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Research Article

Dynamics of Third-Order Nonlinear Neutral Equations

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The aim of this paper is to study oscillatory and asymptotic properties of the third-order nonlinear neutral equation with continuously distributed delays of the form $(r(t)([x(t)+\int_a^b p(t,\mu)\,x[\tau(t,\mu)]\,d\mu]'')^{\alpha})'+\int_c^d q(t,\xi)\,f^{\alpha}(x[g(t,\xi)])\,d\xi=0$. Applying suitable generalized Riccati transformation and integral averaging technique, we present new criteria for oscillation or certain asymptotic behavior of nonoscillatory solutions of this equation. Obtained results essentially improve and complement earlier ones.

1. Introduction

In recent years, the dynamics theory such as oscillation theory and asymptotic behavior of differential equations and their applications have been and still are receiving intensive attention [1–4]. In fact, in the last few years several monographs and hundreds of research papers have been written; see, for example, the monograph [5]. Determining oscillation criteria for particular second-order differential equations has received a great deal of attention in the last few years [6–8]. For example, [9] considered

$$\left(r(t)\left[x(t)+p(t)\,x(t-\tau)\right]'\right)' + \int_a^b q(t,\xi)\,x\left[g(t,\xi)\right]d\sigma(\xi)=0 \tag{1}$$

and obtained oscillatory criteria of Philos type. In [10], by means of Riccati transformation technique, Han et al. established some new oscillation criteria for the second-order Emden Fowler delay dynamic equations on a time scale T:

$$x''(t)+p(t)\,x^{\gamma}(\tau(t))=0. \tag{2}$$

However, compared to second-order differential equations, the study of oscillation and asymptotic behavior of third-order differential equations has received considerably less attention in the literature [11–15]. In [16], Qiu investigated the oscillation criteria for the third-order neutral differential equations taking the following form:

$$\left(r(t)\left[x(t)+p(t)\,x(t)\right]''\right)' + \int_a^b q(t,\xi)\,x\left[g(t,\xi)\right]d\sigma(\xi)=0, \quad t\ge t_0. \tag{3}$$

By using a generalized Riccati transformation and integral averaging technique, Zhang et al. [17] established some new sufficient conditions which ensure that every solution of the following equation oscillates or converges to zero:

$$\left(r(t)\left(\left[x(t)+\int_a^b p(t,\mu)\,x\left[\tau(t,\mu)\right]d\mu\right]''\right)\right)' + \int_c^d q(t,\xi)\,f\left(x\left[g(t,\xi)\right]\right)d\xi=0. \tag{4}$$

As we know, the dynamics theory such as oscillation theory and asymptotic behavior of the following equation have not been investigated up to now:

$$\left(r(t)\left(\left[x(t)+\int_a^b p(t,\mu)\,x\left[\tau(t,\mu)\right]d\mu\right]''\right)^{\alpha}\right)' + \int_c^d q(t,\xi)\,f^{\alpha}\left(x\left[g(t,\xi)\right]\right)d\xi=0. \tag{5}$$

With the help of a generalized Riccati transformation and integral averaging technique, this paper aims to establish some new sufficient conditions of Philos type which ensure that every solution of (5) oscillates or converges to zero. Our results improve and complement the corresponding results in [6, 11–17]. We should point out that, in this paper, α is any quotient of odd positive integers and $\alpha \le 1$; it is more general than that reported in [17] where $\alpha = 1$.

We are interested in (5) in the case of $t \ge t_0$. Throughout this paper, we assume that the following hypotheses hold:

(H_1) $r(t) \in C^1([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty}(1/r(t))dt = \infty$;

(H_2) $P(t,\mu) \in C([t_0,\infty)\times[a,b],R)$, $0 \le p(t) \equiv \int_a^b p(t,\mu)d\mu \le p < 1$;

(H_3) $\tau(t,\mu) \in C([t_0,\infty)\times[a,b],R)$ is not a decreasing function for ξ, and $\tau(t,\mu) \le t$, $\lim_{t\to\infty}\min_{\xi\in[a,b]}\tau(t,\mu) = \infty$;

(H_4) $q(t,\xi) \in C([t_0,\infty)\times[c,d],(0,\infty))$;

(H_5) $g(t,\xi) \in C([t_0,\infty)\times[c,d],R)$ is not a decreasing function for ξ, such that $g(t,\xi) \le t$, $\lim_{t\to\infty}\min_{\xi\in[c,d]}g(t,\xi) = \infty$;

(H_6) $f(x) \in C(R,R)$, $(f(x)/x^{\alpha}) \ge \delta >$, $x \ne 0$.

We also define the following function:

$$z(t) = x(t) + \int_a^b p(t,\mu)\,x[\tau(t,\mu)]\,d\mu. \tag{6}$$

As far as a solution of (5) is concerned, we mean a nontrivial function $x(t) \in C'([T_x,\infty),R)$, $T_x \ge t_0$, which has the property $r(t)z''(t) \in C'([T_x,\infty))$ and satisfies (5) on $[T_x,\infty)$.

We restrict our attention to those solutions of (5) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T > T_x$. A solution of (5) is said to be oscillatory on $[T_x,\infty)$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory.

The rest of this paper is organized as follows. In Section 2, we will present some lemmas which are useful for the proof of our main results. In Section 3, we present new criteria of Philos type for oscillation or certain asymptotic behavior of nonoscillatory solutions of (5).

2. Several Lemmas

Lemma 1. *Let $x(t)$ be a positive solution of (5), and $r'(t) \ge 0$, $z'''(t) < 0$. Then $z(t)$ which is defined as in (6) has only one of the following two properties:*

(I) $z(t) > 0$, $z'(t) >$, $z''(t) > 0$;

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$.

Proof. Letting $x(t)$ be a positive solution of (5) on $[t_0,\infty)$, from (6), we have $z(t) > x(t) > 0$ and $(r(t)(z''(t))^{\alpha})' = -\int_c^d q(t,\xi)f^{\alpha}(x[g(t,\xi)])d\xi < 0$. Then $r(t)(z''(t))^{\alpha}$ is a decreasing function and of one sign, and following $\alpha \in (0,1)$ and $\alpha = p/q$ where p and q are odd positive integers, we have that $(z''(t))^{\alpha}$ and $z''(t)$ have the same sign, so $z''(t)$ is either eventually positive or eventually negative on $t \ge t_1 \ge t_0$; that is, $z''(t) < 0$ or $z''(t) > 0$. If $z''(t) < 0$, then there exists a constant $M > 0$, such that $r(t)z'' \le -M < 0$. By integrating from t_1 to t, we get

$$z'(t) \le z'(t_1) - M\int_{t_1}^{t}\frac{1}{r(s)}ds. \tag{7}$$

Letting $t \to \infty$ and using (H_1), we have $z'(t) \to -\infty$. Thus $z'(t) < 0$ eventually; since $z''(t) < 0$ and $z'(t) < 0$, we have $z(t) < 0$, which contradicts assumption $z(t) > 0$, so $z''(t) > 0$. Therefore, $z(t)$ has only one of the two properties (I) and (II). □

Lemma 2. *Let $x(t)$ be a positive solution of (5), and correspondingly $z(t)$ has property (II). Assume that*

$$\int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty}\left(\int_c^d q(s,\xi)\,d\xi\right)ds\right]^{1/\alpha}du\,dv = \infty. \tag{8}$$

Then

$$\lim_{t\to\infty}x(t) = \lim_{t\to\infty}z(t) = 0. \tag{9}$$

Proof. Let $x(t)$ be a positive solution of (5). Since $z(t)$ has property (II), then there exists finite limit $\lim_{t\to\infty}z(t) = l$. We assert that $l = 0$. Assuming that $l > 0$, then we have $l < z(t) < l+\epsilon$, for all $\epsilon > 0$. Choosing $\epsilon \in (0, l(1-p)/p)$, we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\mu)\,x[\tau(t,\mu)]\,d\mu \\ &> l - \int_a^b p(t,\mu)\,x[\tau(t,\mu)]\,d\mu \\ &\ge l - p(t)\,z[\tau(t,a)] \\ &\ge l - p(l+\epsilon) \\ &= k(l+\epsilon) \\ &> kz(t), \end{aligned} \tag{10}$$

where $k = (l-p(l+\epsilon))/(l+\epsilon) > 0$. Using (H_6) and $x(t) > kz(t)$, from (5), we find that

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \le -k\delta\int_c^d q(t,\xi)\,z^{\alpha}(x[g(t,\xi)])\,d\xi. \tag{11}$$

Note that $z(t)$ has property (II) and (H_5); we have

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \le -k\delta z^{\alpha}(x[g(t,d)])\int_c^d q(t,\xi)\,d\xi \tag{12}$$

$$= -q_1(t)\,z^{\alpha}[g_1(t)], \tag{13}$$

where $q_1(t) = k\delta\int_c^d q(t,\xi)d\xi$, $g_1(t) = g(t,d)$. Integrating inequality (13) from t to ∞, we get

$$\begin{aligned} \left(r(t)\left(z''(t)\right)^{\alpha}\right) &\ge \int_t^{\infty}q_1(s)\,z^{\alpha}[g_1(s)]\,ds, \\ z''(t) &\ge \left[\frac{1}{r(t)}\int_t^{\infty}q_1(s)\,z^{\alpha}[g_1(s)]\,ds\right]^{1/\alpha}. \end{aligned} \tag{14}$$

Using $z[g_1(t)] \geq l$, then we have

$$z''(t) \geq \left[\frac{l^\alpha}{r(t)} \int_t^\infty q_1(s)\, ds\right]^{1/\alpha}. \tag{15}$$

Integrating inequality (15) from t to ∞, we have

$$-z'(t) \geq l \int_t^\infty \left[\frac{1}{r(u)} \left(\int_u^\infty q_1(s)\, ds\right)\right]^{1/\alpha} du. \tag{16}$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geq l \int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \left(\int_u^\infty q_1(s)\, ds\right)\right]^{1/\alpha} du\, dv; \tag{17}$$

we have a contradiction with (8) and so it follows that $\lim_{t\to\infty} x(t) = 0$. □

Lemma 3 (see [18]). *Let $z(t) > 0, z'(t) > 0, z''(t) \leq 0, t > t_0$. Then, for each $\beta \in (0,1)$, there exists $T_\beta \geq t_0$ such that*

$$z(g(t)) \geq \beta \frac{g(t)}{t} \cdot z(t), \quad t \geq T_\beta. \tag{18}$$

Lemma 4 (see [19]). *Letting $z(t) > 0$, $z'(t) > 0$, $z'' \geq 0$, $r' > 0$, $z'''(t) \leq 0$, $t \geq T_\beta$, then there exist $\gamma \in (0,1)$ and $T_\gamma \geq T_\beta$ such that*

$$z(t) \geq \gamma t z'(t), \quad t \geq T_\gamma. \tag{19}$$

Lemma 5. *For all $\alpha > 0$, then for all $A > 0, B > 0$, one has*

$$Bu - Au^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}}{A^\alpha}. \tag{20}$$

Proof. Let $u \geq 0$, $\alpha > 0$. We investigate the maximal value and minimal value of the function $f(u) = Bu - Au^{(\alpha+1)/\alpha}$.

At first, for all $A > 0$, $B > 0$, the derivative of function $f(u) = Bu - Au^{(\alpha+1)/\alpha}$ is $f'(u) = B - A((\alpha+1)/\alpha)u^{1/\alpha}$. It is clear that when $u > (B/A)^\alpha \cdot (\alpha/(\alpha+1))^\alpha$, we have $f'(u) < 0$, and when $u < (B/A)^\alpha \cdot (\alpha/(\alpha+1))^\alpha$, we have $f'(u) > 0$. Hence the function $f(u) = Bu - Au^{(\alpha+1)/\alpha}$ attains its maximum value $(\alpha^\alpha/(\alpha+1)^{\alpha+1}) \cdot (B^{\alpha+1}/A^\alpha)$ at $u = (B/A)^\alpha \cdot (\alpha/(\alpha+1))^\alpha$. This completes the proof. □

3. Main Result

Theorem 6. *Assume that the condition of Lemma 2 holds, and there exists $\rho \in C^1([t_0,\infty),(0,\infty))$, such that $\rho' > 0$ and*

$$\lim_{t\to\infty} \int_T^t \left[Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}(s)}{A^\alpha(s)}\right] ds = \infty, \tag{21}$$

where

$$\begin{gathered} Q(s) = [\beta\gamma\delta(1-p)]^\alpha \cdot \frac{\rho(t)}{t^\alpha} g^{2\alpha}(t,c) \int_c^d q(t,\xi)\, d\xi, \\ B(s) = \frac{\alpha}{[\rho(s) r(s)]^{1/\alpha}}, \\ A(s) = \frac{\rho'(s)}{\rho(s)}. \end{gathered} \tag{22}$$

Then every solution $x(t)$ of (5) either is oscillatory or converges to zero.

Proof. Assume that (5) has a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t) > 0, t \geq t_1$, $x[\tau(t,\mu)] > 0$, $(t,\mu) \in [t_1,\infty) \times [a,b]$; $x[g(t,\xi)] > 0$, $(t,\xi) \in [t_1,\infty) \times [c,d]$, and $z(t)$ is defined as in (6). By Lemma 1, we have that $z(t)$ has property (I) or property (II). At first, when $z(t)$ has property (I), we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\mu)\, x[\tau(t,\mu)]\, d\mu \\ &\geq z(t) - \int_a^b p(t,\mu)\, z[\tau(t,\mu)]\, d\mu \\ &\geq z(t) - z[\tau(t,b)] \int_a^b p(t,\mu)\, d\mu \\ &\geq \left(1 - \int_a^b p(t,\mu)\, d\mu\right) z(t) \\ &\geq (1-p)\, z(t). \end{aligned} \tag{23}$$

Using (H_5) and (H_6), we get

$$\begin{aligned} &\left(r(t)\left(z''(t)\right)^\alpha\right)' \\ &\quad \leq -\delta^\alpha (1-p)^\alpha \int_c^d q(t,\xi)\, z^\alpha[g(t,\xi)]\, d\xi \\ &\quad \leq -\delta^\alpha (1-p)^\alpha z^\alpha[g(t,c)] \int_c^d q(t,\xi)\, d\xi \\ &\quad \equiv -q_2(t)\, z^\alpha[g_2(t)], \end{aligned} \tag{24}$$

where

$$q_2(t) = \delta^\alpha (1-p)^\alpha \int_c^d q(t,\xi)\, d\xi, \qquad g_2(t) = g(t,c). \tag{25}$$

Let

$$w(t) = \rho(t)\, r(t) \left(\frac{z''(t)}{z'(t)}\right)^\alpha, \quad t \geq t_1. \tag{26}$$

Then

$$\frac{w(t)\left(z'(t)\right)^\alpha}{\rho(t)} = r(t)\left(z''(t)\right)^\alpha, \tag{27}$$

so

$$\begin{gathered} \left(\frac{w(t)\left(z'(t)\right)^\alpha}{\rho(t)}\right)' = \left(r(t)\left(z''(t)\right)^\alpha\right)' \leq -q_2(t)\, z^\alpha[g_2(t)], \\ w'(t) \leq -q_2(t)\, \rho(t) \left[\frac{z[g_2(t)]}{z'}\right]^\alpha \\ \quad + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha}{[\rho(t)\, r(t)]^{1/\alpha}} w^{(\alpha+1)/\alpha}(t). \end{gathered} \tag{28}$$

Letting $u(t) = z'(t)$, from Lemma 3, we obtain

$$\frac{1}{z'(t)} \geq \frac{\beta g_2(t)}{t z'[g_2(t)]}, \quad t \geq T_\beta \geq t_1. \tag{29}$$

Using Lemma 4, we get

$$z[g_2(t)] \geq \gamma g_2(t) z'[g_2(t)], \quad t \geq T_\gamma \geq T_\beta. \tag{30}$$

Hence

$$w'(t) \leq -Q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha}{[\rho(t) r(t)]^{1/\alpha}} w^{(\alpha+1)/\alpha}(t), \quad t \geq T_\gamma, \tag{31}$$

where $Q(t)$ is defined as (21). Letting $A(t) = \rho'(t)/\rho(t)$, $B(t) = \alpha/(\rho(t)r(t))^{1/\alpha}$, we have that

$$w'(t) \leq -Q(t) + A(t) w(t) - B(t) w^{(\alpha+1)/\alpha}(t), \tag{32}$$

and, from Lemma 5, we obtain

$$w'(t) \leq -Q(t) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}(t)}{A^\alpha(t)}. \tag{33}$$

Integrating inequality (33) from T to t,

$$\int_T^t w'(s)\, ds \leq -\int_T^t \left(Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}(s)}{A^\alpha(s)} \right) ds, \tag{34}$$

we obtain

$$0 < w(t) \leq w(T) - \int_T^t \left(Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}(s)}{A^\alpha(s)} \right) ds, \tag{35}$$

which contradicts (21). If $z(t)$ has property (II), since (8) holds, then the conditions in Lemma 2 are satisfied. Hence $\lim_{t\to\infty} x(t) = 0$.

This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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