12, pp. 828–833, 2010.

[16] C. S. Ryoo, T. Kim, and R. P. Agarwal, "A numerical investigation of the roots of q-polynomials," *International Journal of Computer Mathematics*, vol. 83, no. 2, pp. 223–234, 2006.

[17] Y. Simsek, "Generating functions of the twisted Bernoulli Numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–257, 2008.

[18] Y. Simsek, "Twisted *(h, q)*-Bernoulli numbers and polynomials related to twisted *(h, q)*-zeta function and L-function," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 790–804, 2006.

[19] Y. Simsek, V. Kurt, and D. Kim, "New approach to the complete sum of products of the twisted *(h, q)*-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 44–56, 2007.