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Research Article

Optimized Hybrid Methods for Solving Oscillatory Second Order Initial Value Problems

N. Senu,[1] F. Ismail,[1] S. Z. Ahmad,[2] and M. Suleiman[1]

[1] *Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia*
[2] *Department of Mathematics, Universiti Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to N. Senu; norazak@upm.edu.my

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Two-step optimized hybrid methods of order five and order six are developed for the integration of second order oscillatory initial value problems. The optimized hybrid method (OHMs) are based on the existing nonzero dissipative hybrid methods. Phase-lag, dissipation or amplification error, and the differentiation of the phase-lag relations are required to obtain the methods. Phase-fitted methods based on the same nonzero dissipative hybrid methods are also constructed. Numerical results show that OHMs are more accurate compared to the phase-fitted methods and some well-known methods appeared in the scientific literature in solving oscillating second order initial value problems. It is also found that the nonzero dissipative hybrid methods are more suitable to be optimized than phase-fitted methods.

1. Introduction

Many differential equations which appear in practice are system of second-order initial value problem (IVP), in which the first derivative does not appear explicitly:

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{1}$$

This type of ordinary differential equations (ODEs) often appears in many scientific areas of engineering and applied sciences such as celestial mechanics, molecular dynamics, quantum mechanics, and theoretical physics. Quite often the solution of (1) exhibits a pronounced oscillatory character. Oscillatory problems are usually harder to solve than the nonoscillatory problems. Coleman [1] in his paper developed the order conditions of hybrid method up to order nine. With the order condition developed by Coleman [1], Franco [2] constructed explicit two-step hybrid methods of fourth, fifth, and sixth order for solving second order IVPs.

To obtain a more efficient process for solving oscillatory problems, numerical methods are constructed by taking into account the nature of the problem. This results in methods in which the coefficients depend on the problem. Some important classes of numerical methods, such as exponentially fitted or phase-fitted methods, can be obtained if a good estimate of the period is known.

Phase-lag or dispersion error is the angle between the true and the approximated solutions. The analysis of phase-lag or dispersion error was first introduced by Brusa and Nigro [3]. Several authors such as van der Houwen and Sommeijer [4, 5] and Thomas [6] studied in detail the phase-lag of numerical methods for solving (1). Several authors in their papers [6–9] have developed hybrid methods with the purpose of making the phase-lag of the method smaller. The technique of vanishing the phase-lag, the amplification error, and their first integral of phase-lag was introduced by Papadopoulos and Simos [10]. Numerical results indicated that such technique produced methods which are very efficient for solving Schrodinger equation. Kosti et al. [11] in their work developed an optimized explicit Runge-Kutta Nyström method with four stages and fifth algebraic order for the numerical solution of orbital and related periodic initial value problems. Other than phase-lag much research is also focused on methods having high dissipative order; dissipation is the distance of the computed solution from the standard cyclic solution. Hence, for solving oscillatory problems, it is the aim of every researcher to derive numerical method which has high algebraic order, phase-lag order, and dissipative order. Methods having dissipative order infinity are called a zero-dissipative methods.

In this work, we will derive methods in which the coefficients depend on the problems. The first two methods are called optimized methods which are developed using the same approach introduced by Kosti et al. [11]. They are optimized by imposing the phase-lag, dissipative or amplification error, and the first derivative of the phase-lag relation. The second two methods derived are called phase-fitted methods which are obtained by minimizing the phase-lag. The new optimized and phase-fitted hybrid methods are based on the nonzero-dissipative, four-stage fifth-order and five-stage sixth-order hybrid methods developed by Franco [2].

The comparison of the new methods with other methods in the scientific literature has shown that the optimized methods are more efficient for solving problems of oscillating in nature.

2. Analysis Phase-Lag of the Methods

An s-stage two-step hybrid method for numerical integration of the IVP (1) is in the form

$$Y_i = (1+c_i)\, y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h, Y_j\right), \quad i = 1, \dots, s,$$
$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f\left(t_n + c_i h, Y_i\right), \tag{2}$$

where b_i, c_i, and a_{ij} can be represented in Butcher notation by the table of coefficients.

s-stage hybrid methods are as follows:

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & \cdots & a_{s,s} \\ \hline & b_1 & \cdots & b_s \end{array} \tag{3}$$

The methods of the form (2) are defined by

$$Y_1 = y_{n-1}, \qquad Y_2 = y_n,$$
$$Y_i = (1+c_i)\, y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f\left(t_n + c_j h, Y_j\right), \quad i = 3, \dots, s, \tag{4}$$
$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left[b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f\left(t_n + c_i h, Y_i\right) \right],$$

where $f_{n-1} = f(t_{n-1}, y_{n-1})$, $f_n = f(t_n, y_n)$, and $h = \Delta t = t_{n+1} - t_n$ and the first two nodes are $c_1 = -1$ and $c_2 = 0$. The method only requires to evaluate $f(t_n, y_n)$, $f(t_n + c_3 h, Y_3), \dots, f(t_n + c_s h, Y_s)$ for each step after starting procedure. This method is considered as two-step hybrid method with $s-1$ stages per step and can be written in Butcher tableau; s-stage explicit hybrid methods are as follows:

$$\begin{array}{c|ccccc} -1 & & & & & \\ 0 & & & & & \\ c_3 & a_{3,1} & a_{3,2} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} \tag{5}$$

In order to construct the new method, we use the test equation

$$y''(x) = -v^2 y(x) \quad \text{for } v > 0. \tag{6}$$

By replacing $f(x, y) = -v^2 y$ into (2) we have

$$Y_i = (1+c_i)\, y_n - c_i y_{n-1} - h^2 \sum_{j=1}^{s} a_{ij} \lambda^2 y, \quad i = 1, \dots, s$$
$$y_{n+1} = 2y_n - y_{n-1} - h^2 \sum_{i=1}^{s} b_i \lambda^2 y. \tag{7}$$

Define $z = vh$, so (7) can also be written as

$$Y_i = (1+c_i)\, y_n - c_i y_{n-1} - z^2 \sum_{j=1}^{s} a_{ij} y, \quad i = 1, \dots, s$$
$$y_{n+1} = 2y_n - y_{n-1} - z^2 \sum_{i=1}^{s} b_i y. \tag{8}$$

Alternatively (8) can be written in vector form as follows:

$$\mathbf{Y} = (\mathbf{e} + \mathbf{c})\, y_n - \mathbf{c} y_{n-1} - z^2 \mathbf{A} \mathbf{Y} \tag{9}$$

$$y_{n+1} = 2y_n - y_{n-1} - z^2 \mathbf{b}^{\mathbf{T}} \mathbf{Y}, \tag{10}$$

where $\mathbf{Y} = (Y_1, \dots, Y_s)^T$, $\mathbf{c} = (c_1, \dots, c_s)^T$, $\mathbf{e} = (1, \dots, 1)^T$,

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}. \tag{11}$$

From (9) we obtain

$$\mathbf{Y} = \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} (\mathbf{e} + \mathbf{c})\, y_n - \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} \mathbf{c} y_{n-1}. \tag{12}$$

Substituting (12) into (10) gives

$$y_{n+1} = \left(2 - z^2 \mathbf{b}^{\mathbf{T}} \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} (\mathbf{e} + \mathbf{c})\right) y_n - \left(1 - z^2 \mathbf{b}^{\mathbf{T}} \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} \mathbf{c}\right) y_{n-1}. \tag{13}$$

Rewrite (13) and then the following recursion is obtained

$$P(\xi, z) = \xi^2 - S(z^2)\xi + T(z^2) = 0, \tag{14}$$

where

$$\begin{aligned} S(z^2) &= 2 - z^2 \mathbf{b}^{\mathbf{T}} \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} (\mathbf{e} + \mathbf{c}), \\ T(z^2) &= 1 - z^2 \mathbf{b}^{\mathbf{T}} \left(\mathbf{I} + z^2 \mathbf{A}\right)^{-1} \mathbf{c}. \end{aligned} \tag{15}$$

The following definition is given by van der Houwen and Sommeijer [4] for Runge-Kutta Nyström method and has been used by Franco [2] for hybrid method.

Definition 1. Apply the hybrid method (2) to (6). Next we define the phase-lag, $\varphi(z)$ and dissipation, and $d(z)$

$$\varphi(z) = z - \cos^{-1}\left(\frac{S(z^2)}{2\sqrt{T(z^2)}}\right), \qquad d(z) = 1 - \sqrt{T(z^2)}. \tag{16}$$

If $\varphi(z) = O(z^{q+1})$, then the hybrid method is said to have phase-lag order q. If $d(z) = O(z^{r+1})$, then the hybrid method is said to have dissipation order r.

3. Derivation of the New Hybrid Methods

In this section, we construct optimized hybrid methods of four-stage fifth order and five-stage sixth order. First, let us denote (15) in polynomials form as follows:

$$\begin{aligned} S(z^2) &= 2 - \alpha_1 z^2 + \alpha_2 z^4 - \alpha_3 z^6 + \cdots + \alpha_i z^{2i}, \\ T(z^2) &= 1 - \beta_1 z^2 + \beta_2 z^4 - \beta_3 z^6 + \cdots + \beta_i z^{2i}. \end{aligned} \tag{17}$$

The relations between phase-lag (dispersion), dissipation (amplification error), and the derivative of phase-lag are taken into consideration throughout obtaining the new methods. On the other hand, phase-fitted methods which are based on the same hybrid methods are also developed. The effectiveness of the new optimized methods is compared with that of the phase-fitted method as well as other existing methods in the literature.

3.1. Optimized Hybrid Methods. Polynomials (17) for hybrid method which satisfied algebraic order conditions up to order six can be written in these expressions:

(i) for $s = 5$,

$$\begin{aligned} S(z^2) = 2 &+ (-b_1 - b_1c_1 - b_3 - b_3c_3 \\ &\quad - b_2 - b_2c_2 - b_4 - b_4c_4) z^2 \\ &+ (b_3a_{31} + b_4a_{41} + c_1b_3a_{31} + c_1b_4a_{41} \\ &\quad + b_4a_{43} + b_4a_{43}c_3 + b_3a_{32} + b_4a_{42} \\ &\quad + c_2b_3a_{32} + c_2b_4a_{42}) z^4 \\ &+ (-b_4a_{32}a_{43} - b_4a_{32}a_{43}c_2 - b_4a_{31}a_{43} \\ &\quad - b_4a_{31}a_{43}c_1) z^6 \\ T(z^2) = 1 &+ (-b_1c_1 - b_3c_3 - b_2c_2 - b_4c_4) z^2 \\ &+ (c_1b_3a_{31} + c_1b_4a_{41} + b_4a_{43}c_3 \\ &\quad + c_2b_3a_{32} + c_2b_4a_{42}) z^4 \\ &+ (-b_4a_{32}a_{43}c_2 - b_4a_{31}a_{43}c_1) z^6 \end{aligned} \tag{18}$$

(ii) for $s = 6$,

$$\begin{aligned} S(z^2) = 2 &+ (-b_2 - b_4 - b_4c_4 - b_3 \\ &\quad - b_3c_3 - b_5 - b_5c_5) z^2 \\ &+ (b_3a_{32} + b_4a_{42} + b_5a_{54} + b_5a_{54}c_4 \\ &\quad + b_5a_{52} + b_4a_{43} + b_5a_{53} \\ &\quad + c_3b_4a_{43} + c_3b_5a_{53}) z^4 \\ &+ (-b_5a_{53}a_{32} - b_5a_{54}a_{42} - b_5a_{43}a_{54} \\ &\quad - b_5a_{43}a_{54}c_3 - b_4a_{43}a_{32}) z^6 \\ &+ b_5a_{54}a_{32}a_{43}z^8 \\ T(z^2) = 1 &+ (b_1 - b_4c_4 - b_3c_3 - b_5c_5) z^2 \\ &+ (-b_3a_{31} - b_4a_{41} + b_5a_{54}c_4 - b_5a_{51} \\ &\quad + c_3b_4a_{43} + c_3b_5a_{53}) z^4 \\ &+ (b_5a_{53}a_{31} + b_5a_{54}a_{41} - b_5a_{43}a_{54}c_3 \\ &\quad + b_4a_{43}a_{31}) z^6 \\ &- b_5a_{54}a_{31}a_{43}z^8. \end{aligned} \tag{19}$$

In order to develop optimized hybrid method the following relations must hold:

the phase-lag condition,

$$\varphi(z) = z - \cos^{-1}\left(\frac{S(z^2)}{2\sqrt{T(z^2)}}\right) = 0, \tag{20}$$

dissipation condition,

$$d(z) = 1 - \sqrt{T(z^2)} = 0, \tag{21}$$

and the first derivative of (20),

$$\varphi'(z) = 0. \tag{22}$$

The hybrid method in Franco (2006) [2], which is four-stage fifth order with dispersion order eight and dissipation order five or which we called nonzero dissipative method since the order of the dissipation is finite, is then substituted into the nullifying equations (20)–(22), and the equations are solved numerically. The coefficients of the method with a_{41}, a_{42}, and a_{43} taken as free parameters can be written in Butcher tableau as follows (see (26)).

The optimized hybrid method is obtained with a_{41}, a_{42}, and a_{43} given by

$$\begin{aligned} a_{41} = {} & \Big[\left(74646684 - 102396(\cos(z))^2\right)z^6 \\ & + (33746796\cos(z) - 911163492)z^4 \\ & + (33746796\cos(z)\sin(z) \\ & \quad - 911163492\sin(z))z^3 \\ & + \big(-2781245376\cos(z) - 23312031984 \\ & \quad + 134987184(\cos(z))^2\big)z^2 \\ & + (998395776\cos(z)\sin(z) \\ & \quad - 26956685952\sin(z))z \\ & + 107826743808 - 111820326912\cos(z) \\ & + 3993583104(\cos(z))^2\Big] \\ & \cdot\left(\left[6625\left(z^6\left((\sin(z))^2 + 728\right)\right)\right]\right)^{-1}. \end{aligned} \tag{23}$$

$$\begin{aligned} a_{42} = {} & \Big[\left(950688900 - 1304100(\cos(z))^2\right)z^6 \\ & + \big(15918300(\cos(z))^2 - 23208881400 \\ & \quad + 429794100\cos(z)\big)z^4 \\ & + (-11604440700\sin(z) \\ & \quad + 429794100\cos(z)\sin(z))z^3 \\ & + \big(31836600(\cos(z))^3 - 72344216376\cos(z) \\ & \quad + 1687339800(\cos(z))^2 + 96583330152\big)z^2 \\ & + (26956685952\sin(z) \\ & \quad - 998395776\cos(z)\sin(z))z \\ & - 107826743808 - 3993583104(\cos(z))^2 \\ & + 111820326912\cos(z)\Big] \\ & \cdot\left(\left[3125\left(z^6\left((\sin(z))^2 + 728\right)\right)\right]\right)^{-1} \end{aligned} \tag{24}$$

$$\begin{aligned} a_{43} = {} & 27955081728 \\ & \cdot\left(\sin(z)h - 4 + z^2 + 4\cos(z)\right)(\cos(z) - 27) \\ & \cdot\left(165625z^6\left((\sin(z))^2 + 728\right)\right)^{-1}. \end{aligned} \tag{25}$$

An optimized four stage fifth-order hybrid method is as follows:

$$\begin{array}{c|cccc} -1 & & & & \\ 0 & & & & \\ \dfrac{25}{28} & \dfrac{1325}{43904} & \dfrac{35775}{43904} & & \\ -\dfrac{23}{5} & a_{41} & a_{42} & a_{43} & \\ \hline & \dfrac{173}{1908} & \dfrac{2791}{3450} & \dfrac{307328}{3056775} & -\dfrac{125}{636732} \end{array} \tag{26}$$

and the free parameters in Taylor expansion are given by

$$\begin{aligned} a_{41} = {} & \frac{16744}{33125} - \frac{17687}{74200}z^2 + \frac{14096539}{519400000}z^4 \\ & - \frac{977401307}{8638660800000}z^6 - \frac{22049905351}{571722278400000}z^8 \\ & + \frac{1535705806157}{1760904617472000000}z^{10} + O\left(z^{12}\right) \end{aligned} \tag{27}$$

$$\begin{aligned} a_{42} = {} & \frac{383111}{15625} - \frac{53061}{875000}z^2 - \frac{11620359}{245000000}z^4 \\ & + \frac{17015477671}{814968000000}z^6 - \frac{646287207193}{988827840000000}z^8 \\ & - \frac{12921642015637}{830615385600000000}z^{10} + O\left(z^{12}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} a_{43} = {} & -\frac{13866608}{828125} + \frac{247618}{828125}z^2 - \frac{336053}{16562500}z^4 \\ & - \frac{44765797}{55093500000}z^6 - \frac{3154671007}{200540340000000}z^8 \\ & + \frac{3137160877}{2079677600000000}z^{10} + O\left(z^{12}\right). \end{aligned} \tag{29}$$

For the construction of five-stage sixth-order optimized hybrid methods, we substitute (18) into (20)–(22) and using the coefficients of the five-stage sixth-order method which is nonzero dissipative given in Franco (2006) [2] (see (30)).

An optimized five-stage sixth-order hybrid method is as follows:

$$
\begin{array}{c|ccccc}
-1 & & & & & \\
0 & & & & & \\
-\frac{1}{5} & -\frac{4}{125} & -\frac{6}{125} & & & \\
-\frac{2}{5} & -\frac{113}{3000} & -\frac{13}{750} & -\frac{7}{120} & & \\
\frac{2}{3} & \frac{5200}{6561} & a_{52} & a_{53} & a_{54} & \\
\hline
 & \frac{1}{60} & \frac{23}{24} & -\frac{125}{156} & \frac{125}{192} & \frac{729}{4160}
\end{array} \tag{30}
$$

Choosing a_{52}, a_{53}, and a_{54} as the free parameters, the optimized sixth-order hybrid method is obtained:

$$
\begin{aligned}
a_{52} = 5 \Big[& \left(-31528 (\sin(z))^2 - 39410\right) z^{10} \\
& + \left(- 337800 \cos(z) + 756895 + 200156 (\sin(z))^2 \right) z^8 \\
& + \left(30085590 \cos(z) + 6920388 (\sin(z))^2 + 1677312 \cos(z) (\sin(z))^2 - 33332940 \right) z^6 \\
& + (22089600 \cos(z) \sin(z) - 33134400 \sin(z)) z^5 \\
& + \left(20966400 \cos(z) (\sin(z))^2 - 546624000 + 265824000 \cos(z) - 149760000 (\sin(z))^2 \right) z^4 \\
& + (-673920000 \sin(z) + 449280000 \cos(z) \sin(z)) z^3 \\
& + \left(- 2021760000 \cos(z) - 1864512000 (\sin(z))^2 - 44928000 \cos(z) (\sin(z))^2 + 617760000 \right) z^2 \\
& + (2808000000 \cos(z) \sin(z) - 4212000000 \sin(z)) z \\
& - 28080000000 \cos(z) - 11232000000 (\sin(z))^2 + 28080000000 \Big] \\
& \cdot \Big(13122 \Big[z^6 \left(\left(56 (\sin(z))^2 + 70\right) z^4 + \left(-25 + 700 (\sin(z))^2 + 600 \cos(z)\right) z^2 \right. \\
& \left. - 7500 + 3750 \cos(z) - 1500 (\sin(z))^2 \right) \Big] \Big)^{-1}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
a_{53} = -1625 \Big[& \left(-490 - 392 (\sin(z))^2\right) z^{10} \\
& + \left(-5425 - 9380 (\sin(z))^2 - 4200 \cos(z)\right) z^8 \\
& + \left(- 479340 + 272310 \cos(z) - 56700 (\sin(z))^2 \right) z^6 \\
& + (-483840 \sin(z) + 322560 \cos(z) \sin(z)) z^5 \\
& + \left(-11289600 - 1290240 (\sin(z))^2 + 6451200 \cos(z) \right) z^4 \\
& + (9676800 \cos(z) \sin(z) - 14515200 \sin(z)) z^3 \\
& + \left(-38707200 (\sin(z))^2 - 6912000 - 27648000 \cos(z)\right) z^2 \\
& + (-103680000 \sin(z) + 69120000 \cos(z) \sin(z)) z \\
& + 691200000 - 276480000 (\sin(z))^2 - 691200000 \cos(z) \Big] \\
& \cdot \Big(52488 \Big[z^6 \left(\left(56 (\sin(z))^2 + 70\right) z^4 + \left(-25 + 700 (\sin(z))^2 + 600 \cos(z)\right) z^2 \right. \\
& \left. - 7500 + 3750 \cos(z) - 1500 (\sin(z))^2 \right) \Big] \Big)^{-1}
\end{aligned} \tag{32}
$$

$$a_{54} = 650000\left[\left(-5-4(\sin(z))^2\right)z^6 + (-864+576\cos(z))z^4 + (-864\sin(z)+576\cos(z)\sin(z))z^3 + \left(360-2160\cos(z)-2304(\sin(z))^2\right)z^2 + (3600\cos(z)\sin(z)-5400\sin(z))z - 4400(\sin(z))^2+36000-36000\cos(z)\right] \cdot \left(2187\left[z^6\left(\left(56(\sin(z))^2+70\right)z^4 + \left(-25+700(\sin(z))^2+600\cos(z)\right)z^2 - 7500+3750\cos(z) - 1500(\sin(z))^2\right)\right]\right)^{-1} \quad (33)$$

and the free parameters in Taylor expansion are given as follows:

$$a_{52} = \frac{4175}{4374} + \frac{23530}{137781}z^2 + \frac{95173}{4133430}z^4 - \frac{645749}{1364031900}z^6 - \frac{270131383}{358058373750}z^8 - \frac{234666525343}{1718680194000000}z^{10} + O\left(z^{12}\right) \quad (34)$$

$$a_{53} = -\frac{2275}{1944} - \frac{47060}{137781}z^2 - \frac{15977}{413343}z^4 + \frac{82927}{48715425}z^6 + \frac{261389377}{179029186875}z^8 + \frac{2697629171}{11160261000000}z^{10} + O\left(z^{12}\right), \quad (35)$$

$$a_{54} = \frac{5200}{6561} + \frac{23530}{137781}z^2 + \frac{2249}{118098}z^4 - \frac{1666769}{1364031900}z^6 - \frac{280717231}{358058373750}z^8 - \frac{210021906751}{1718680194000000}z^{10} + O\left(z^{12}\right). \quad (36)$$

3.2. Phase-Fitted Hybrid Methods. In this paper, we also develop the phase-fitted the original hybrid methods. To develop phase-fitted method, (20) must hold. Equation (18) is substituted into (20), choosing a_{31} as the free parameter and using the same coefficients as in (26), together with

$$a_{41} = \frac{16744}{33125}, \qquad a_{42} = \frac{383111}{15625}, \qquad a_{43} = -\frac{13866608}{828125}. \quad (37)$$

We obtain a phase-fitted hybrid method of four-stage fifth-order with the solution of a_{31} which is given by

$$a_{31} = -265\left[34726809600(\cos(z))^2 + 34726809600z^2 - 11575603200z^4 + 105369600z^4(\cos(z))^2 - 34726809600 + 3875760z^{10} - 62289z^{12} + 1539968640z^6 - 106798720z^8\right] \cdot \left(\left[39337984z^4(\cos(z))^2\left(-23520+769z^2\right)\right]\right)^{-1} \quad (38)$$

and the Taylor expansion of the free parameter is

$$a_{31} = \frac{1325}{43904} + \frac{529841}{14869757952}z^6 + \frac{67815450029}{3847103777341440}z^8 + \frac{8656887188933513}{11762904509599186944 00}z^{10} + O\left(z^{12}\right). \quad (39)$$

For the construction of five-stage sixth-order phase-fitted hybrid method, we substitute (19) into (20) and set a_{52} as the free parameter. We use the coefficients given in (30) together with

$$a_{53} = -\frac{2275}{1944}, \qquad a_{54} = \frac{5200}{6561}. \quad (40)$$

Choosing a_{52} as the free parameter, we get a phase-fitted hybrid method of five-stage sixth-order and a_{52} is given by

$$a_{52} = \left[-3744000 + 1872000z^2 + 157125z^4 + 5200z^6 - 728z^8 + 4160\sqrt{810000-210z^8}\cos(z)\right] \cdot \left(328050z^4\right)^{-1} \quad (41)$$

and the Taylor expansion of a_{52} is

$$a_{52} = \frac{4175}{4374} - \frac{2353}{688905}z^4 + \frac{15223}{20667150}z^6 - \frac{168103}{2728063800}z^8 + \frac{130789}{63654822000}z^{10} + O\left(z^{12}\right). \quad (42)$$

4. Problems Tested and Numerical Results

In this section, we compare the optimized methods of order five and order six with the phase-fitted methods together

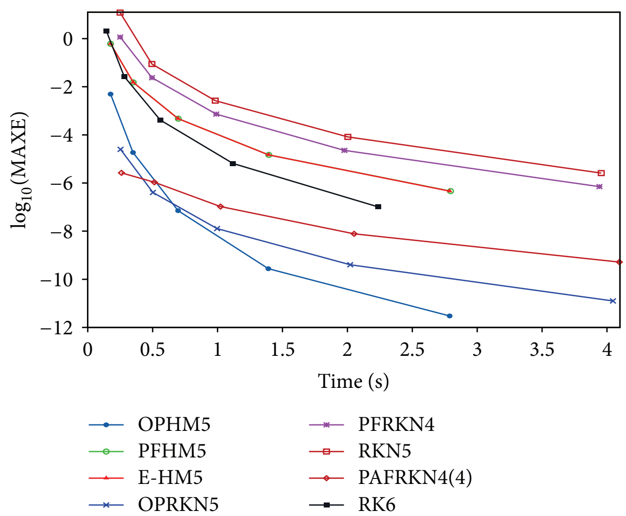

Figure 1: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 2 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

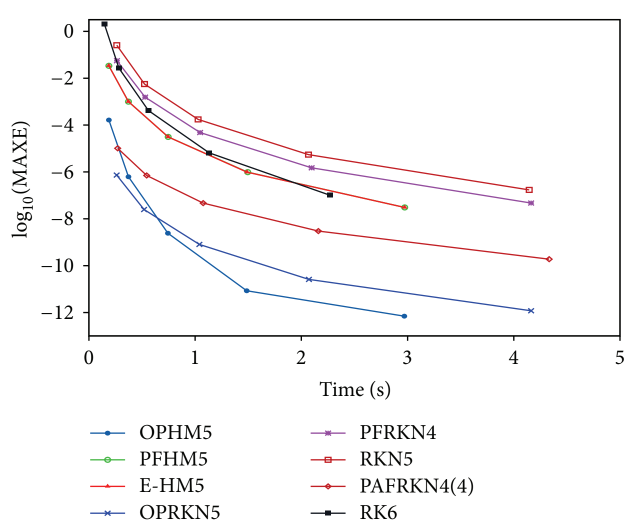

Figure 2: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 3 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

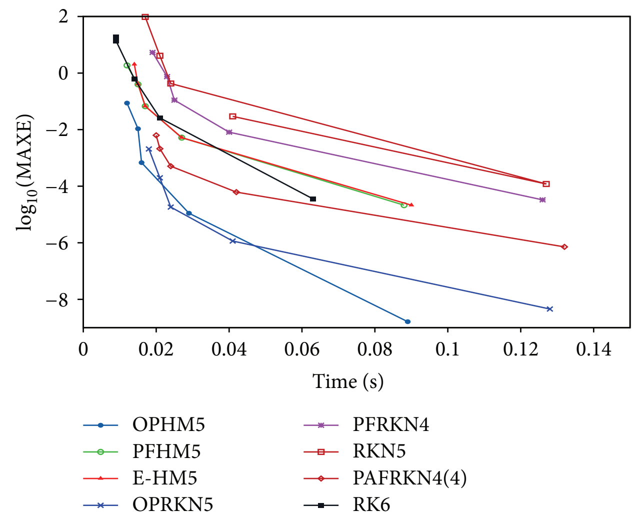

Figure 3: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 4 with $t_{\text{end}} = 10^4$ and $h = 1 - 0.125i$, $i = 1, \ldots, 5$.

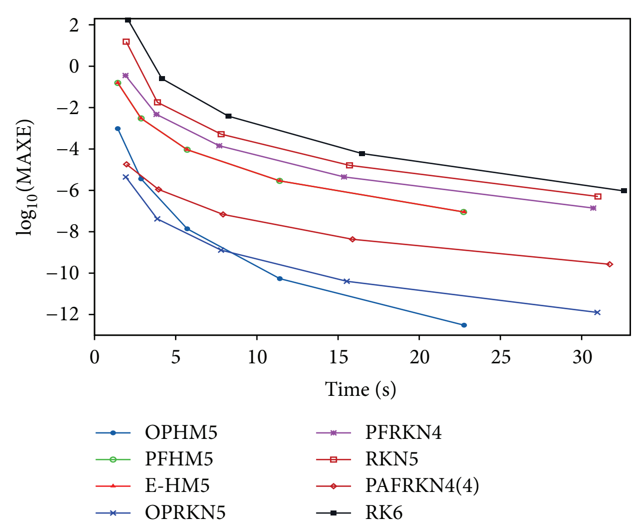

Figure 4: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 5 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 2, \ldots, 6$.

with a few existing methods in the literature in order to determine the accuracy of the new optimized methods. The existing methods are the original hybrid method by Franco [2], optimized Runge-Kutta Nyström method order five developed by Kosti et al. [11], phase-fitted Runge-Kutta Nyström method of order four developed by Papadopoulos et al. [12], Runge-Kutta Nyström method of order five by Hairer et al. [13], Runge-Kutta method of order six given in Butcher [14], and phase-fitted and amplification-fitted Runge-Kutta Nyström method of order four developed in Papadopoulos et al. [12]. Efficiency curves of the fifth and sixth order methods are given in Figures 1–7 and 8–14, respectively. The test problems are listed below.

Problem 2 (Chawla and Rao [15]). Consider

$$y''(t) = -100y(t), \qquad y(0) = 1, \qquad y'(0) = -2, \quad (43)$$

and the fitted frequency, $v = 10$.

Exact solution is $y = -(1/5)\sin(10t) + \cos(10t)$.

Problem 3 (Attili et al. [16]). Consider

$$y''(t) = -64y(t), \qquad y(0) = \frac{1}{4}, \qquad y'(0) = -\frac{1}{2}, \quad (44)$$

and the fitted frequency, $v = 8$. Exact solution is as follows: $y = (\sqrt{17}/16)\sin(8t + \theta)$ and $\theta = \pi - \tan^{-1}(4)$.

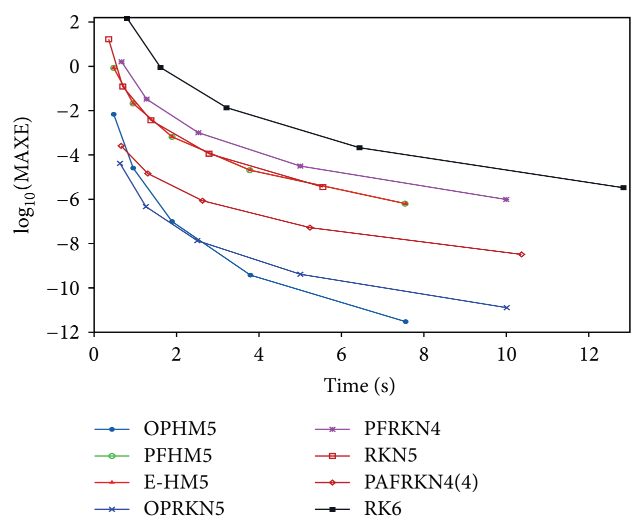

FIGURE 5: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 6 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

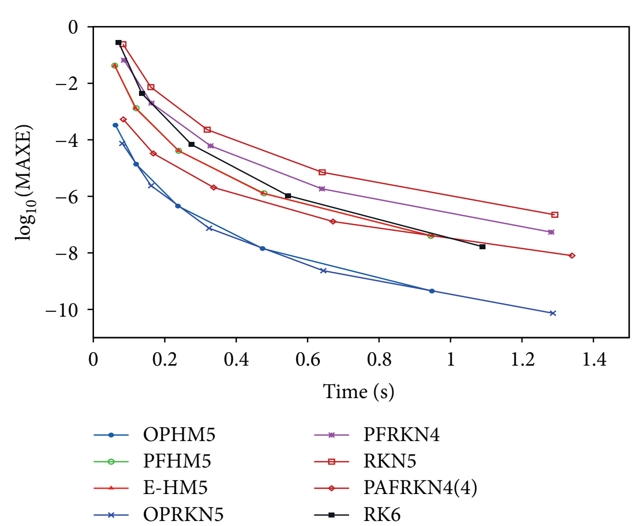

FIGURE 6: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 7 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

Problem 4 (Allen Jr. and Wing [17]). Consider

$$y''(t) = -y(t) + t, \qquad y(0) = 1, \qquad y'(0) = 2, \tag{45}$$

and the fitted frequency, $v = 1$. Exact solution is $y = \sin(t) + \cos(t) + t$.

Problem 5 (Lambert and Watson [18]). Consider

$$\frac{d^2 y_1(t)}{dt^2} = -v^2 y_1(t) + v^2 f(t) + f''(t)$$
$$\frac{d^2 y_2(t)}{dt^2} = -v^2 y_2(t) + v^2 f(t) + f''(t)$$

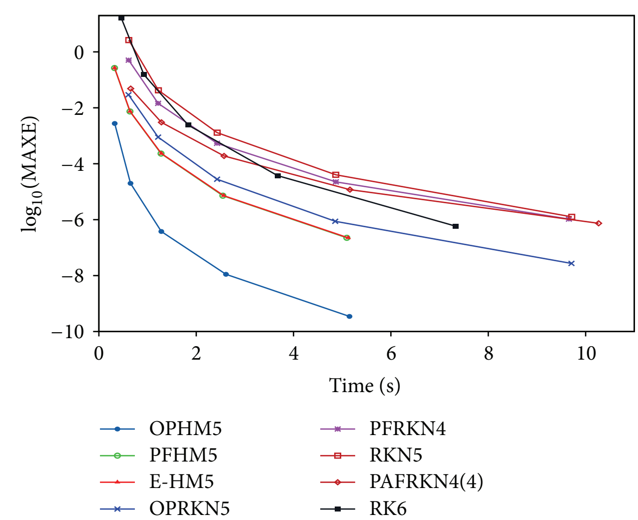

FIGURE 7: The efficiency curves of the optimized and phase-fitted order five methods and their comparisons for Problem 8 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

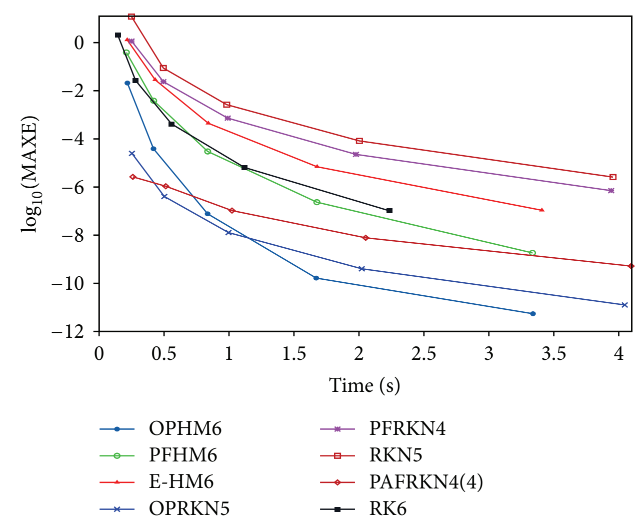

FIGURE 8: The efficiency curves of the optimized order six method and their comparisons for Problem 2 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i$, $i = 1, \ldots, 5$.

$$y_1(0) = a + f(0), \qquad y_1'(0) = f'(0),$$
$$y_2(0) = f(0), \qquad y_2'(0) = va + f'(0). \tag{46}$$

Exact solution is as folllows: $y_1(t) = a\cos(vt) + f(t)$, $y_2(t) = a\sin(vt) + f(t)$, $f(t)$ is chosen to be $e^{-0.05t}$, and parameters v and a are 20 and 0.1, respectively.

Problem 6 (Papadopoulos et al. [19]). Consider

$$y''(t) = -v^2 y(t) + \left(v^2 - 1\right)\sin(t),$$
$$y(0) = 1, \qquad y(0) = v + 1, \qquad v = 10. \tag{47}$$

The analytical solution is $y(t) = \cos(vt) + \sin(vt) + \sin(t)$.

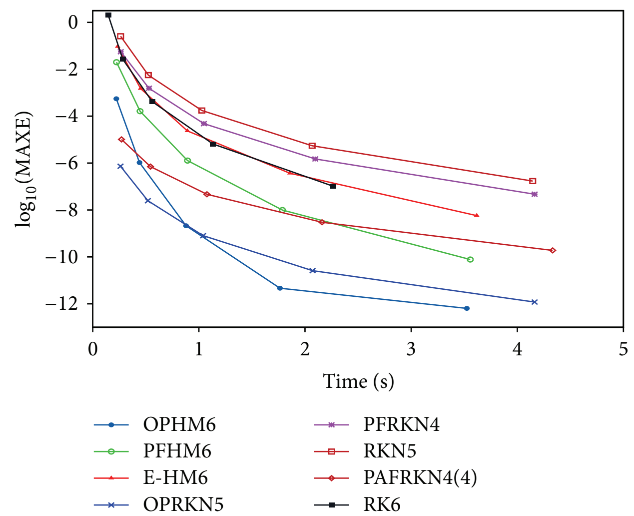

FIGURE 9: The efficiency curves of the optimized order six method and their comparisons for Problem 3 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i, i = 1, \ldots, 5$.

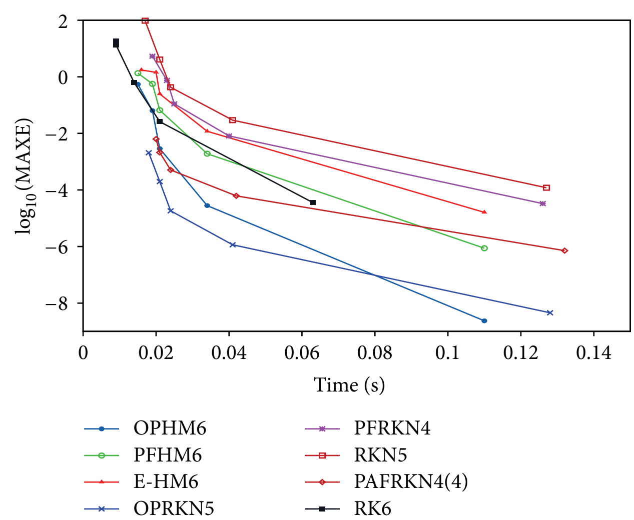

FIGURE 10: The efficiency curves of the optimized order six method and their comparisons for Problem 4 with $t_{\text{end}} = 10^4$ and $h = 1 - 0.125i, i = 1, \ldots, 5$.

Problem 7 (Stiefel and Bettis [20]). Consider

$$\begin{aligned} y_1''(t) + y_1(t) &= 0.001\cos(t), \\ y_1(0) = 1, \quad y_1'(0) &= 0, \\ y_2''(t) + y_2(t) &= 0.001\sin(t), \\ y_2(0) = 0, \quad y_2'(0) &= 0.9995, \\ \text{for } v &= 1. \end{aligned} \tag{48}$$

Exact solution is

$$\begin{aligned} y_1 &= \cos(t) + 0.0005t\sin(t) \\ y_2 &= \sin(t) - 0.0005t\cos(t). \end{aligned} \tag{49}$$

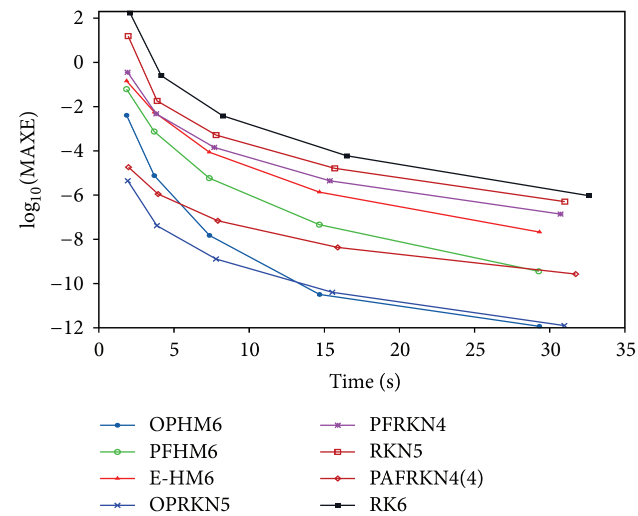

FIGURE 11: The efficiency curves of the optimized order six method and their comparisons for Problem 5 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i, i = 2, \ldots, 6$.

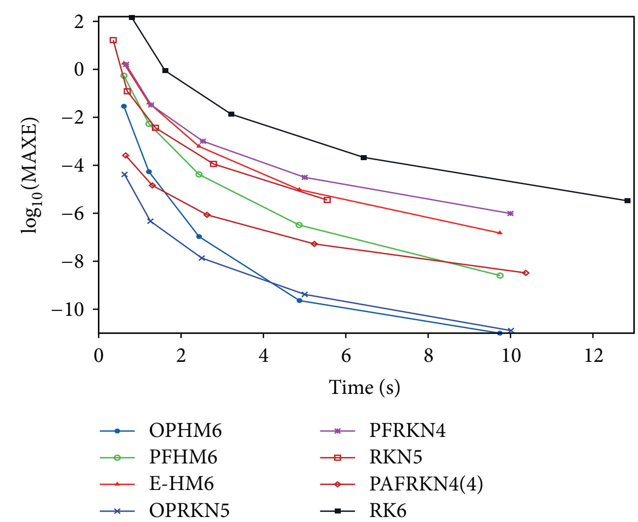

FIGURE 12: The efficiency curves of the optimized order six method and their comparisons for Problem 6 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i, i = 1, \ldots, 5$.

Problem 8 (Franco [2]). Consider

$$\begin{aligned} y''(t) + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} \qquad y(t) &= \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}, \\ y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad y'(0) &= \begin{pmatrix} -4 \\ 8 \end{pmatrix} \\ g_1(t) &= 9\cos(2t) - 12\sin(2t), \\ g_2(t) &= -12\cos(2t) + 9\sin(2t) \end{aligned} \tag{50}$$

and whose analytic solution is given by

$$y(t) = \begin{pmatrix} \sin(t) - \sin(5t) + \cos(2t) \\ \sin(t) + \sin(5t) + \sin(2t) \end{pmatrix} \quad \text{for } v = 5. \tag{51}$$

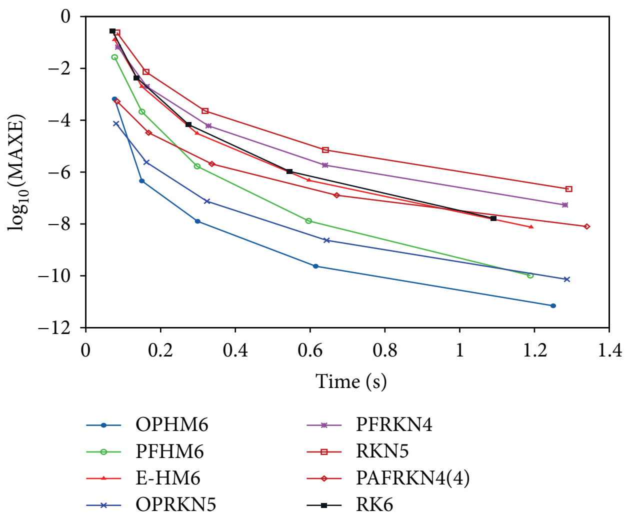

FIGURE 13: The efficiency curves of the optimized order six method and their comparisons for Problem 7 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i, i = 1, \ldots, 5$.

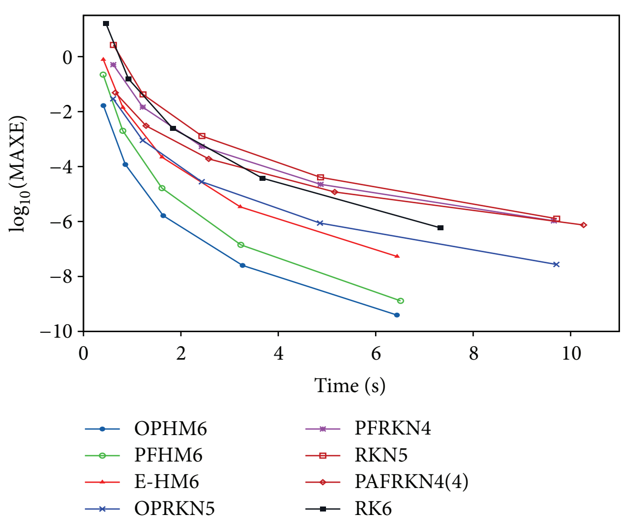

FIGURE 14: The efficiency curves of the optimized order six method and their comparisons for Problem 8 with $t_{\text{end}} = 10^4$ and $h = 0.125/2^i, i = 1, \ldots, 5$.

The following notations are used in Figures 1–14:

(i) *OPHM5*: new optimized hybrid method four-stage fifth order derived in this paper;

(ii) *OPHM6*: new optimized hybrid method five-stage sixth order derived in this paper;

(iii) *PFHM5*: new phase-fitted hybrid method four-stage fifth order derived in this paper;

(iv) *PFHM6*: new phase-fitted hybrid method five-stage sixth order derived in this paper;

(v) *E-HM5*: explicit hybrid method of four-stage fifth-order with dispersion of order eight and dissipation of order five developed by Franco [2];

(vi) *E-HM6*: explicit hybrid method of five-stage sixth-order developed by Franco [2];

(vii) *OPRKN5*: Optimized Runge-Kutta Nyström method of four-stage fifth-order developed by Kosti et al. [11];

(viii) *PFRKN4*: Phase-fitted Runge-Kutta Nyström method of four-stage fourth-order by Papadopoulos et al. [19];

(ix) *RKN5*: A classical Runge-Kutta Nyström method order five in Hairer et al. [13].

(x) *RK6*: A Runge-Kutta method of order six with seven stages by Butcher [14].

(xi) *PAFRKN4(4)*: A four-stage fourth-order phase-fitted and amplification-fitted Runge-Kutta Nyström method developed by Papadopoulos et al. [12].

A measure of the accuracy is examined using absolute error which is defined by

$$\text{Absolute error} = \max \{ |y(x_n) - y_n| \}, \tag{52}$$

where $y(x_n)$ is the exact solution and y_n is the computed solution.

From the observation on efficiency curves in Figures 1–5, it is shown that OPHM5 is more accurate compared to other methods in the literature followed by OPRKN5 and then PAFRKN4(4) method. However, for Problem 7, OPHM5 is just as good as OPRKN5, but still OPHM5 required less time to integrate the problem till the end of the interval, while for Problem 8, OPHM5 is the most accurate method followed by PFHM5 or E-HM5 and then OPRKN5. From the efficiency curves we noticed that the phase-fitted version of the hybrid method did not improve the accuracy of the method, whereas optimized method improved the accuracy of the method; hence it performed much better than the original hybrid method and other existing methods.

For method of order six the efficiency curves are shown in Figures 8–14. In Figures 8 and 9, OPHM6 is better in accuracy compared to OPRKN5 followed closely by PAFRKN4(4) and PFHM6. In Figures 10–12, as the time increases OPHM6 is gradually better in accuracy compared to OPRKN5 followed by PAFRKN4(4), PFHM6, and the rest of the methods. In **Figure** 13, again OPHM6 is the most efficient method followed by OPRKN5 PFHM6 and PAFRKN4(4) and other methods in the literature, while in **Figure** 4, the most efficient method is OPHM6, followed by PFHM6, E-HM6, and OPRKN5 methods. Consequently, examining the results in Figures 8–14 the optimized sixth order method is very efficient in solving oscillating IVPs. Phase-fitted method improved the accuracy of the method slightly compared to the original hybrid method but could not compete with the optimized method.

5. Conclusion

In this paper, optimized hybrid methods of order five and order six are constructed based on the existing hybrid methods which have high phase-lag or dispersion order but are not zero-dissipative, originally developed by Franco [2]. Phase-fitted methods based on the same existing methods are also constructed, and numerical results are shown in Figures 1–14. From the numerical results we conclude that the new optimized methods are more efficient for integrating oscillatory initial value problems of second order ODEs compared to the phase-fitted methods and other well-known existing methods in the scientific literature.

Furthermore, from the numerical results, it is proven that in solving oscillating special second order ODEs, the nonzero dissipative hybrid methods are more suitable to be optimized than phase-fitted methods. The results are not quite true for zero-dissipative methods (see Ahmad et al. [21]) in which it is found that the phase-fitted method is very efficient in solving oscillating problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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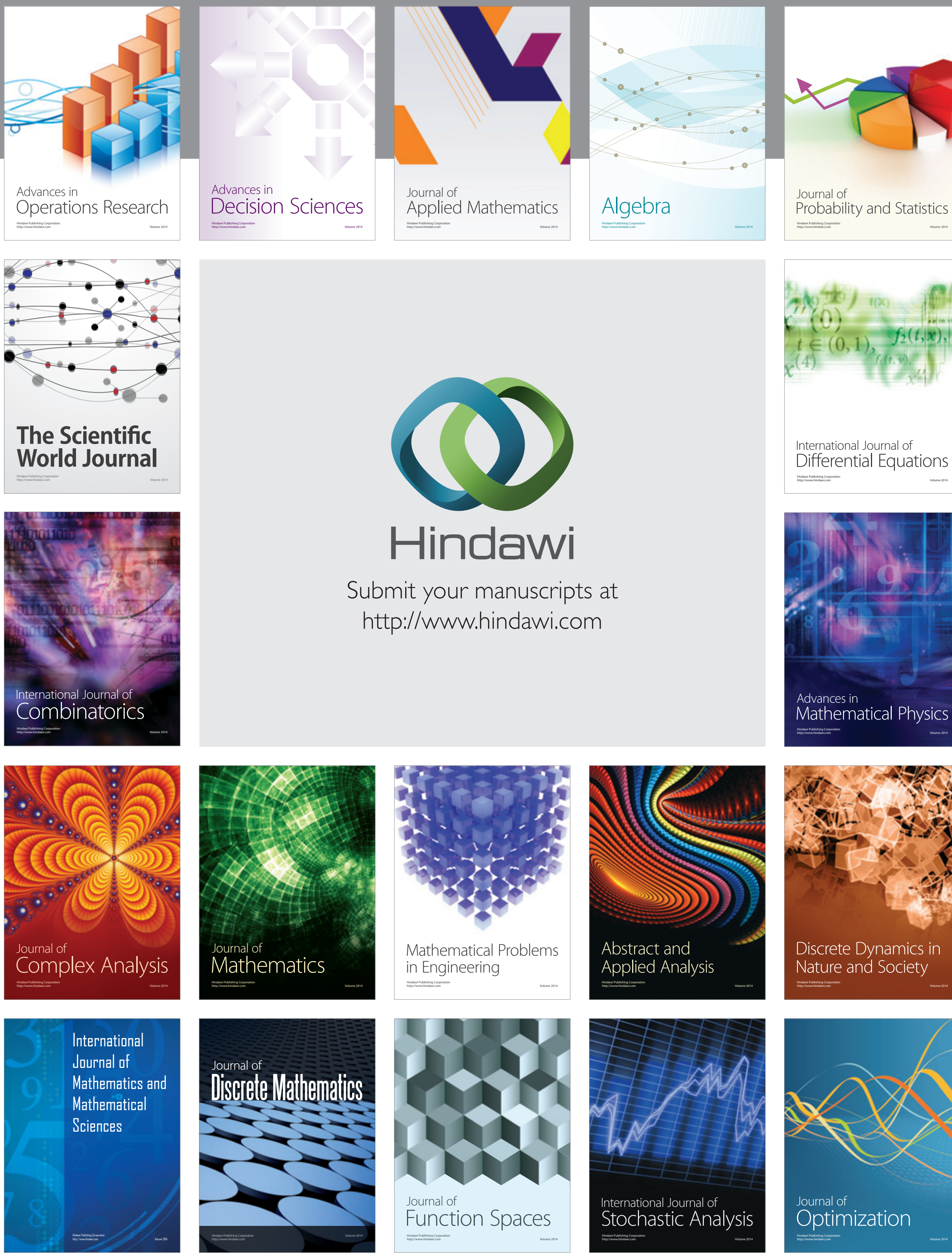
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