

## Research Article

# On Third Order Stable Difference Scheme for Hyperbolic Multipoint Nonlocal Boundary Value Problem

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This paper presents a third order of accuracy stable difference scheme for the approximate solution of multipoint nonlocal boundary value problem of the hyperbolic type in a Hilbert space with self-adjoint positive definite operator. Stability estimates for solution of the difference scheme are obtained. Some results of numerical experiments that support theoretical statements are presented.

## 1. Introduction

Hyperbolic partial differential equations (PDEs) are of growing interest in several areas of engineering and natural sciences such as acoustic, electromagnetic, hydrodynamic, elasticity, fluid mechanics, and other areas of physics (see, e.g., [1–5] and the references given therein). In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological, or environmental processes often involve nonclassical conditions. Such conditions are usually identified as nonlocal boundary conditions and project situations when the data on the domain boundary cannot be measured directly or when the data on the boundary depend on the data inside the domain. Many researchers study nonlocal boundary value problems (NBVPs) for hyperbolic and mixed types of partial differential equations. There are many results (see [6–10]) on integral inequalities with two dependent limits to theory of integral-differential equations of the hyperbolic type and of difference schemes for the approximate solution of these problems. Stability has been actively studied in the development of numerical methods for the approximate solutions of PDEs (see [5–31]). In particular, an appropriate model for the analysis of stability is provided by a suitable unconditionally stable difference scheme with an unbounded operator.

It is known (see [29, 30]) that various multipoint NBVPs for hyperbolic equations can be reduced to the problem

$$\begin{aligned} \frac{d^2 u(t)}{dt^2} + Au(t) &= f(t) \quad 0 \leq t \leq 1, \\ u(0) &= \sum_{r=1}^n \alpha_r u(\lambda_r) + \varphi, \\ u_t(0) &= \sum_{r=1}^n \beta_r u_t(\lambda_r) + \psi, \\ 0 &< \lambda_1 < \lambda_2 < \dots < \lambda_n \leq 1. \end{aligned} \quad (1)$$

Here  $A$  is a self-adjoint positive definite operator with domain  $D(A)$  in a Hilbert space  $H$ .

A function  $u(t)$  is called a solution of problem (1) if the following conditions are satisfied:

- (i)  $u(t)$  is twice continuously differentiable on the segment  $[0, 1]$ . The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.
- (ii) The element  $u(t)$  belongs to  $D(A)$  for all  $t \in [0, 1]$  and the function  $Au(t)$  is continuous on the segment  $[0, 1]$ .
- (iii)  $u(t)$  satisfies the equations and the initial conditions (1).

In the last decades, many scientists worked and published scientific papers in the field of finite difference method for the numerical solutions of hyperbolic PDEs. In this field, first order and two types of second order stable difference schemes for the solution of abstract hyperbolic problem (1) were presented in [17] and high order of accuracy difference schemes for the solution of the same problem were presented in [18]. However, the difference schemes presented in these references are generated by square roots of  $A$ . This action is not good for realization. Therefore, in spite of theoretical results the role of their application to a numerical solution of (1) is not great.

In the present paper, a third order of accuracy unconditionally stable difference scheme generated by the integer power of  $A$  for the approximate solution of the multipoint NBVP (1) is presented. Stability estimates for the solution of this difference scheme are established.

Some results of this paper, without proof, are published in [16].

Note that many scientists have studied the solutions of boundary value problems such as parabolic equations, elliptic equations, and equations of mixed types extensively (see, e.g., [6–10, 21–31] and the references therein).

## 2. Third-Order Accurate Stable Difference Scheme

In the present section a third order of accuracy stable difference scheme for the approximate solution of multipoint NBVP (1) is presented. We associate problem (1) with the corresponding third order of accuracy difference scheme:

$$\tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3} Au_k + \frac{1}{6} A (u_{k+1} + u_{k-1})$$

$$+ \frac{1}{12} \tau^2 A^2 u_{k+1} = f_k,$$

$$f_k = \frac{2}{3} f(t_k) + \frac{1}{6} (f(t_{k+1}) + f(t_{k-1})) - \frac{1}{12}$$

$$\cdot \tau^2 (-Af(t_{k+1}) + f''(t_{k+1})),$$

$$t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1,$$

$$(I - i\tau A^2) u_0 = \sum_{k=1}^n \alpha_k \left\{ u_{[\lambda_k/\tau]} \right.$$

$$+ \tau^{-1} (u_{[\lambda_k/\tau]} - u_{[\lambda_k/\tau]-1}) \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)$$

$$+ \frac{3}{2} (f_{[\lambda_k/\tau]} - Au_{[\lambda_k/\tau]}) \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2$$

$$+ \frac{7}{6} (f'_{[\lambda_k/\tau]} - \tau^{-1} A (u_{[\lambda_k/\tau]} - u_{[\lambda_k/\tau]-1}))$$

$$\cdot \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \} + \varphi,$$

$$(I + i\tau A^2) \tau^{-1} (u_1 - u_0) = \sum_{k=1}^n \beta_k$$

$$\cdot \left\{ \tau^{-1} (u_{[\lambda_k/\tau]} - u_{[\lambda_k/\tau]-1}) + (f_{[\lambda_k/\tau]} - Au_{[\lambda_k/\tau]}) \right.$$

$$\cdot \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)$$

$$+ \frac{1}{2!} (f'_{[\lambda_k/\tau]} - \tau^{-1} A (u_{[\lambda_k/\tau]} - u_{[\lambda_k/\tau]-1}))$$

$$\cdot \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2$$

$$+ \frac{1}{3!} (f''_{[\lambda_k/\tau]} - Af_{[\lambda_k/\tau]} + A^2 u_{[\lambda_k/\tau]})$$

$$\cdot \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \} + \psi + f_{1,1} \tau.$$

(2)

Here

$$f_{1,1} = \left\{ f(0) + (-f(0) + \tau f'(0)) \frac{1}{2} - 2f'(0) \frac{\tau}{6} \right\}. \quad (3)$$

Note that throughout this paper for clarity  $\lambda_1 > 2\tau$  and  $\lambda_n < 1$  will be considered.

We are interested in studying the stability of solution of difference scheme (2) under assumption

$$\begin{aligned} & \sum_{k=1}^n |\alpha_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right. \\ & + \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \} + \sum_{k=1}^n |\beta_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\ & + \frac{1}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 + \frac{1}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \} + \frac{1}{2} \sum_{k=1}^n |\alpha_k| \quad (4) \\ & \cdot \sum_{k=1}^n |\beta_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 + \frac{7}{12} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^4 \right. \\ & \left. + \frac{7}{36} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^6 \right\} < 1. \end{aligned}$$

Throughout this section for simplicity we denote

$$\begin{aligned}
 B_n^\tau &= (I + \tau^2 A^4)^{-1} \left\{ 1 - \frac{1}{2} \sum_{k=1}^n \alpha_k \left( (\tilde{R}_1 R^{[\lambda_k/\tau]-1} - R_1 \tilde{R}^{[\lambda_k/\tau]-1}) \left( -\frac{1}{\tau} \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right) + \frac{7}{6\tau} A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \right) \right. \right. \\
 &\quad + \left. \left. (\tilde{R}_1 R^{[\lambda_k/\tau]} - R_1 \tilde{R}^{[\lambda_k/\tau]}) \left( 1 + \frac{1}{\tau} \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right) - \frac{3}{2} A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2 - \frac{7}{6\tau} A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \right) \right) - \frac{1}{2} \right. \\
 &\quad \cdot R_2 \sum_{k=1}^n \beta_k \left( (\tilde{R}^{[\lambda_k/\tau]-1} - R^{[\lambda_k/\tau]-1}) \left( \frac{1}{\tau} - A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right) - \frac{1}{2\tau} A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2 + \frac{1}{6} A^2 \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \right) \right. \\
 &\quad + \left. \left. (\tilde{R}^{[\lambda_k/\tau]} - R^{[\lambda_k/\tau]}) \left( \frac{1}{\tau} - A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right) - \frac{1}{2\tau} A \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2 + \frac{1}{6} A^2 \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^3 \right) \right) \right) + \frac{1}{4} \\
 &\quad \cdot R_2 \sum_{k=1}^n \sum_{k=1}^n \alpha_k \beta_k \left( -\frac{1}{\tau} + A \tau \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^2 + \frac{7}{12} A^2 \tau^3 \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^4 - \frac{7}{36} A^3 \tau^5 \left( \lambda_k - \left[ \frac{\lambda_k}{\tau} \right] \tau \right)^6 \right) (\tilde{R}_1 - R_1) \\
 &\quad \cdot \left[ \tilde{R}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-1} (R - I) - R^{[\lambda_k/\tau]} \tilde{R}^{[\lambda_k/\tau]-1} (\tilde{R} - I) \right] \left. \right\}.
 \end{aligned} \tag{5}$$

First, let us present some lemmas. We will consider the following operators:

$$\begin{aligned}
 R &= \left( I - \frac{1}{3} \tau^2 A + i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \\
 &\quad \cdot \left( I + \frac{1}{6} \tau^2 A + \frac{1}{12} \tau^4 A^2 \right)^{-1}
 \end{aligned} \tag{6}$$

and their conjugate  $\tilde{R}$

$$\begin{aligned}
 R_1 &= \left( -\frac{5\tau^4}{144} A^2 + \frac{7\tau^6}{216} A^3 \right. \\
 &\quad - i\tau A^{1/2} \left( I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \sqrt{I + \frac{1}{72} \tau^4 A^2} \\
 &\quad \cdot \left( -i\tau A^{1/2} \left( \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \right. \\
 &\quad \left. \left. \cdot \left( I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \right) \right)^{-1},
 \end{aligned} \tag{7}$$

and their conjugate  $\tilde{R}_1$ ,

$$\begin{aligned}
 R_2 &= \left( I - \frac{\tau^2}{12} A \right) \left( I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right) \\
 &\quad \cdot \left( -i\tau A^{1/2} \left( I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \sqrt{I + \frac{1}{72} \tau^4 A^2} \right)^{-1},
 \end{aligned}$$

$$\begin{aligned}
 R_3 &= \left( I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right) \left( \left( I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \right. \\
 &\quad \left. \cdot \left( -i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \right)^{-1}, \\
 R_4 &= \left( I + \frac{\tau^2}{3} A + \frac{\tau^4}{9} A^2 + \frac{\tau^6}{72} A^3 \right) \\
 &\quad \cdot \left( -iA^{1/2} \left( \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \left( I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right) \right. \\
 &\quad \left. \cdot \left( I + \frac{\tau^2}{6} A \right) \right)^{-1}, \\
 R_5 &= \left( -\frac{\tau^2}{2} A - \frac{\tau^4}{12} A^2 + i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \left( I + \frac{\tau^2}{6} \right. \\
 &\quad \left. \cdot A + \frac{\tau^4}{12} A^2 \right)^{-1},
 \end{aligned} \tag{8}$$

and their conjugate  $\tilde{R}_5$ ,

$$\begin{aligned}
 R_6 &= \left( I - \frac{1}{3} \tau^2 A + i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \\
 &\quad \cdot \left( \frac{\tau^2}{2} A + \frac{\tau^4}{12} A^2 - i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right)^{-1},
 \end{aligned} \tag{9}$$

and their conjugate  $\tilde{R}_6$ ,

$$\begin{aligned}
 R_7 &= \left( -2i\tau A^{1/2} \sqrt{I + \frac{1}{72} \tau^4 A^2} \right) \left( I + \frac{\tau^2}{6} A + \frac{\tau^4}{12} A^2 \right)^{-1}.
 \end{aligned} \tag{10}$$

Now we consider the following lemma which was presented before in [14].

**Lemma 1.** *The following estimates hold:*

$$\begin{aligned}
\|R\|_{H \rightarrow H} &\leq 1, \\
\|\tilde{R}\|_{H \rightarrow H} &\leq 1, \\
\|R_1\|_{H \rightarrow H} &\leq 1, \\
\|\tilde{R}_1\|_{H \rightarrow H} &\leq 1, \\
\|A^{1/2}R_2\|_{H \rightarrow H} &\leq 1, \\
\|\tau A^{1/2}R_3\|_{H \rightarrow H} &\leq 1, \\
\|A^{1/2}R_4\|_{H \rightarrow H} &\leq 1, \\
\|A^{-1/2}R_5\|_{H \rightarrow H} &\leq \tau, \\
\|A^{-1/2}\tilde{R}_5\|_{H \rightarrow H} &\leq \tau, \\
\|\tau A^{1/2}R_6\|_{H \rightarrow H} &\leq 1, \\
\|\tau A^{1/2}\tilde{R}_6\|_{H \rightarrow H} &\leq 1, \\
\|A^{-1/2}\tilde{R}_7\|_{H \rightarrow H} &\leq \tau.
\end{aligned} \tag{11}$$

*Proof.* Using the spectral property of self-adjoint positive definite operators, we write

$$\begin{aligned}
&\left\| \left( I - \frac{1}{3}\tau^2 A \pm i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \right. \\
&\quad \cdot \left. \left( I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-1} \right\|_{H \rightarrow H} \\
&\leq \sup_{\delta \leq \lambda < \infty} \left| \frac{1 - (1/3)\tau^2 \lambda + i\tau \lambda^{1/2} \sqrt{1 + (1/72)\tau^4 \lambda^2}}{1 + (1/6)\tau^2 \lambda + (1/12)\tau^4 \lambda^2} \right|.
\end{aligned} \tag{12}$$

Since

$$\begin{aligned}
&\frac{(1 - (1/3)\tau^2 \lambda)^2 + (\tau \lambda^{1/2} \sqrt{1 + (1/72)\tau^4 \lambda^2})^2}{(1 + (1/6)\tau^2 \lambda + (1/12)\tau^4 \lambda^2)^2} \\
&= \frac{1 + (1/3)\tau^2 \lambda + (1/9)\tau^4 \lambda^2 + (1/72)\tau^6 \lambda^3}{1 + (1/3)\tau^2 \lambda + (7/36)\tau^4 \lambda^2 + (1/36)\tau^6 \lambda^3 + (1/144)\tau^8 \lambda^4} \\
&\leq 1,
\end{aligned} \tag{13}$$

we have that

$$\begin{aligned}
\|R\|_{H \rightarrow H} &\leq 1, \\
\|\tilde{R}\|_{H \rightarrow H} &\leq 1.
\end{aligned} \tag{14}$$

In exactly the same manner, one can easily obtain all the other estimates of Lemma 1.  $\square$

**Lemma 2.** *Suppose that assumption (4) holds. Then, the operator  $I - B_n^\tau$  has an inverse  $T_\tau = (I - B_n^\tau)^{-1}$ . From the symmetry and positivity properties of the operator  $A$  the following estimate holds:*

$$\|T_\tau\|_{H \rightarrow H} \leq M. \tag{15}$$

*Proof.* Using the definitions of  $B_n^\tau$ ,  $R$ ,  $\tilde{R}$ , estimates in (11), and the triangle inequality, we get

$$\begin{aligned}
B_n^\tau &\leq \frac{1}{2} \sum_{k=1}^n |\alpha_k| \left\{ \left( \|\tilde{R}_1 R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \right. \\
&\quad \cdot \|R_5 (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \\
&\quad \left. \left. + \|R_1 \tilde{R}^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|\tilde{R}_5 (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right) \right. \\
&\quad \cdot \left| \frac{\lambda_k}{\tau} - \left\lfloor \frac{\lambda_k}{\tau} \right\rfloor \right| + \frac{7}{6} \left( \|\tilde{R}_1 R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \\
&\quad \cdot \|R_5 \tau^2 A (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \\
&\quad \left. \left. + \|R_1 \tilde{R}^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|\tilde{R}_5 \tau^2 A (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right) \right. \\
&\quad \cdot \left| \frac{\lambda_k}{\tau} - \left\lfloor \frac{\lambda_k}{\tau} \right\rfloor \right|^3 + \|\tau^2 A (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \\
&\quad \cdot \left( \|\tilde{R}_1 R^{[\lambda_k/\tau]}\|_{H \rightarrow H} + \|R_1 \tilde{R}^{[\lambda_k/\tau]}\|_{H \rightarrow H} \right) \\
&\quad \left. + \frac{3}{2} \|\tau^2 A (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right. \\
&\quad \cdot \left. \left( \|\tilde{R}_1 R^{[\lambda_k/\tau]}\|_{H \rightarrow H} + \|R_1 \tilde{R}^{[\lambda_k/\tau]}\|_{H \rightarrow H} \right) \left| \frac{\lambda_k}{\tau} - \left\lfloor \frac{\lambda_k}{\tau} \right\rfloor \right|^2 \right\} \\
&- \frac{1}{2} \sum_{k=1}^n |\beta_k| \|A^{1/2} R_2\|_{H \rightarrow H} \left\{ \|(I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right. \\
&\quad \cdot \left( \|\tilde{R}^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|A^{-1/2} \tau^{-1} \tilde{R}_5\|_{H \rightarrow H} \right. \\
&\quad \left. \left. + \|R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|A^{-1/2} \tau^{-1} R_5\|_{H \rightarrow H} \right) \right. \\
&\quad \left. + \frac{1}{2} \left( \|\tilde{R}^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|\tilde{R}_5 \tau A^{1/2} (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right. \right. \\
&\quad \left. \left. + \|R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \|R_5 \tau A^{1/2} (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right) \right. \\
&\quad \cdot \left| \frac{\lambda_k}{\tau} - \left\lfloor \frac{\lambda_k}{\tau} \right\rfloor \right|^2 + \|A^{1/2} \tau (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \\
&\quad \cdot \left( \|\tilde{R}^{[\lambda_k/\tau]}\|_{H \rightarrow H} + \|R^{[\lambda_k/\tau]}\|_{H \rightarrow H} \right) \left| \frac{\lambda_k}{\tau} - \left\lfloor \frac{\lambda_k}{\tau} \right\rfloor \right| \\
&\quad \left. + \frac{1}{6} \|\tau^3 A^{3/2} (I + \tau^2 A^4)^{-1}\|_{H \rightarrow H} \right\}
\end{aligned}$$

$$\begin{aligned}
 & \cdot \left( \|\bar{R}^{[\lambda_k/\tau]}\|_{H \rightarrow H} + \|R^{[\lambda_k/\tau]}\|_{H \rightarrow H} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big\} \\
 & - \frac{1}{4} \sum_{k=1}^n \sum_{k=1}^n |\alpha_k \beta_k| \|A^{1/2} R_2\|_{H \rightarrow H} \\
 & \cdot \left\{ \left( \|\bar{R}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \right. \\
 & \cdot \left\| (\bar{R}_1 - R_1) R_5 \tau^{-1} A^{-1/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \\
 & + \left\| R^{[\lambda_k/\tau]} \bar{R}^{[\lambda_k/\tau]-1} \right\|_{H \rightarrow H} \\
 & \cdot \left\| (\bar{R}_1 - R_1) \bar{R}_5 \tau^{-1} A^{-1/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \Big) \\
 & \cdot \left( \|\bar{R}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \\
 & \cdot \left\| (\bar{R}_1 - R_1) R_5 \tau^{-1} A^{1/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \\
 & + \left\| R^{[\lambda_k/\tau]} \bar{R}^{[\lambda_k/\tau]-1} \right\|_{H \rightarrow H} \\
 & \cdot \left\| (\bar{R}_1 - R_1) \bar{R}_5 \tau A^{1/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \Big) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 + \frac{7}{12} \\
 & \cdot \left( \|\bar{R}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \\
 & \cdot \left\| (\bar{R}_1 - R_1) R_5 A^{3/2} \tau^3 (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \\
 & + \left\| R^{[\lambda_k/\tau]} \bar{R}^{[\lambda_k/\tau]-1} \right\|_{H \rightarrow H} \\
 & \cdot \left\| (\bar{R}_1 - R_1) \bar{R}_5 \tau^3 A^{3/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \Big) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^4 + \frac{7}{36} \\
 & \cdot \left( \|\bar{R}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-1}\|_{H \rightarrow H} \right. \\
 & \cdot \left\| (\bar{R}_1 - R_1) R_5 \tau^5 A^{5/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \\
 & + \left\| R^{[\lambda_k/\tau]} \bar{R}^{[\lambda_k/\tau]-1} \right\|_{H \rightarrow H} \\
 & \cdot \left\| (\bar{R}_1 - R_1) \bar{R}_5 \tau^5 A^{5/2} (I + \tau^2 A^4)^{-1} \right\|_{H \rightarrow H} \Big) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^6 \Big\} \leq q,
 \end{aligned}$$

(16)

where

$$\begin{aligned}
 q = & \sum_{k=1}^n |\alpha_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right. \\
 & + \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big\} + \sum_{k=1}^n |\beta_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 + \frac{1}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big\} + \frac{1}{2} \sum_{k=1}^n |\alpha_k| \\
 & \cdot \sum_{k=1}^n |\beta_k| \left\{ 1 + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 + \frac{7}{12} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^4 \right. \\
 & + \left. \frac{7}{36} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^6 \right\}.
 \end{aligned}$$

(17)

Since  $q < 1$ , the operator  $I - B_n^T$  has a bounded inverse and

$$\|(I - B_n^T)^{-1}\|_{H \rightarrow H} \leq \frac{1}{1 - q}. \tag{18}$$

Lemma 2 is proved.  $\square$

Note that the stability estimates in the sequel are obtained by the method presented in the proofs of Lemmas 1 and 2 to obtain stability estimates. For clarity and avoiding long equations the proofs of estimates are not put.

Now, applying the following formula (see [14])

$$\begin{aligned}
 u_0 &= \mu, \\
 u_1 &= \left( I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right)^{-1} \\
 & \cdot \left( \left( I - \frac{5}{12} \tau^2 A + \frac{\tau^4}{144} A^2 \right) \mu \right. \\
 & + \left. \tau \left( I - \frac{\tau^2}{12} A \right) \omega + \tau^2 f_{1,1} \right), \\
 u_k &= \frac{1}{2} [\bar{R}_1 R^k - R_1 \bar{R}^k] \mu + \frac{1}{2} [\bar{R}^k - R^k] R_2 \omega + \frac{1}{2} [\bar{R}^k \\
 & - R^k] R_3 \tau^2 f_{1,1} + \frac{1}{2} R_4 \sum_{s=1}^{k-1} [\bar{R}^{k-s} - R^{k-s}] f_s \tau^2,
 \end{aligned}$$

(19)

and the multipoint nonlocal boundary conditions in (2), we obtain

$$\begin{aligned}
 \mu &= T_\tau (I + i\tau^2 A^4)^{-1} \left\{ \left( I - \frac{1}{2} R_2 \right. \right. \\
 & \cdot \sum_{k=1}^n \beta_k \left\{ \tau^{-1} (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \right. \\
 & + \frac{1}{2} \tau A (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 - A \tau (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
 & + \left. \left. \frac{1}{6} \tau^3 A^2 [\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}] \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \right\}
 \end{aligned}$$

$$\begin{aligned}
& \cdot \left( \sum_{k=1}^n \frac{1}{2} \alpha_k R_3 \tau^2 f_{1,1} \left\{ \left( \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5 \right) \right. \right. \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} A \tau^2 \left( R^{[\lambda_k/\tau]-1} R_5 + \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 \right) \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 + \left[ \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right] \\
& \left. - \frac{3}{2} \tau^2 A \left[ \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right] \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right\} \\
& + \frac{1}{2} R_4 \sum_{k=1}^n \alpha_k \sum_{s=1}^{[\lambda_k/\tau]-2} f_s \tau^2 \left\{ \left( \bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5 \right) \right. \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
& \left. - \frac{7}{6} A \tau^2 \left( \bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5 \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
& + \frac{1}{2} R_4 \sum_{k=1}^n \alpha_k \sum_{s=1}^{[\lambda_k/\tau]-1} f_s \tau^2 \left( \bar{R}^{[\lambda_k/\tau]-s} + R^{[\lambda_k/\tau]-s} \right) \\
& \cdot \left( 1 - \frac{3}{2} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) \frac{1}{2} R_4 \\
& \cdot \sum_{k=1}^n \alpha_k R_7 \left\{ \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
& \cdot f_{[\lambda_k/\tau]-1} \tau^2 + \sum_{k=1}^n \alpha_k \left\{ \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f_{[\lambda_k/\tau]} \tau^2 \right. \\
& \left. + \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f'_{[\lambda_k/\tau]} \tau^3 \right\} + \varphi \\
& - \left( -\frac{1}{2} R_2 \sum_{k=1}^n \alpha_k \left\{ \left( \bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 \right) \right. \right. \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} \tau^2 A \left( \bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 \right) \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 + \left( \bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]} \right) \\
& \left. - \frac{3}{2} A \tau^2 \left( \bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right\} \\
& \cdot \left( R_3 \sum_{k=1}^n \frac{1}{2} \beta_k f_{1,1} \tau^2 \left\{ \tau^{-1} \left( \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5 \right) \right. \right. \\
& - \frac{1}{2} \tau A \left( \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5 \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \\
& - A \tau \left( \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
& \left. + \frac{1}{6} A^2 \tau^3 \left( \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
& + \frac{1}{2} \sum_{k=1}^n \sum_{s=1}^{[\lambda_k/\tau]-2} R_4 \beta_k \left( \bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5 \right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \left( \frac{1}{\tau} - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_s \tau^2 \\
& + \frac{1}{2} \sum_{k=1}^n \sum_{s=1}^{[\lambda_k/\tau]-1} R_4 \beta_k \left( \bar{R}^{[\lambda_k/\tau]-s} + R^{[\lambda_k/\tau]-s} \right) \\
& \cdot \left( -A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{1}{6} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right) f_s \tau^2 \\
& + \frac{1}{2} \sum_{k=1}^n R_4 \beta_k R_7 \left( \frac{1}{\tau} - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_{[\lambda_k/\tau]-1} \tau^2 \\
& + \sum_{k=1}^n \beta_k \left\{ -\frac{1}{6} A \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f_{[\lambda_k/\tau]} \right. \\
& + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| f_{[\lambda_k/\tau]} \tau + \frac{1}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f'_{[\lambda_k/\tau]} \tau^2 \\
& \left. + \frac{1}{6} \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f''_{[\lambda_k/\tau]} \right\} + \psi \Bigg\},
\end{aligned}$$

$$\omega = T_\tau (I + i\tau^2 A^4)^{-1}$$

$$\begin{aligned}
& \cdot \left\{ \left( 1 - \frac{1}{2} \sum_{k=1}^n a_k \left\{ \left( \bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 \right) \right. \right. \right. \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} A \tau^2 \left( \bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 \right) \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 + \left( \bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]} \right) \\
& \left. \cdot \left( 1 - \frac{3}{2} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) \right\} \\
& \cdot \left( R_3 \sum_{k=1}^n \frac{1}{2} \beta_k f_{1,1} \tau^2 \left\{ \tau^{-1} \left( \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5 \right) \right. \right. \\
& - \frac{1}{2} \tau A \left( \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5 \right) \\
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 - A \tau \left( \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
& \left. + \frac{1}{6} A^2 \tau^3 \left( \bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]} \right) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
& + \frac{1}{2} \sum_{k=1}^n \sum_{s=1}^{[\lambda_k/\tau]-2} R_4 \beta_k f_s \tau^2 \left( \bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5 \right) \\
& \cdot \left( \frac{1}{\tau} - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) \\
& + \frac{1}{2} \sum_{k=1}^n \sum_{s=1}^{[\lambda_k/\tau]-1} R_4 \beta_k \left( \bar{R}^{[\lambda_k/\tau]-s} + R^{[\lambda_k/\tau]-s} \right) \\
& \cdot \left( -A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{1}{6} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right) f_s \tau^2 \\
& + \frac{1}{2} \sum_{k=1}^n R_4 \beta_k R_7 \left( \frac{1}{\tau} - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_{[\lambda_k/\tau]-1} \tau^2
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^n \beta_k \left\{ -\frac{1}{6} A \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f_{[\lambda_k/\tau]} \right. \\
 & + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| f_{[\lambda_k/\tau]} \tau \\
 & + \frac{1}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f'_{[\lambda_k/\tau]} \tau^2 \\
 & \left. + \frac{1}{6} \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f''_{[\lambda_k/\tau]} \right\} + \psi \Bigg\} \\
 & - \left( -\frac{1}{2} \sum_{k=1}^n \beta_k \left\{ \frac{1}{\tau} (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \right. \right. \\
 & + \frac{1}{2} A \tau (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \\
 & - A \tau (\bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
 & \left. + \frac{1}{6} A^2 \tau^3 (\bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \Bigg) \\
 & \cdot \left( \sum_{k=1}^n \frac{1}{2} \alpha_k R_3 \tau^2 f_{1,1} \left\{ (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \right. \right. \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{7}{6} A \tau^2 (R^{[\lambda_k/\tau]-1} R_5 + \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 + [\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}] \\
 & \left. - \frac{3}{2} \tau^2 A [\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}] \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right\} \\
 & + \frac{1}{2} R_4 \sum_{k=1}^n \alpha_k \sum_{s=1}^{[\lambda_k/\tau]-2} f_s \tau^2 \left\{ (\bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5) \right. \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{7}{6} A \tau^2 (\bar{R}^{[\lambda_k/\tau]-s-1} \bar{R}_5 + R^{[\lambda_k/\tau]-s-1} R_5) \\
 & \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Bigg\} + \frac{1}{2} R_4 \\
 & \cdot \sum_{k=1}^n \alpha_k \sum_{s=1}^{[\lambda_k/\tau]-1} f_s \tau^2 (\bar{R}^{[\lambda_k/\tau]-s} + R^{[\lambda_k/\tau]-s}) \\
 & \cdot \left( 1 - \frac{3}{2} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) \frac{1}{2} R_4 \\
 & \cdot \sum_{k=1}^n \alpha_k R_7 \left\{ \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
 & \cdot f_{[\lambda_k/\tau]-1} \tau^2 + \sum_{k=1}^n \alpha_k \left\{ \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f_{[\lambda_k/\tau]} \tau^2 \right. \\
 & \left. + \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f'_{[\lambda_k/\tau]} \tau^3 \right\} + \varphi \Bigg). \tag{20}
 \end{aligned}$$

Formulas (19) and (20) give a solution of problem (2).

**Theorem 3.** Suppose that assumptions (4) and (11) hold and  $\varphi \in D(A^{3/2})$  and  $\psi \in D(A^{1/2})$ . Then, for the solution of difference scheme (2), the following stability estimates hold:

$$\begin{aligned}
 \max_{1 \leq k \leq N} \|u_k\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|A^{-1/2} f_s\|_H \tau + \|\varphi\|_H \right. \\
 & \left. + \|A^{-1/2} \psi\|_H + \tau \|A^{-1/2} f_{1,1}\|_H \right\}, \\
 \max_{1 \leq k \leq N} \|A^{1/2} u_k\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|f_s\|_H \tau + \|A^{1/2} \varphi\|_H \right. \\
 & \left. + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}, \tag{21} \\
 \max_{1 \leq k \leq N} \|A u_k\|_H & \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H \right. \\
 & \left. + \|A \varphi\|_H + \|A^{1/2} \psi\|_H + \tau \|A^{1/2} f_{1,1}\|_H \right\},
 \end{aligned}$$

where  $M$  does not depend on  $\tau, \varphi, \psi, f_{1,1}$  and  $f_s, 1 \leq s \leq N-1$ .

*Proof.* Using formulas (20), estimates (11) and (15), and the method of Lemmas 1 and 2, we get

$$\begin{aligned}
 \|\mu\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|A^{-1/2} f_s\|_H \tau + \|\varphi\|_H + \|A^{-1/2} \psi\|_H \right. \\
 & \left. + \tau \|A^{-1/2} f_{1,1}\|_H \right\}, \\
 \|A^{-1/2} \omega\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|A^{-1/2} f_s\|_H \tau + \|\varphi\|_H \right. \\
 & \left. + \|A^{-1/2} \psi\|_H + \tau \|A^{-1/2} f_{1,1}\|_H \right\}. \tag{22}
 \end{aligned}$$

Applying  $A^{1/2}$  to formulas (20), we obtain

$$\begin{aligned}
 \|A^{1/2} \mu\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|f_s\|_H \tau + \|A^{1/2} \varphi\|_H + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}, \\
 \|\omega\|_H & \leq M \left\{ \sum_{s=1}^{k-1} \|f_s\|_H \tau + \|A^{1/2} \varphi\|_H + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}. \tag{23}
 \end{aligned}$$

Now, applying Abel's formula to (20), we get

$$\begin{aligned}
\mu &= T_\tau (I + i\tau^2 A^4)^{-1} \\
&\cdot \left\{ \left( I - \frac{1}{2} R_2 \sum_{k=1}^n \beta_k \left\{ \tau^{-1} (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \right. \right. \right. \\
&+ \frac{1}{2} \tau A (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \\
&- A\tau (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
&+ \frac{1}{6} \tau^3 A^2 (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left. \right\} \\
&\cdot \left( \sum_{k=1}^n \frac{1}{2} \alpha_k R_3 \tau^2 f_{1,1} \right. \\
&\cdot \left\{ (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
&- \frac{7}{6} A\tau^2 (R^{[\lambda_k/\tau]-1} R_5 + \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \\
&+ [\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}] \\
&- \frac{3}{2} \tau^2 A (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \left. \right\} \\
&+ \frac{1}{2} R_4 \tau^2 \sum_{k=1}^n \alpha_k \\
&\cdot \left[ \sum_{s=2}^{[\lambda_k/\tau]-2} \left( (R_5 R_6 R^{[\lambda_k/\tau]-s-1} - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1}) \right. \right. \\
&\cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
&- \frac{7}{6} A\tau^2 (R_5 R_6 R^{[\lambda_k/\tau]-s-1} - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1}) \\
&\cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left. \right) (f_s - f_{s-1}) \\
&+ \left( (\bar{R}_5 \bar{R}_6 - R_5 R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
&- \frac{7}{6} A\tau^2 (\bar{R}_5 \bar{R}_6 - R_5 R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left. \right) f_{[\lambda_k/\tau]-2} \\
&- \left( (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
&+ \frac{7}{6} A\tau^2 (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \\
&\cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left. \right) f_1 + \frac{1}{2} R_4 \tau^2 \sum_{k=1}^n \alpha_k \\
&\cdot \left[ \sum_{s=2}^{[\lambda_k/\tau]-1} \left( (R_6 R^{[\lambda_k/\tau]-s} - \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s}) \right. \right.
\end{aligned}$$

$$\begin{aligned}
&- \frac{3}{2} A\tau^2 (R_6 R^{[\lambda_k/\tau]-s} - \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s}) \\
&\cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \left. \right) (f_s - f_{s-1}) \\
&+ \left( (\bar{R}_6 - R_6) - \frac{3}{2} A\tau^2 (\bar{R}_6 - R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_{[\lambda_k/\tau]-1} \\
&- \left( (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) \right. \\
&+ \frac{3}{2} A\tau^2 (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \left. \right) f_1 \\
&\cdot \frac{1}{2} R_4 \sum_{k=1}^n \alpha_k R_7 \left\{ \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| - \frac{7}{6} A\tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} \\
&\cdot f_{[\lambda_k/\tau]-1} \tau^2 + \sum_{k=1}^n \alpha_k \left\{ \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f_{[\lambda_k/\tau]} \tau^2 \right. \\
&+ \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f'_{[\lambda_k/\tau]} \tau^3 \left. \right\} + \varphi \\
&- \left( -\frac{1}{2} R_2 \sum_{k=1}^n \alpha_k \right. \\
&\cdot \left\{ (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
&- \frac{7}{6} \tau^2 A (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \\
&+ (\bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]}) \\
&- \frac{3}{2} A\tau^2 (\bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \left. \right\} \\
&\cdot \left( R_3 \sum_{k=1}^n \frac{1}{2} \beta_k f_{1,1} \tau^2 \right. \\
&\cdot \left\{ \tau^{-1} (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \right. \\
&- \frac{1}{2} \tau A (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \\
&- A\tau (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
&+ \frac{1}{6} A^2 \tau^3 (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left. \right\} \\
&+ \frac{1}{2} R_4 \tau^2 \sum_{k=1}^n \beta_k \\
&\cdot \left[ \sum_{s=2}^{[\lambda_k/\tau]-2} \left( \frac{1}{\tau} (R_5 R_6 R^{[\lambda_k/\tau]-s-1} - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1}) \right. \right. \\
&- \frac{1}{2} A\tau (R_5 R_6 R^{[\lambda_k/\tau]-s-1} - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1})
\end{aligned}$$



$$\begin{aligned}
& \cdot \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \Big) (f_s - f_{s-1}) \\
& + \left( \frac{1}{\tau} (\bar{R}_5 \bar{R}_6 - R_5 R_6) - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right. \\
& \cdot (\bar{R}_5 \bar{R}_6 - R_5 R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \Big) f_{[\lambda_k/\tau]-2} \\
& - \left( \frac{1}{\tau} (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \right. \\
& - \frac{1}{2} A \tau (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_1 \\
& + \frac{1}{2} R_4 \tau^2 \sum_{k=1}^n \beta_k \\
& \cdot \left[ \sum_{s=2}^{[\lambda_k/\tau]-1} \left( -A \tau (R_6 R^{[\lambda_k/\tau]-s} - \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s}) \right. \right. \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| + \frac{1}{6} A \tau^2 (R_6 R^{[\lambda_k/\tau]-s} - \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s}) \right. \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big) (f_s - f_{s-1}) \right. \\
& + \left( -A \tau (\bar{R}_6 - R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
& + \frac{1}{6} A \tau^2 (\bar{R}_6 - R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big) f_{[\lambda_k/\tau]-1} \\
& - \left( -A \tau (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
& + \frac{1}{6} A \tau^2 (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big) f_1 \\
& + \frac{1}{2} \sum_{k=1}^n R_4 \beta_k R_7 \left( \frac{1}{\tau} - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_{[\lambda_k/\tau]-1} \tau^2 \\
& + \sum_{k=1}^n \beta_k \\
& \cdot \left\{ -\frac{1}{6} A \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f_{[\lambda_k/\tau]} + \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| f_{[\lambda_k/\tau]} \tau \right. \\
& + \frac{1}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f'_{[\lambda_k/\tau]} \tau^2 \\
& \left. + \frac{1}{6} \tau^3 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \left\{ f''_{[\lambda_k/\tau]} + \psi \right\} \right\},
\end{aligned}$$

(24)

$$\omega = T_\tau (I + i\tau^2 A^4)^{-1}$$

$$\cdot \left\{ \left( 1 - \frac{1}{2} \sum_{k=1}^n a_k \right. \right.$$

$$\begin{aligned}
& \left. \left\{ (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \right. \right. \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
& - \frac{7}{6} A \tau^2 (\bar{R}_1 R^{[\lambda_k/\tau]-1} R_5 + R_1 \bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5) \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right. \\
& + (\bar{R}_1 R^{[\lambda_k/\tau]} + R_1 \bar{R}^{[\lambda_k/\tau]}) \\
& \cdot \left. \left( 1 - \frac{3}{2} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) \right\} \\
& \cdot \left( R_3 \sum_{k=1}^n \frac{1}{2} \beta_k f_{1,1} \tau^2 \right. \\
& \cdot \left. \left\{ \tau^{-1} (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \right. \right. \\
& - \frac{1}{2} \tau A (\bar{R}^{[\lambda_k/\tau]-1} \bar{R}_5 + R^{[\lambda_k/\tau]-1} R_5) \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right. \\
& - A \tau (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \\
& + \frac{1}{6} A^2 \tau^3 (\bar{R}^{[\lambda_k/\tau]} + R^{[\lambda_k/\tau]}) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \Big\} \\
& + \frac{1}{2} R_4 \tau^2 \sum_{k=1}^n \beta_k \\
& \cdot \left[ \sum_{s=2}^{[\lambda_k/\tau]-2} \left( \frac{1}{\tau} (R_5 R_6 R^{[\lambda_k/\tau]-s-1} \right. \right. \\
& - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1}) \\
& - \frac{1}{2} A \tau (R_5 R_6 R^{[\lambda_k/\tau]-s-1} - \bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-s-1}) \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) (f_s - f_{s-1}) \right. \\
& + \left( \frac{1}{\tau} (\bar{R}_5 \bar{R}_6 - R_5 R_6) \right. \\
& - \frac{1}{2} A \tau \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \\
& \cdot (\bar{R}_5 \bar{R}_6 - R_5 R_6) \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \Big) f_{[\lambda_k/\tau]-2} \\
& - \left( \frac{1}{\tau} (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \right. \\
& - \frac{1}{2} A \tau (\bar{R}_5 \bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_5 R_6 R^{[\lambda_k/\tau]-1}) \\
& \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_1 \Big]
\end{aligned}$$



$$\begin{aligned}
 & + \left( (\bar{R}_6 - R_6) - \frac{3}{2}A\tau^2 (\bar{R}_6 - R_6) \right. \\
 & \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_{[\lambda_k/\tau]-1} \\
 & - \left( (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) - \frac{3}{2}A\tau^2 \right. \\
 & \cdot (\bar{R}_6 \bar{R}^{[\lambda_k/\tau]-1} - R_6 R^{[\lambda_k/\tau]-1}) \\
 & \cdot \left. \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 \right) f_1 \\
 & \cdot \frac{1}{2} R_4 \sum_{k=1}^n \alpha_k R_7 \\
 & \cdot \left\{ \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right| \right. \\
 & \left. - \frac{7}{6} A \tau^2 \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 \right\} f_{[\lambda_k/\tau]-1} \tau^2 \\
 & + \sum_{k=1}^n \alpha_k \left\{ \frac{3}{2} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^2 f_{[\lambda_k/\tau]} \tau^2 \right. \\
 & \left. + \frac{7}{6} \left| \frac{\lambda_k}{\tau} - \left[ \frac{\lambda_k}{\tau} \right] \right|^3 f'_{[\lambda_k/\tau]} \tau^3 + \varphi \right\}.
 \end{aligned} \tag{25}$$

Let us obtain estimates for  $\|A\mu\|_H$  and  $\|A^{1/2}\omega\|_H$  using the method of lemmas. First, applying  $A$  to formula (24), using estimates (11) and (15) and the triangle inequality, we obtain

$$\begin{aligned}
 \|A\mu\|_H & \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\
 & \left. + \|A^{1/2}\psi\|_H + \tau \|A^{1/2}f_{1,1}\|_H \right\}.
 \end{aligned} \tag{26}$$

Second, applying  $A^{1/2}$  to formula (25), using estimates (11) and (15) and the triangle inequality, we obtain

$$\begin{aligned}
 \|A^{1/2}\omega\|_H & \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\
 & \left. + \|A^{1/2}\psi\|_H + \tau \|A^{1/2}f_{1,1}\|_H \right\}.
 \end{aligned} \tag{27}$$

The estimates (21) are proved for  $k = 0, 1$ . Now, we give the proofs of estimates (21) which were also presented in [14] for  $k \geq 2$ . Using formula (19), estimates (11), (22), and (23), the triangle inequality, and method of Lemmas 1 and 2, we obtain

$$\begin{aligned}
 \|u_k\|_H & \leq M \left\{ \sum_{s=1}^{N-1} \|A^{-1/2}f_s\|_H \tau + \|\varphi\|_H + \|A^{-1/2}\psi\|_H \right. \\
 & \left. + \tau \|A^{-1/2}f_{1,1}\|_H \right\}.
 \end{aligned} \tag{28}$$

Applying  $A^{1/2}$  to (19), we get

$$\begin{aligned}
 \|A^{1/2}u_k\|_H & \leq M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau \right. \\
 & \left. + \|A^{1/2}(I + i\tau A^{1/2})\varphi\|_H + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}.
 \end{aligned} \tag{29}$$

Applying Abel's formula to (19), we get

$$\begin{aligned}
 u_k & = \frac{1}{2} [\bar{R}_1 R^k - R_1 \bar{R}^k] \mu + \frac{1}{2} [\bar{R}^k - R^k] R_2 \omega + \frac{1}{2} [\bar{R}^k \\
 & - R^k] R_3 \tau^2 f_{1,1} + \tau^2 R_4 \\
 & \cdot \frac{1}{2} \left( \sum_{s=2}^{k-1} [R_6 R^{k-s} - \bar{R}_6 \bar{R}^{k-s}] (f_s - f_{s-1}) \right. \\
 & \left. + (\bar{R}_6 - R_6) f_{k-1} - [\bar{R}_6 \bar{R}^{k-1} - R_6 R^{k-1}] f_1 \right),
 \end{aligned} \tag{30}$$

$2 \leq k \leq N$ .

Applying  $A$  to formula (30), using estimates (11) and the triangle inequality, we obtain

$$\begin{aligned}
 \|Au_k\|_H & \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H \right. \\
 & + \|A(I + i\tau A^{1/2})\varphi\|_H + \|A^{1/2}\psi\|_H \\
 & \left. + \tau \|A^{1/2}f_{1,1}\|_H \right\}.
 \end{aligned} \tag{31}$$

Theorem 3 is proved.  $\square$

We denote the mesh function of approximation by  $a^\tau = \{a_k\}_{k=0}^N$ . Then  $\|(I + \tau^2 A^4)^{-1} a_1\|_H \sim \|a_1\|_H = o(\tau)$  if we assume that  $\tau \|A a_1\|_H$  tends to 0 as  $\tau \rightarrow 0$  not slower than  $\|a_1\|_H$ . It takes place in applications by supplementary restriction of the smooth property of the data of space variables. Let us give applications of the abstract results presented in Theorem 3.

First, the mixed multipoint NBVP for hyperbolic equation

$$\begin{aligned}
 u_{tt} - (a(x) u_x)_x + \delta u & = f(t, x), \\
 0 < t < 1, \quad 0 < x < 1, \\
 u(0, x) & = \sum_{j=1}^n \alpha_j u(\lambda_j, x) + \varphi(x), \\
 0 \leq x \leq 1,
 \end{aligned}$$

$$\begin{aligned}
u_t(0, x) &= \sum_{j=1}^n \beta_j u_t(\lambda_j, x) + \psi(x), \\
0 \leq x \leq 1, \\
u(t, 0) &= u(t, 1), \\
u_x(t, 0) &= u_x(t, 1), \\
0 \leq t \leq 1
\end{aligned} \tag{32}$$

is considered. Problem (32) has a unique smooth solution  $u(t, x)$ ,  $\delta > 0$ , and the smooth functions  $a(x) \geq a > 0$  ( $a(0) = a(1)$ ,  $x \in (0, 1)$ ),  $\varphi(x)$ ,  $\psi(x)$  ( $x \in [0, 1]$ ) and  $f(t, x)$  ( $t, x \in [0, 1]$ ). This allows us to reduce mixed problem (32) to multipoint NBVP (1) in a Hilbert space  $H = L_2[0, 1]$  with a self-adjoint positive definite operator  $A^x$  defined by (32).

The discretization of problem (32) is carried out in two steps. In the first step, we present the grid space

$$[0, 1]_h = \{x : x_r = rh, 0 \leq r \leq K, Kh = 1\} \tag{33}$$

in a Hilbert space  $L_{2h} = L_2([0, 1]_h)$ ,  $W_{2h}^1 = W_{2h}^1([0, 1]_h)$  and  $W_{2h}^2 = W_{2h}^2([0, 1]_h)$  of the grid functions  $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$  defined on  $[0, 1]_h$ , equipped with the norms

$$\begin{aligned}
\|\varphi^h\|_{L_{2h}} &= \left( \sum_{r=1}^{K-1} |\varphi^h(x)|^2 h \right)^{1/2}, \\
\|\varphi^h\|_{W_{2h}^1} &= \|\varphi^h\|_{L_{2h}} + \left( \sum_{r=1}^{K-1} |(\varphi^h)_{\bar{x},j}|^2 h \right)^{1/2}, \\
\|\varphi^h\|_{W_{2h}^2} &= \|\varphi^h\|_{L_{2h}} + \left( \sum_{r=1}^{K-1} |(\varphi^h)_{\bar{x},j}|^2 h \right)^{1/2} \\
&\quad + \left( \sum_{r=1}^{K-1} |(\varphi^h)_{\bar{x}\bar{x},j}|^2 h \right)^{1/2},
\end{aligned} \tag{34}$$

respectively. To the differential operator  $A$  generated by the problem (32), we assign the difference operator  $A_h^x$  by the formula

$$A_h^x \varphi^h(x) = \{-(a(x) \varphi_{\bar{x}})_{x,r} + \delta \varphi_r\}_1^{K-1}, \tag{35}$$

acting in the space of grid functions  $\varphi^h(x) = \{\varphi_r\}_0^K$  satisfying the conditions  $\varphi_0 = \varphi_K$ ,  $\varphi_1 - \varphi_0 = \varphi_K - \varphi_{K-1}$ . With the help of  $A_h^x$  we arrive at the multipoint NBVP

$$\begin{aligned}
\frac{d^2 v^h(t, x)}{dt^2} + A_h^x v^h(t, x) &= f^h(t, x), \\
0 < t < 1, \quad x \in [0, 1]_h,
\end{aligned}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (36) by difference scheme (37):

$$\begin{aligned}
\tau^{-2} (u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) &+ \frac{2}{3} A_h^x u_k^h(x) + \frac{1}{6} \\
\cdot A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) &+ \frac{1}{12} \tau^2 (A_h^x)^2 u_{k+1}^h(x) \\
&= f_k^h(x), \\
f_k^h(x) &= \frac{2}{3} f^h(t_k, x) + \frac{1}{6} (f^h(t_{k+1}, x) + f^h(t_{k-1}, x)) \\
&\quad - \frac{1}{12} \tau^2 (-A f^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)), \\
x \in [0, 1]_h, \quad t_k &= k\tau, \quad N\tau = 1, \quad 1 \leq k \leq N-1,
\end{aligned}$$

$$\begin{aligned}
u_0^h(x) &= \sum_{j=1}^n \alpha_j \left\{ u_{[\lambda_j/\tau]}^h(x) \right. \\
&\quad + \tau^{-1} (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right) \\
&\quad + \frac{3}{2} (f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^2 \\
&\quad + \frac{7}{6} (f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x))) \\
&\quad \cdot \left. \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^3 \right\} + \varphi^h(x), \quad x \in [0, 1]_h,
\end{aligned}$$

$$\begin{aligned}
(I + \tau^2 (A_h^x)^4) \tau^{-1} (u_1^h(x) - u_0^h(x)) &= \sum_{j=1}^n \beta_j \\
\cdot \left\{ \tau^{-1} (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)) \right. \\
&\quad + (f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right) \\
&\quad + \frac{1}{2!} (f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x))) \\
&\quad \cdot \left. \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^2 \right.
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{3!} \left( f''_{[\lambda_j/\tau]} - A_h^x f_{[\lambda_j/\tau]} + (A_h^x)^2 u_{[\lambda_j/\tau]}^h(x) \right) \\
 & \cdot \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^3 + \psi^h(x), \quad x \in [0, 1]_h, \\
 f_{1,1}^h(x) & = \frac{1}{2} f^h(0, x) + \frac{\tau}{6} f_t^h(0, x).
 \end{aligned} \tag{37}$$

**Theorem 4.** *Let  $\tau$  and  $h$  be sufficiently small numbers. Then, the solution of difference scheme (37) satisfies the following stability estimates:*

$$\begin{aligned}
 \max_{0 \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^1} & \leq M_1 \left[ \max_{1 \leq k \leq N-1} \|f_k^h\|_{L_{2h}} \right. \\
 & \left. + \|\psi^h\|_{L_{2h}} + \|\varphi^h\|_{W_{2h}^1} + \tau \|\varphi^h\|_{W_{2h}^2} + \tau \|f_{1,1}^h\|_{L_{2h}} \right], \\
 \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^2} & \tag{38} \\
 & \leq M_1 \left[ \|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|\tau^{-1} (f_k^h - f_{k-1}^h)\|_{L_{2h}} \right. \\
 & \left. + \|\psi^h\|_{W_{2h}^1} + \|\varphi^h\|_{W_{2h}^2} + \tau \|\varphi^h\|_{W_{2h}^3} + \tau \|f_{1,1}^h\|_{W_{2h}^1} \right].
 \end{aligned}$$

Here  $M_1$  does not depend on  $\tau$ ,  $h$ ,  $\varphi^h(x)$ ,  $\psi^h(x)$ ,  $f_{1,1}^h$  and  $f_k^h$ ,  $1 \leq k < N$ .

The proof of Theorem 4 is based on the proof of abstract Theorem 3 and the symmetry property of the operator  $A_h^x$  defined by (35).

Second, let  $\Omega$  be the unit open cube in the  $m$ -dimensional Euclidean space  $\mathbb{R}^m \{x = (x_1, \dots, x_m) : 0 < x_j < 1, 1 \leq j \leq m\}$  with boundary  $S$ ,  $\bar{\Omega} = \Omega \cup S$ . In  $[0, 1] \times \Omega$ , the mixed boundary value problem for the multidimensional hyperbolic equation

$$\begin{aligned}
 \frac{\partial^2 u(t, x)}{\partial t^2} - \sum_{r=1}^m (a_r(x) u_{x_r})_{x_r} & = f(t, x), \\
 x & = (x_1, \dots, x_m) \in \Omega_h, \quad 0 < t < 1, \\
 u^h(0, x) & = \sum_{j=1}^n \alpha_j u^h(\lambda_j, x) + \varphi^h(x), \quad x \in \bar{\Omega}_h, \tag{39} \\
 u_t^h(0, x) & = \sum_{j=1}^n \beta_j u_t^h(\lambda_j, x) + \psi^h(x), \quad x \in \bar{\Omega}_h, \\
 u^h(t, x) & = 0, \quad x \in S
 \end{aligned}$$

is considered. Here  $a_r(x)$ , ( $x \in \Omega$ ),  $\varphi(x)$ ,  $\psi(x)$  ( $x \in \bar{\Omega}_h$ ), and  $f(t, x)$  ( $t \in (0, 1)$ ,  $x \in \Omega_h$ ) are given smooth functions and  $a_r(x) \geq a > 0$ .

We introduce the Hilbert space  $L_2(\bar{\Omega})$  of all square integrable functions defined on  $\bar{\Omega}$ , equipped with the norm

$$\|f\|_{L_2(\bar{\Omega})} = \left\{ \int \cdots \int_{x \in \bar{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \right\}^{1/2}. \tag{40}$$

Problem (39) has a unique smooth solution  $u(t, x)$ , and  $\varphi(x)$ ,  $\psi(x)$ ,  $a_r(x)$ , and  $f(t, x)$  are smooth functions. This allows us to reduce the mixed problem (39) to multipoint NBVP (1) in a Hilbert space  $H = L_2(\bar{\Omega})$  with a self-adjoint positive definite operator  $A^x$  defined by (39).

The discretization of problem (39) is carried out in two steps. In the first step, we define the grid sets

$$\begin{aligned}
 \bar{\Omega}_h & = \{x = x_r = (h_1 r_1, \dots, h_m r_m), \quad r = (r_1, \dots, r_m), \quad 0 \\
 & \leq r_j \leq N_j, \quad h_j N_j = 1, \quad j = 1, \dots, m\}, \tag{41} \\
 \Omega_h & = \bar{\Omega}_h \cap \Omega, \\
 S_h & = \bar{\Omega}_h \cap S.
 \end{aligned}$$

We introduce the Banach space  $L_{2h} = L_2(\bar{\Omega}_h)$ ,  $W_{2h}^1 = W_{2h}^1(\bar{\Omega}_h)$ , and  $W_{2h}^2 = W_{2h}^2(\bar{\Omega}_h)$  of the grid functions  $\varphi^h(x) = \{\varphi^h(h_1 r_1, \dots, h_m r_m)\}$  defined on  $\bar{\Omega}_h$ , equipped with norms

$$\begin{aligned}
 \|\varphi^h\|_{L_2(\bar{\Omega}_h)} & = \left( \sum_{x \in \bar{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_m \right)^{1/2}, \\
 \|\varphi^h\|_{W_{2h}^1} & = \|\varphi^h\|_{L_{2h}} \\
 & + \left( \sum_{x \in \bar{\Omega}_h} \sum_{r=1}^m |(\varphi^h)_{\bar{x}_r, j_r}|^2 h_1 \cdots h_m \right)^{1/2}, \tag{42} \\
 \|\varphi^h\|_{W_{2h}^2} & = \|\varphi^h\|_{L_{2h}} \\
 & + \left( \sum_{x \in \bar{\Omega}_h} \sum_{r=1}^m |(\varphi^h)_{\bar{x}_r}|^2 h_1 \cdots h_m \right)^{1/2} \\
 & + \left( \sum_{x \in \bar{\Omega}_h} \sum_{r=1}^m |(\varphi^h)_{x_r, \bar{x}_r, j_r}|^2 h_1 \cdots h_m \right)^{1/2},
 \end{aligned}$$

respectively. To the differential operator  $A$  generated by problem (39), we assign the difference operator  $A_h^x$  by the formula

$$A_h^x u_x^h = - \sum_{r=1}^m (a_r(x) u_{\bar{x}_r}^h)_{x_r, j_r} \tag{43}$$

acting in the space of grid functions  $u^h(x)$ , satisfying the conditions  $u^h(x) = 0$  for all  $x \in S_h$ . It is known that  $A_h^x$  is

a self-adjoint positive definite operator in  $L_2(\bar{\Omega}_h)$ . With the help of  $A_h^x$  we arrive at the multipoint NBVP:

$$\begin{aligned} \frac{d^2 v^h(t, x)}{dt^2} + A_h^x v^h(t, x) &= f^h(t, x), \\ 0 < t < 1, \quad x &\in \Omega_h, \\ v^h(0, x) &= \sum_{j=1}^n \alpha_j v^h(\lambda_j, x) + \varphi^h(x), \\ x &\in \bar{\Omega}_h, \\ \frac{dv^h(0, x)}{dt} &= \sum_{j=1}^n \beta_j v_t^h(\lambda_j, x) + \psi^h(x), \\ x &\in \bar{\Omega}_h \end{aligned} \quad (44)$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (44) by difference scheme (45):

$$\begin{aligned} \tau^{-2} (u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + \frac{2}{3} A_h^x u_k^h(x) + \frac{1}{6} \\ \cdot A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) + \frac{1}{12} \tau^2 (A_h^x)^2 u_{k+1}^h(x) \\ = f_k^h(x), \\ f_k^h(x) = \frac{2}{3} f^h(t_k, x) + \frac{1}{6} (f^h(t_{k+1}, x) + f^h(t_{k-1}, x)) \\ - \frac{1}{12} \tau^2 (-A f^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)), \\ t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1, \quad x \in \Omega_h, \\ u_0^h(x) = \sum_{j=1}^n \alpha_j \left\{ u_{[\lambda_j/\tau]}^h(x) \right. \\ + \tau^{-1} (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right) \\ + \frac{3}{2} (f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^2 \\ + \frac{7}{6} (f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x))) \\ \cdot \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^3 \left. \right\} + \varphi^h(x), \quad x \in \Omega_h, \\ (I + \tau^2 (A_h^x)^4) \tau^{-1} (u_1^h(x) - u_0^h(x)) = \sum_{j=1}^n \beta_j \\ \cdot \left\{ \tau^{-1} (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)) \right. \end{aligned}$$

$$\begin{aligned} + (f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)) \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right) \\ + \frac{1}{2!} (f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x (u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x))) \\ \cdot \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^2 \\ + \frac{1}{3!} (f''_{[\lambda_j/\tau]} - A_h^x f_{[\lambda_j/\tau]} + (A_h^x)^2 u_{[\lambda_j/\tau]}^h(x)) \\ \cdot \left( \lambda_j - \left[ \frac{\lambda_j}{\tau} \right] \tau \right)^3 \left. \right\} + \psi^h(x), \quad x \in \Omega_h, \\ f_{1,1}^h(x) = \frac{1}{2} f^h(0, x) + \frac{\tau}{6} f_t^h(0, x). \end{aligned} \quad (45)$$

**Theorem 5.** *Let  $\tau$  and  $|h|$  be sufficiently small numbers. Then, the solution of difference scheme (45) satisfies the following stability estimates:*

$$\begin{aligned} \max_{0 \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^1} \leq M_1 \left[ \max_{1 \leq k \leq N-1} \|f_k^h\|_{L_{2h}} \right. \\ + \|\psi^h\|_{L_{2h}} + \|\varphi^h\|_{W_{2h}^1} + \tau \|\varphi^h\|_{W_{2h}^2} + \tau \|f_{1,1}^h\|_{L_{2h}} \left. \right], \\ \max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^2} \\ \leq M_1 \left[ \|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|\tau^{-1} (f_k^h - f_{k-1}^h)\|_{L_{2h}} \right. \\ + \|\psi^h\|_{W_{2h}^1} + \|\varphi^h\|_{W_{2h}^2} + \tau \|\varphi^h\|_{W_{2h}^3} + \tau \|f_{1,1}^h\|_{W_{2h}^1} \left. \right]. \end{aligned} \quad (46)$$

Here  $M_1$  does not depend on  $\tau, h, \varphi^h(x), \psi^h(x), f_{1,1}^h$ , and  $f_k^h, 1 \leq k < N$ .

Proof of Theorem 5 is based on the proof of abstract Theorem 3, the symmetry property of the operator  $A_h^x$  defined by formula (43), and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in  $L_{2h}$ .

**Theorem 6.** *For the solutions of the elliptic difference problem*

$$\begin{aligned} A_h^x u^h(x) &= \omega^h(x), \quad x \in \Omega_h, \\ u^h(x) &= 0, \quad x \in S_h \end{aligned} \quad (47)$$

the following coercivity inequality holds [25]:

$$\sum_{r=1}^m \|u_{x_r, \bar{x}_r, j_r}^h\|_{L_{2h}} \leq M \|\omega^h\|_{L_{2h}}. \quad (48)$$

Here  $M$  does not depend on  $h, \omega^h(x)$ .

TABLE 1: Errors for the approximate solutions of problem (49).

Method	$N = 20,$ $M = 20$	$N = 40,$ $M = 40$	$N = 80,$ $M = 80$
Difference scheme solution	0,0005962	0,0001554	0,00003946

### 3. Numerical Results

In this section some numerical results are presented. The multipoint NBVP

$$\frac{\partial^2 u(t, x)}{\partial t^2} - \frac{\partial^2 u(t, x)}{\partial x^2} = (t^2 - 2t + 3) \sin x,$$

$$0 < t < 1, 0 < x < \pi,$$

$$u(0, x) = \frac{1}{8}u(1, x) + \frac{1}{8}u\left(\frac{1}{2}, x\right) + \frac{31}{32} \sin x,$$

$$0 \leq x \leq \pi, \tag{49}$$

$$u_t(0, x) = \frac{1}{8}u_t(1, x) + \frac{1}{8}u_t\left(\frac{1}{2}, x\right) - \frac{15}{8} \sin x,$$

$$0 \leq x \leq \pi,$$

$$u(t, 0) = u(t, \pi) = 0, \quad 0 \leq t \leq 1$$

for one-dimensional hyperbolic equation is considered. The exact solution of (49) is

$$u(t, x) = (1 - t)^2 \sin x. \tag{50}$$

For the approximate solution of problem (49) the third order of accuracy difference scheme (2) is used, and the set  $[0, 1]_\tau \times [0, \pi]_h$  of a family of grid points depending on the small parameters  $\tau$  and  $h$  is considered:  $[0, 1]_\tau \times [0, \pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \leq k \leq N, N\tau = 1, x_n = nh, 0 \leq n \leq M, Mh = \pi\}$ . The implementations of the numerical experiments are carried out by Matlab.

The errors are computed by the following formula:

$$E_M^N = \max_{1 \leq k \leq N-1, 1 \leq n \leq M-1} |u(t_k, x_n) - u_n^k|. \tag{51}$$

Here,  $u(t_k, x_n)$  represents the exact solution and  $u_n^k$  represents the numerical solution at  $(t_k, x_n)$ . Errors are presented in Table 1.

In the table, the results are presented for different  $M$  and  $N$  values which are the step numbers for time and space variables, respectively. The results in Table 1 indicate that error decreases as  $M$  and  $N$  values get larger. Therefore it can be verified by the experimental results (Table 1) that the difference scheme is stable.

### 4. Conclusion

In this paper a third order of accuracy unconditionally stable difference scheme for the approximate solution of hyperbolic multipoint NBVP in a Hilbert space with self-adjoint positive definite operator is presented. The stability is established

without any assumptions in respect of the grid steps  $\tau$  and  $h$ . Numerical results verifying the theoretical statements are presented. It is not easy to obtain arbitrary order stable difference schemes. Nevertheless one can obtain fourth order or higher order stable difference schemes using our results and method. It is possible to consider the same problem with fractional derivative or with small parameter  $\epsilon$ .

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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