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Research Article

On Global Attractors for a Class of Reaction-Diffusion Equations on Unbounded Domains with Some Strongly Nonlinear Weighted Term

Jin Zhang[1] and Chengkui Zhong[2]

[1] *Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China*
[2] *Department of Mathematics, Nanjing University, Nanjing 210093, China*

Correspondence should be addressed to Chengkui Zhong; ckzhong@nju.edu.cn

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We consider the existence and properties of the global attractor for a class of reaction-diffusion equation $\partial u/\partial t - \Delta u - u + \kappa(x)|u|^{p-2}u + f(u) = 0$, in $\mathbb{R}^n \times \mathbb{R}^+$; $u(x,0) = u_0(x)$, in $\mathbb{R}^n$. Under some suitable assumptions, we first prove that the problem has a global attractor $\mathcal{A}$ in $L^2(\mathbb{R}^n)$. Then, by using the Z_2-index theory, we verify that $\mathcal{A}$ is an infinite dimensional set and it contains infinite distinct pairs of equilibrium points.

1. Introduction

In this paper, we are mainly concerned with the long-time behaviour of solutions for the following reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - \Delta u - u + \kappa(x)|u|^{p-2}u + f(u) = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$
$$u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $p > 2$ and $\kappa(x) \geq \alpha_0 > 0$ satisfies

$$\int_{\mathbb{R}^n} \left(\frac{1}{\kappa(x)}\right)^{2/(p-2)} dx < \infty \tag{2}$$

(e.g., $\kappa(x) = (1+|x|)^r$ with $r > (p-2)n/2$). The function $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following assumptions:

(f_1) $f(s)$ is odd; that is, $f(-s) = -f(s)$, for all $s \in \mathbb{R}$;

(f_2) there exists a constant α, $0 < \alpha < p-1$, such that

$$\sup \lim_{|s|\to\infty} \frac{|f(s)|}{|s|^\alpha} = 0,$$
$$\sup \lim_{|s|\to 0} \frac{|f(s)|}{|s|} = 0; \tag{3}$$

(f_3) there exists a positive constant l such that $f'(s) \geq -l$;

(f_4) there exists a positive constant β such that $0 \leq \beta F(s) \leq f(s)s$, where

$$F(s) := \int_0^s f(s)\,ds. \tag{4}$$

As we know, the basic problems to consider the long-time behaviour of solutions for the above equation are to prove the existence of global attractors for the semigroup of solutions and discuss some properties of the global attractors, such as the dimension property and the existence of multiple equilibrium points.

Those problems for the equations in bounded domains have been studied extensively by many authors and have been rather well understood; see, for example, [1–7]. However, the solution for the equation is different in unbounded domain. The main difference is the fact that, in contrast to the case of bounded domains, the global attractors for the reaction-diffusion equations in unbounded domains admit finite dimension under some specific assumptions and infinite dimension under general assumptions.

For the kind of equation

$$\frac{du}{dt} - \Delta u + f(x, u) = h \quad (5)$$

in unbounded domains. In pioneering work [8], the authors used weighted spaces instead of the usual spaces to prove the existence of the global attractors; further details can be found in [9–11]. In [12–15], the authors have developed some new ideas and methods to deal with more general cases in unbounded domains, including uniformly local Sobolev spaces, locally compact attractor, and the so-called entropy theory, and have obtained the existence of the locally compact global attractors for the semigroups associated with the equations. Under some structural assumptions on the term f (i.e., $f(x,u)u \geq 0$ or $f(x,u)u \geq \alpha|u|^p + \psi(x)$), the authors in [16, 17] prove the existence of global attractor for the equation in unbounded domain in usual space $L^2(\mathbb{R}^n)$.

On the other hand, we have noticed that the Z_2-index is a powerful method to find multiple critical points of some even functional. The authors in [18] used the Z_2-index to obtain the existence of infinite dimensional global attractor for a class of p-Laplacian equation in bounded domain, for which $p > 2$ is necessary. Additional information about other attractor problems can be found in [19–23].

Motivated by the above papers, in this paper, we are interested in finding a semigroup associated with a reaction-diffusion equation in unbounded domain, such that the semigroup has a global attractor in the usual space; furthermore the dimension of the global attractor is infinite.

The main results of this paper can be stated as follows.

Theorem 1. *Assuming that $n \geq 3$, $p > 2$, and $\kappa(x) \geq \alpha_0 > 0$ satisfies condition (2) and the nonlinear term f satisfies $(f_1) \sim (f_4)$, then reaction-diffusion equation (1) has a global attractor $\mathcal{A}$ in $L^2(\mathbb{R}^n)$.*

Theorem 2. *Assume that $n \geq 3$, $p > 2$, and $\kappa(x) \geq \alpha_0 > 0$ satisfies condition (2) and the nonlinear term f satisfies $(f_1) \sim (f_4)$. Let $\mathcal{A}$ be the global attractor of (1). Then, for any $m \in \mathbb{N}^+$, there exists a neighborhood $\mathcal{O}(0)$ of the origin, such that $\gamma(\mathcal{A} \setminus \mathcal{O}(0)) \geq m$, where $\gamma(\mathcal{A} \setminus \mathcal{O}(0))$ denotes the Z_2-index of the set $\mathcal{A} \setminus \mathcal{O}(0)$.*

We recall that, from [24], any compact set E, with fractal dimension $\dim(E) = n$, can be mapped into spaces $\mathbb{R}^{2n+1}$ by a linear odd Hölder continuous one-to-one projector. Thus, we obtained the following corollary.

Corollary 3. *Under the assumptions of Theorem 2, the fractal dimension of the global attractor $\mathcal{A}$ is infinite.*

Theorem 4. *Assume that $n \geq 3$, $p > 2$, and $\kappa(x) \geq \alpha_0 > 0$ satisfies condition (2) and the nonlinear term f satisfies $(f_1) \sim (f_4)$. Let $\mathcal{A}$ be the global attractor of (1). Then, $\mathcal{A}$ contains infinite distinct pairs of equilibrium points.*

The proofs of the above theorems are, respectively, given in Sections 3 and 4. Some preliminaries and associate lemmas can be found in Section 2.

2. Some Preliminaries

Initially, backgrounds about global attractors and Z_2-index theory are reviewed. Proofs are then given for the lemmas and the existence of solution for (1).

In this paper, we define the following space:

$$\begin{aligned} H^1_\kappa(\mathbb{R}^n) &\triangleq D^{1,2}(\mathbb{R}^n) \cap L^p_\kappa(\mathbb{R}^n) \\ &= \left\{u \mid \|u\|_{p,\kappa} + \|u\|_{D^{1,2}} < +\infty\right\}, \end{aligned} \quad (6)$$

with the corresponding norm

$$\|u\|_{H^1_\kappa} = \|u\|_{p,\kappa} + \|u\|_{D^{1,2}}, \quad (7)$$

where

$$\begin{aligned} \|u\|_{D^{1,2}} &= \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right)^{1/2}, \\ \|u\|_{p,\kappa} &= \left(\int_{\mathbb{R}^n} \kappa(x) |u|^p \, dx\right)^{1/p}. \end{aligned} \quad (8)$$

Let V be a Banach space, and define $\Sigma = \{A \subset V \mid A \text{ closed}, A = -A\}$ as the class of closed symmetric subsets of V. Based on this, the formal definition of Z_2-index can be given.

Definition 5 (see [25]). Let $A \in \Sigma$, $A \neq \emptyset$. The Z_2-index or Krasnoselskii genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \begin{cases} \inf\left\{m : \exists h \in C^0(A, \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\right\}, \\ \infty, \quad \text{if } \left\{m : \exists h \in C^0(A, \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\right\} = \emptyset. \end{cases} \quad (9)$$

In particular, if $0 \in A$, $\gamma(A) = \infty$, then define $\gamma(\emptyset) = 0$.

The properties of Z_2-index $\gamma(A)$ are provided in the following lemma.

Lemma 6 (see [25]). *Let $h \in C^0(V, V)$ be an odd map and $A, A_1, A_2 \in \Sigma$. Then the Z_2-index $\gamma(A)$ on V satisfies the following properties:*

(A_1) $\gamma(A) \geq 0$, $\gamma(A) = 0 \Leftrightarrow A = \emptyset$;

(A_2) $A_1 \subset A_2 \Rightarrow \gamma(A_1) \leq \gamma(A_2)$;

(A_3) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$;

(A_4) $\gamma(A) \leq \gamma(\overline{h(A)})$, *for all $A \in \Sigma$, and $h : V \to V$ is odd and continuous;*

(A_5) *If $A \in \Sigma$ is compact and $0 \notin A$, then $\gamma(A) < \infty$ and there exists a neighborhood $\mathcal{N}$ of A in V such that $\overline{\mathcal{N}} \in \Sigma$ and $\gamma(A) = \gamma(\overline{\mathcal{N}})$;*

(A_6) *For any bounded symmetric neighborhood Ω of the origin in $\mathbb{R}^n$ there holds $\gamma(\partial\Omega) = n$.*

Applying the index theory to an even functional E on some Banach space V, we can obtain a sequence of minimax values. Moreover, if E satisfies the $(P.S.)$ condition, the sequence of minimax values must be the critical values of the functional E.

Definition 7 (see [25]). Let V be a Banach space, $E \in C^1(V, \mathbb{R})$, and $d \in \mathbb{R}$. The functional E is said to satisfy the $(P.S.)$ condition if any sequence $\{u_n\} \subset V$ such that

$$\{E(u_n)\} \text{ is bounded and } E'(u_n) \longrightarrow 0 \tag{10}$$

has a convergent subsequence.

Lemma 8 (see [26]). *Suppose V is a Banach space and suppose $E \in C^1(V, \mathbb{R})$, $E(u) = E(-u)$ for all u. E satisfies the following conditions:*

(B_1) *there exists a subspace $V_r \subset V$ with $\dim V_r = r$ and $\rho > 0$, such that*

$$\sup_{u \in V_r \cap S_\rho} E(u) < E(0), \tag{11}$$

where $S_\rho = \{u \in V \mid \|u\| = \rho\}$;

(B_2) *there exists a closed subspace $W \subset V$ with $\operatorname{codim} W = s$, such that*

$$\inf_{u \in W} E(u) > -\infty; \tag{12}$$

(B_3) *$E(u)$ satisfies the $(P.S.)$ condition.*

Then if $r > s$, the functional E possesses at least $r - s$ pairs of critical points.

Following the proof in [8], we will prove the existence of a unique weak solution of (1) for any initial data $u_0 \in L^2(\mathbb{R}^n)$.

Firstly, we consider the problem in the bounded domain. We denote $\Omega_R = \{x \in \mathbb{R}^n : |x| < R\}$ and the function $\Psi_R \in C^\infty(\mathbb{R}^n)$ with $0 \le \Psi_R(x) \le 1$, $|\Psi_R'(x)| \le 2$, satisfying

$$\Psi_R(x) = \begin{cases} 1, & |x| \le R-1, \\ 0, & |x| \ge R. \end{cases} \tag{13}$$

It is well known that (see, e.g., [5, 6])

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u - u + \kappa(x)|u|^{p-2}u + f(u) = 0 \\ &\qquad\qquad \text{in } \Omega_R \times \mathbb{R}^+, \\ &u(0) = u_{0,R} \quad \text{in } \Omega_R, \\ &u(t)|_{\partial\Omega_R} = 0 \\ &\qquad\qquad t \in (0, \infty), \end{aligned} \tag{14}$$

has a unique solution u_R, where $u_{0,R} = \Psi_R u_0$. And for every $T > 0$ it satisfies

$$\begin{aligned} u_R \in\ & C\left([0,T], L^2(\Omega_R)\right) \cap L^2\left(0, T, H_0^1(\Omega_R)\right) \\ &\cap L^p\left(0, T, L_\kappa^p(\Omega_R)\right). \end{aligned} \tag{15}$$

The following lemmas give some estimates for solution u_R of the bounded problem (14).

Lemma 9. *Let $n \ge 3$, $p > 2$, $\kappa(x) \ge \alpha_0 > 0$ satisfy condition (2), and f satisfies $(f_1) \sim (f_4)$; let u_R be a solution of problem (14). Then, for any $T > 0$, the following estimates hold:*

$$\int_{\Omega_R} |u_R(t)|^2\,dx \le C_1, \qquad t \in [0,T], \tag{16}$$

$$\int_0^T \|u_R\|_{H_0^1(\Omega_R)}^2\,dt + \int_0^T \|u_R\|_{L_\kappa^p(\Omega_R)}^p\,dt \le C_2, \tag{17}$$

where the constants C_1, C_2 depend on data T, $u_{0,R}$, and κ but are independent of R.

Proof. Firstly, for any $u \in L_\kappa^p(\Omega_R)$, utilizing Hölder inequality, we have

$$\begin{aligned} \int_{\Omega_R} |u|^2\,dx &= \int_{\Omega_R} \left(\frac{1}{\kappa}\right)^{2/p} \cdot \kappa^{2/p}|u|^2\,dx \\ &\le \left(\int_{\Omega_R} \left(\frac{1}{\kappa}\right)^{2/p \cdot p/(p-2)} dx\right)^{(p-2)/p} \cdot \left(\int_{\Omega_R} \kappa|u|^p\,dx\right)^{2/p} \\ &\le \left(\int_{\mathbb{R}^n} \left(\frac{1}{\kappa}\right)^{2/(p-2)} dx\right)^{(p-2)/p} \cdot \left(\int_{\Omega_R} \kappa|u|^p\,dx\right)^{2/p} \\ &= M_1(\kappa)\|u\|_{L_\kappa^p(\Omega_R)}^2. \end{aligned} \tag{18}$$

Then utilizing Young's inequality, we have

$$\int_{\Omega_R} |u|^2\,dx \le M_2(\kappa) + \frac{1}{p}\|u\|_{L_\kappa^p(\Omega_R)}^p, \tag{19}$$

where

$$\begin{aligned} M_1(\kappa) &= \left(\int_{\mathbb{R}^n} \left(\frac{1}{\kappa}\right)^{2/(p-2)} dx\right)^{(p-2)/p}, \\ M_2(\kappa) &= 2^{2/p} \cdot \frac{p-2}{p} \int_{\mathbb{R}^n} \left(\frac{1}{\kappa}\right)^{2/(p-2)} dx. \end{aligned} \tag{20}$$

Multiplying (14) by u_R and integrating over Ω_R, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R}|u_R|^2\,dx+\int_{\Omega_R}|\nabla u_R|^2\,dx-\int_{\Omega_R}|u_R|^2\,dx+\int_{\Omega_R}\kappa|u_R|^p\,dx+\int_{\Omega_R}f(u_R)\,u_R dx\le 0, \tag{21}$$

and it follows from (19) and $f(u_R)u_R\ge 0$ that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R}|u_R|^2\,dx+\int_{\Omega_R}|\nabla u_R|^2\,dx+\frac{p-1}{p}\int_{\Omega_R}\kappa|u_R|^p\,dx\le M_2(\kappa). \tag{22}$$

Integrating t between 0 and T yields

$$\frac{1}{2}\|u_R(T)\|^2_{L^2(\Omega_R)}-\frac{1}{2}\|u_{0,R}\|^2_{L^2(\Omega_R)}+\int_0^T\|u_R\|^2_{H_0^1(\Omega_R)}dt+\frac{p-1}{p}\int_0^T\|u_R\|^p_{L^p_\kappa(\Omega_R)}dt\le TM_2(\kappa). \tag{23}$$

It follows that

$$\int_0^T\|u_R\|^2_{H_0^1(\Omega_R)}dt+\frac{p-1}{p}\int_0^T\|u_R\|^p_{L^p_\kappa(\Omega_R)}dt\le TM_2(\kappa)+\frac{1}{2}\|u_{0,R}\|^2_{L^2(\Omega_R)}, \tag{24}$$

which implies second estimate (17).

On the other hand, it follows from (21) and $f(u_R)u_R\ge 0$ that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R}|u_R|^2\,dx\le\int_{\Omega_R}|u_R|^2\,dx. \tag{25}$$

Referring to Gronwall's inequality, first estimate (16) can be easily obtained. □

Lemma 10. *Let $n\ge 3$, $p>2$, and $\kappa(x)\ge\alpha_0>0$ satisfy condition (2), and f satisfies $(f_1)\sim(f_4)$; let u_R be a solution of problem (14). Then, for any $T>0$, the following estimate holds:*

$$\left\|\frac{\partial u_R}{\partial t}\right\|_{L^{p/(p-1)}(0,T,(H_\kappa^1(\Omega_R))^*)}\le C_3, \tag{26}$$

where $(H_\kappa^1(\Omega_R))^*\triangleq(H_0^1(\Omega_R)\cap L^p_\kappa(\Omega_R))^*$ and the constant C_3 depends on data T, $u_{0,R}$, and κ but independent of R.

Proof. For any $v\in H_0^1(\Omega_R)\cap L^p_\kappa(\Omega_R)$, we have

$$\left|\left\langle\frac{\partial u_R}{\partial t},v\right\rangle\right|\le\int_{\Omega_R}|\nabla u_R|\cdot|\nabla v|\,dx+\int_{\Omega_R}|u_Rv|\,dx+\int_{\Omega_R}\kappa\left|u_R^{p-1}v\right|dx+\int_{\Omega_R}f(u_R)\,v\,dx. \tag{27}$$

By (f_2) we find that

$$|f(u)|\le C\left(|u|+|u|^{p-1}\right), \tag{28}$$

so we get

$$\begin{aligned}\int_{\Omega_R}f(u_R)\,v\,dx&\le C\int_{\Omega_R}|u_Rv|\,dx+C\int_{\Omega_R}\left|u_R^{p-1}v\right|dx\\&\le C\int_{\Omega_R}|u_Rv|\,dx+\frac{C}{\alpha_0}\int_{\Omega_R}\kappa\left|u_R^{p-1}v\right|dx.\end{aligned} \tag{29}$$

Applying the Hölder inequality to each term, it follows that

$$\begin{aligned}&\int_{\Omega_R}|\nabla u_R|\cdot|\nabla v|\,dx\le\|u_R\|_{H_0^1(\Omega_R)}\|v\|_{H_0^1(\Omega_R)},\\&\int_{\Omega_R}|u_Rv|\,dx\le\|u_R\|_{L^2(\Omega_R)}\|v\|_{L^2(\Omega_R)}\le M_1^2(\kappa)\|u_R\|_{L^p_\kappa(\Omega_R)}\|v\|_{L^p_\kappa(\Omega_R)},\\&\int_{\Omega_R}\kappa\left|u_R^{p-1}v\right|dx\le\|u_R\|^{p-1}_{L^p_\kappa(\Omega_R)}\|v\|_{L^p_\kappa(\Omega_R)}.\end{aligned} \tag{30}$$

Substituting into inequality (27), there exists a constant $\widetilde{C}_3>0$, such that

$$\left\|\frac{\partial u_R}{\partial t}\right\|_{(H_\kappa^1(\Omega_R))^*}\le\widetilde{C}_3\left(\|u_R\|_{H_0^1(\Omega_R)}+\|u_R\|_{L^p_\kappa(\Omega_R)}+\|u_R\|^{p-1}_{L^p_\kappa(\Omega_R)}\right). \tag{31}$$

Then, referring to Lemma 9, the estimate

$$\left(\int_0^T\left\|\frac{\partial u_R}{\partial t}\right\|^{p/(p-1)}_{(H_\kappa^1(\Omega_R))^*}dt\right)^{(p-1)/p}\le C_3 \tag{32}$$

yields the conclusion. □

It is worth noting that both estimates in Lemmas 9 and 10 are independent of R, so we let $R\to+\infty$, providing the existence and uniqueness of the solution of problem (1). Before giving the proof of the existence theorem, we first state the following two lemmas.

Lemma 11 (see [5]). *Let $V\subset\subset H\subset Y$ be Banach spaces, with V reflexive. Suppose that $\{u_n\}$ is a sequence uniformly bounded in $L^2(0,T,V)$ and $\{du_n/dt\}$ is uniformly bounded in $L^p(0,T,Y)$, for $p>1$. Then there is a subsequence that converges strongly in $L^2(0,T,H)$.*

Lemma 12 (see [27]). *Let $x,y\in\mathbb{R}^N$, $p\ge 2$, and $\langle\cdot,\cdot\rangle$ be the standard scalar product in $\mathbb{R}^N$. Then, there exists a constant $\alpha>0$ such that*

$$\left\langle|x|^{p-2}x-|y|^{p-2}y,x-y\right\rangle\ge\alpha|x-y|^p. \tag{33}$$

Theorem 13. *Let $n\ge 3$, $p>2$, $\kappa(x)\ge\alpha_0>0$ satisfy condition (2), and f satisfies $(f_1)\sim(f_4)$, then for any $u_0\in L^2(\mathbb{R}^n)$ and*

$T > 0$, there exists a unique weak solution $u(x,t)$ of (1) which satisfies

$$u \in C\left([0,T], L^2(\mathbb{R}^n)\right) \cap L^2\left(0,T,D^{1,2}(\mathbb{R}^n)\right) \cap L^p\left(0,T,L^p_\kappa(\mathbb{R}^n)\right). \tag{34}$$

Furthermore, $u_0 \mapsto u(t)$ is continuous on $L^2(\mathbb{R}^n)$.

Proof. We choose R_k such that $R_k \to +\infty$ as $k \to +\infty$. Denote that u_{R_k} $(k = 1,2,\ldots)$ are the solutions of boundary problem (14) in the domain Ω_{R_k}.

Now, we extend the functions u_{R_k} $(k = 1,2,\ldots)$ from $L^2(\Omega_R)$ into $L^2(\mathbb{R}^n)$. For each $k \in \mathbb{N}$, define the function u_{R_k} as zero for $|x| \geq R_k$ and multiply by $\Psi_{R_k}(x)$, where $\Psi_{R_k}(x)$ is defined by (13). For simplicity, we denote $u_k = \Psi_{R_k} u_{R_k}$ by the extended functions and $\Omega_k \triangleq \Omega_{R_k}$. Since

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u_k|^2 dx = \int_{\mathbb{R}^n} \left|\Psi_{R_k} u_{R_k}\right|^2 dx \leq \int_{\Omega_k} \left|u_{R_k}\right|^2 dx, \\
&\int_{\mathbb{R}^n} |\nabla u_k|^2 dx \leq 2\int_{\mathbb{R}^n} \left|\nabla \Psi_{R_k}\right|^2 \left|u_{R_k}\right|^2 dx + 2\int_{\mathbb{R}^n} \left|\Psi_{R_k}\right|^2 \left|\nabla u_{R_k}\right|^2 dx \\
&\qquad \leq 8\int_{\Omega_k} \left|u_{R_k}\right|^2 dx + 2\int_{\Omega_k} \left|\nabla u_{R_k}\right|^2 dx, \\
&\int_{\mathbb{R}^n} \kappa |u_k|^p dx \leq \int_{\Omega_k} \kappa \left|u_{R_k}\right|^p dx,
\end{aligned} \tag{35}$$

we obtain that Lemmas 9 and 10 are still valid. It follows that

$$\begin{aligned}
&\{u_k\} \text{ is uniformly bounded in } L^\infty\left([0,T], L^2(\mathbb{R}^n)\right), \\
&\{u_k\} \text{ is uniformly bounded in } L^2\left(0,T,D^{1,2}(\mathbb{R}^n)\right), \\
&\{u_k\} \text{ is uniformly bounded in } L^p\left(0,T,L^p_\kappa(\mathbb{R}^n)\right), \\
&\left\{\frac{\partial u_k}{\partial t}\right\} \text{ is uniformly bounded in } L^{p/(p-1)}\left(0,T,\left(H^1_\kappa(\mathbb{R}^n)\right)^*\right).
\end{aligned} \tag{36}$$

Hence, taking a subsequence of $\{u_k\}$ if necessary there exists $u \in L^\infty([0,T],L^2(\mathbb{R}^n)) \cap L^2(0,T,D^{1,2}(\mathbb{R}^n)) \cap L^p(0,T,L^p_\kappa(\mathbb{R}^n))$ such that

$$\begin{aligned}
&u_k \rightharpoonup u, \quad \text{in } L^2\left(0,T,D^{1,2}(\mathbb{R}^n)\right), \\
&u_k \rightharpoonup u, \quad \text{in } L^p\left(0,T,L^p_\kappa(\mathbb{R}^n)\right), \\
&\frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^{p/(p-1)}\left(0,T,\left(H^1_\kappa(\mathbb{R}^n)\right)^*\right).
\end{aligned} \tag{37}$$

Similarly to the proof in [5, 6], we can obtain

$$f(u_k) \rightharpoonup f(u) \quad \text{in } L^{p/(p-1)}\left(0,T,\left(L^p_\kappa(\mathbb{R}^n)\right)^*\right). \tag{38}$$

In addition, referring to Lemma 11, we have

$$u_k \longrightarrow u \quad \text{in } L^2\left([0,T], L^2(\mathbb{R}^n)\right). \tag{39}$$

Therefore, for any $v \in C_0^\infty([0,T], H^1_\kappa(\mathbb{R}^n))$,

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^n} \left(\frac{\partial u_k}{\partial t} - \Delta u_k - u_k + \kappa |u_k|^{p-2} u_k + f(u_k)\right) v\, dx\, dt \longrightarrow \int_0^T \int_{\mathbb{R}^n} \left(\frac{\partial u}{\partial t} - \Delta u - u + \kappa |u|^{p-2} u + f(u)\right) v\, dx\, dt.
\end{aligned} \tag{40}$$

Thus, u is the weak solution of (1).

In the following, we will prove uniqueness of solution and the continuous dependence. Let u, v be any two solutions of (1) with initial data u_0, v_0; setting $w = u - v$, we have

$$\frac{\partial w}{\partial t} - \Delta w - w + \kappa(x)|u|^{p-2} u - \kappa(x)|v|^{p-2} v + f(u) - f(v) = 0, \tag{41}$$

with initial data $w(0) = u_0 - v_0$. Multiplying by w and integrating on $\mathbb{R}^n$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |w|^2 dx + \int_{\mathbb{R}^n} |\nabla w|^2 dx - \int_{\mathbb{R}^n} |w|^2 dx + \int_{\mathbb{R}^n} (f(u) - f(v))(u - v)\, dx \\
&\quad + \int_{\mathbb{R}^n} \left(\kappa |u|^{p-2} u - \kappa |v|^{p-2} v\right)(u - v)\, dx = 0,
\end{aligned} \tag{42}$$

and it follows from Lemma 12 that

$$\int_{\mathbb{R}^n} \left(\kappa |u|^{p-2} u - \kappa |v|^{p-2} v\right)(u - v)\, dx \geq \alpha \int_{\mathbb{R}^n} \kappa |u - v|^p dx \geq 0, \tag{43}$$

and by condition (f_3) we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} (f(u) - f(v))(u - v)\, dx &= \int_{\mathbb{R}^n} \left(\int_{v(x)}^{u(x)} f'(s)\, ds\right)(u(x) - v(x))\, dx \\
&\geq -l \int_{\mathbb{R}^n} |u - v|^2 dx = -l \int_{\mathbb{R}^n} |w|^2 dx.
\end{aligned} \tag{44}$$

Then note that $\int_{\mathbb{R}^n} |\nabla w|^2 dx \geq 0$; we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |w|^2 dx \leq (l + 1)\int_{\mathbb{R}^n} |w|^2 dx, \tag{45}$$

and integrating this gives

$$\|u(t) - v(t)\|_{L^2} \leq e^{2(l+1)t} \left\|u_0 - v_0\right\|_{L^2}, \tag{46}$$

which implies uniqueness if $u_0 = v_0$ and the continuous dependence on initial data. □

The following theorem shows the existence of global attractors when an absorbing set exists.

Lemma 14 (see [5, 6]). *If a continuous semigroup $S(t)$ has a compact absorbing set B, then there exists a global attractor $\mathscr{A} = \omega(B)$, where $\omega(B)$ is the ω-limit set of the set B.*

3. The Existence of a Global Attractor

In this section, we will prove Theorem 1. Before the proof, we first give the following lemma.

Lemma 15. *Assuming that $n \geq 3$, $p > 2$, and $\kappa(x) \geq \alpha_0 > 0$ satisfies assumption (2), then $H^1_\kappa(\mathbb{R}^n)$ is compactly embedded in $L^2(\mathbb{R}^n)$.*

Proof. Assume that $\{u_n\}$ is a bounded sequence in $H^1_\kappa(\mathbb{R}^n)$. Then there exists a constant $C > 0$, such that

$$\begin{aligned} \|u_n\|_{D^{1,2}(\mathbb{R}^n)} &\leq C, \\ \|u_n\|_{L^p_\kappa(\mathbb{R}^n)} &\leq C, \end{aligned} \tag{47}$$

so it has a subsequence $\{u_{n_k}\}$ satisfying

$$\begin{aligned} u_{n_k} &\rightharpoonup u_0 \quad \text{in } D^{1,2}(\mathbb{R}^n), \\ u_{n_k} &\rightharpoonup u_0 \quad \text{in } L^p_\kappa(\mathbb{R}^n). \end{aligned} \tag{48}$$

For arbitrary $\varepsilon > 0$, choose the constant R sufficiently large, such that

$$\left(\int_{\mathbb{R}^n\setminus\Omega_R} \left(\frac{1}{\kappa}\right)^{2/(p-2)} dx\right)^{(p-2)/p} < \frac{\varepsilon}{C^2}. \tag{49}$$

Note that $\Psi_{R+1}u_{n_k} \rightharpoonup \Psi_{R+1}u_0$ in $H^1_0(\Omega_{R+1})$ and due to the boundedness of the domain Ω_{R+1}, the Sobolev embedding theorem can be used, yielding

$$\Psi_{R+1}u_{n_k} \longrightarrow \Psi_{R+1}u_0 \quad \text{in } L^2(\Omega_{R+1}). \tag{50}$$

Then there exists $K > 0$ sufficiently large such that, for all $n_k \geq K$, we have

$$\int_{\Omega_{R+1}} \left|\Psi_{R+1}u_{n_k} - \Psi_{R+1}u_0\right|^2 dx < \varepsilon, \tag{51}$$

and it follows that

$$\begin{aligned} \left\|u_{n_k} - u_0\right\|^2_{L^2(\mathbb{R}^n)} &= \int_{\Omega_R} \left|u_{n_k} - u_0\right|^2 dx \\ &\quad + \int_{\mathbb{R}^n\setminus\Omega_R} \left|u_{n_k} - u_0\right|^2 dx \\ &\leq \int_{\Omega_{R+1}} \left|\Psi_{R+1}u_{n_k} - \Psi_{R+1}u_0\right|^2 dx \\ &\quad + \int_{\mathbb{R}^n\setminus\Omega_R} \left|u_{n_k} - u_0\right|^2 dx \\ &\leq \varepsilon + \int_{\mathbb{R}^n\setminus\Omega_R} \left|u_{n_k} - u_0\right|^2 dx. \end{aligned} \tag{52}$$

Utilizing the Hölder inequality and inequality (49), we have

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\Omega_R} \left|u_{n_k} - u_0\right|^2 dx \\ &= \int_{\mathbb{R}^n\setminus\Omega_R} \left(\frac{1}{\kappa}\right)^{2/p} \cdot \kappa^{2/p} \left|u_{n_k} - u_0\right|^2 dx \\ &\leq \left(\int_{\mathbb{R}^n\setminus\Omega_R} \left(\frac{1}{\kappa}\right)^{2/p\cdot p/(p-2)} dx\right)^{(p-2)/p} \\ &\quad \cdot \left(\int_{\mathbb{R}^n\setminus\Omega_R} \left[\kappa^{2/p} \left|u_{n_k} - u_0\right|^2\right]^{p/2} dx\right)^{2/p} \\ &= \left(\int_{\mathbb{R}^n\setminus\Omega_R} \left(\frac{1}{\kappa}\right)^{2/(p-2)} dx\right)^{(p-2)/p} \\ &\quad \cdot \left(\int_{\mathbb{R}^n\setminus\Omega_R} \kappa \left|u_{n_k} - u_0\right|^p dx\right)^{2/p} \\ &< \frac{\varepsilon}{C^2} \cdot \left\|u_{n_k} - u_0\right\|^2_{L^p_\kappa(\mathbb{R}^n)} < \frac{\varepsilon}{C^2} \cdot 4C^2 = 4\varepsilon, \end{aligned} \tag{53}$$

which implies

$$\left\|u_{n_k} - u_0\right\|^2_{L^2(\mathbb{R}^n)} < 5\varepsilon, \quad \text{for } n_k \geq K. \tag{54}$$

This completes the proof of Lemma 15. □

Proof of Theorem 1. In order to prove that (1) has a global attractor, referring to Lemma 14, it is sufficient to show the existence of a compact absorbing set in $L^2(\mathbb{R}^n)$.

Let u be the solution of (1); multiplying the first equation of (1) by u and integrating on $\mathbb{R}^n$, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx - \int_{\mathbb{R}^n} |u|^2 dx \\ &\quad + \int_{\mathbb{R}^n} \kappa(x)|u|^p dx + \int_{\mathbb{R}^n} f(u) u\, dx = 0. \end{aligned} \tag{55}$$

Since $\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq 0$ and $f(u)u \geq 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n} |u|^2 dx \\ &\quad \leq 2\int_{\mathbb{R}^n} |u|^2 dx - \int_{\mathbb{R}^n} \kappa(x)|u|^p dx. \end{aligned} \tag{56}$$

Similar to estimate (19), we have

$$\int_{\mathbb{R}^n} |u|^2 dx \leq C(\kappa) + \frac{1}{p}\int_{\mathbb{R}^n} \kappa(x)|u|^p dx, \tag{57}$$

and thus

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n} |u|^2 dx \\ &\quad \leq 2C(\kappa) - \left(1 - \frac{2}{p}\right)\int_{\mathbb{R}^n} \kappa(x)|u|^p dx. \end{aligned} \tag{58}$$

Note that $p > 2$ yielded

$$\frac{d}{dt}\int_{\mathbb{R}^n}|u|^2\,dx + 2\int_{\mathbb{R}^n}|u|^2\,dx \le 4C, \tag{59}$$

and then Gronwall's inequality can be applied, yielding

$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} \le 4C, \quad \text{when } t \ge t_0\left(\|u_0\|_{L^2}\right), \tag{60}$$

where $t_0 = (1/2)\ln(\|u_0\|^2_{L^2}/2C)$. Now, combining estimates (55) and (57), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx \\ + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u|^p\,dx + \int_{\mathbb{R}^n}f(u)u\,dx \le C.\end{aligned} \tag{61}$$

Integrating between t and $t+1$, it follows that

$$\begin{aligned}\int_t^{t+1}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(s)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u(s)|^p\,dx \right. \\ \left. + \int_{\mathbb{R}^n}f(u(s))u(s)\,dx\right)ds \le C + \frac{1}{2}\|u(t)\|^2_{L^2}.\end{aligned} \tag{62}$$

When $t > t_0$, it follows from (60) that

$$\begin{aligned}\int_t^{t+1}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(s)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u(s)|^p\,dx \right. \\ \left. + \int_{\mathbb{R}^n}f(u(s))u(s)\,dx\right)ds \le 3C.\end{aligned} \tag{63}$$

Now, multiplying the first equation of (1) by u_t and integrating on $\mathbb{R}^n$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n}|u|^2\,dx \right. \\ \left. + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u|^p\,dx + \int_{\mathbb{R}^n}F(u)\,dx\right) \\ = -\int_{\mathbb{R}^n}|u_t|^2\,dx \le 0,\end{aligned} \tag{64}$$

so it follows from (61) and (64) that

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u|^p\,dx \right. \\ \left. + \int_{\mathbb{R}^n}F(u)\,dx\right) \le C.\end{aligned} \tag{65}$$

Integrating between s and t $(t-1 \le s < t)$, it holds that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(t)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(t)|^p\,dx \\ &\quad + \int_{\mathbb{R}^n}F(u(t))\,dx \le C + \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(s)|^2\,dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(s)|^p\,dx + \int_{\mathbb{R}^n}F(u(s))\,dx \le C \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(s)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(s)|^p\,dx \\ &\quad + \frac{1}{\beta}\int_{\mathbb{R}^n}f(u(s))u(s)\,dx.\end{aligned} \tag{66}$$

Then integrating the equation with respect to s between $t-1$ and t again, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(t)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(t)|^p\,dx \\ &\quad + \int_{\mathbb{R}^n}F(u(t))\,dx \le C + \int_t^{t+1}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(s)|^2\,dx\right. \\ &\quad \left. + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(s)|^p\,dx + \frac{1}{\beta}\int_{\mathbb{R}^n}f(u(s))u(s)\,dx\right)ds,\end{aligned} \tag{67}$$

when $t > t_0$, so it follows from (63) and $\int_{\mathbb{R}^n}F(u(t))dx \ge 0$ that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u(t)|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa|u(t)|^p\,dx \\ \le C + \max\left\{3, \frac{3}{\beta}\right\}C,\end{aligned} \tag{68}$$

which implies that there exists a constant $\widetilde{C}$, such that

$$\|u(t)\|_{H^1_\kappa} \le \widetilde{C}, \quad \forall t > t_0. \tag{69}$$

Finally, referring to Lemma 15, $H^1_\kappa(\mathbb{R}^n)$ is compactly embedded in $L^2(\mathbb{R}^n)$, and we obtain a compact absorbing set in $L^2(\mathbb{R}^n)$ which concludes the proof of Theorem 1. □

4. The Dimension of the Global Attractor and the Equilibrium Points

Next we will estimate the Z_2-index of the global attractor obtained in Theorem 1. Before the formal proof of Theorem 2, we first consider the energy function

$$\begin{aligned}E(u) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 - |u|^2\right)dx + \frac{1}{p}\int_{\mathbb{R}^n}\kappa(x)|u|^p\,dx \\ + \int_{\mathbb{R}^n}F(u)\,dx.\end{aligned} \tag{70}$$

It is well known that functional (70) has an infinite dimensional negative subspace H^- of $H^1_\kappa(\mathbb{R}^n)$; that is, there exists linearly independent nonzero functions $u_1, u_2, \ldots, u_m, \ldots \in H^-$, satisfying

$$\int_{\mathbb{R}^n} |\nabla u_m|^2 \, dx - \int_{\mathbb{R}^n} |u_m|^2 \, dx < 0, \quad m = 1, 2, \ldots. \tag{71}$$

Let $H_m \triangleq \operatorname{span}\{u_1, u_2, \ldots, u_m\}$ be a subspace of H^- with $\dim(H_m) = m$, where $u_1, u_2, \ldots, u_m$ are orthogonal in both $D^{1,2}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Now, we give the proof of Theorem 2.

Proof of Theorem 2. For arbitrary $m \in \mathbb{N}^+$, we first prove that there exists a set $B_m \in H^1_\kappa(\mathbb{R}^n)$ with $\gamma(B_m) \geq m$ and a neighborhood $\mathcal{O}$ of the origin, such that

$$\omega(B_m) \subset \mathcal{A} \setminus \mathcal{O}. \tag{72}$$

It follows from (64) that

$$\frac{d}{dt}(E(u)) = -\int_{\mathbb{R}^n} |u_t|^2 \, dx \leq 0; \tag{73}$$

that is, for any $u_0 \in H^1_\kappa(\mathbb{R}^n)$, the function $t \rightarrow E(u(t))$ is nonincreasing. For arbitrary $m > 0$ and $u \in H_m \setminus \{0\}$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \int_{\mathbb{R}^n} |u|^2 \, dx < 0. \tag{74}$$

Denoting $A_m \triangleq H_m \cap S_1 = \{u \in H_m : \|u\|_{H^1_\kappa} = 1\}$, then A_m is compact in $H^1_\kappa(\mathbb{R}^n)$; thus, there exists $\delta > 0$, such that for all $u \in A_m$

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \int_{\mathbb{R}^n} |u|^2 \, dx < -\delta < 0. \tag{75}$$

Referring to Lemma 6 (A_6), for every constant $\varepsilon > 0$, we have

$$\gamma(\varepsilon A_m) = \gamma(A_m) = m, \tag{76}$$

where $\varepsilon A_m = \{\varepsilon u : u \in A_m\}$. Thus, for $v = \varepsilon u \in \varepsilon A_m$, it follows that

$$\begin{aligned} E(v) = {} & \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} |u|^2 \, dx \\ & + \frac{\varepsilon^p}{p} \int_{\mathbb{R}^n} \kappa(x) |u|^p \, dx + \int_{\mathbb{R}^n} F(\varepsilon u) \, dx. \end{aligned} \tag{77}$$

Recalling $p > 2$ and condition (f_2), when ε is sufficiently small, we have

$$E(v) \leq -\frac{\varepsilon^2 \delta}{2} + \frac{\varepsilon^p}{p} + o(\varepsilon^2) \leq -\delta_1 < 0, \quad \forall v \in \varepsilon A_m. \tag{78}$$

In addition, since $E(0) = 0$ and $t \rightarrow E(u(t))$ is nonincreasing, then $\omega(\varepsilon A_m) \subset \mathcal{A} \setminus \{0\}$. Since $\omega(\varepsilon A_m)$ is closed and compact, there exists open neighborhood $\mathcal{O}$ of 0, such that

$$\omega(\varepsilon A_m) \subset \mathcal{A} \setminus \mathcal{O}. \tag{79}$$

Let $B_m = \varepsilon A_m$; we have completed the proof of the first step.

Next, we only need to prove $\gamma(\mathcal{A} \setminus \mathcal{O}) \geq \gamma(B_m)$. Referring to Lemma 6 ($A_5$), there exists $\mathcal{N}(\mathcal{A} \setminus \mathcal{O})$, which is a neighborhood of $\mathcal{A} \setminus \mathcal{O}$, satisfying

$$\gamma(\mathcal{N}(\mathcal{A} \setminus \mathcal{O})) = \gamma(\mathcal{A} \setminus \mathcal{O}). \tag{80}$$

In addition, referring to the definition of ω-limit set $\omega(B_m)$, there exists $t_0 > 0$ such that

$$\overline{S(t) B_m} \subset \mathcal{N}(\omega(B_m)) \subset \mathcal{N}(\mathcal{A} \setminus \mathcal{O}). \tag{81}$$

Thus,

$$\gamma(\mathcal{A} \setminus \mathcal{O}) = \gamma(\mathcal{N}(\mathcal{A} \setminus \mathcal{O})) \geq \gamma\left(\overline{S(t) B_m}\right). \tag{82}$$

It is obvious that $S(t)$ is odd since $S(t)(-u_0) = -u(t) = -S(t)u_0$. Then, referring to Lemma 6 (A_4), we have $\gamma(\overline{S(t)B_m}) \geq \gamma(B_m)$, and then $\gamma(\mathcal{A} \setminus \mathcal{O}(0)) \geq \gamma(B_m)$. The proof is complete. □

At last, we want to investigate existence of the multiple equilibrium points of the equation, that is, solutions of the following elliptic equation:

$$-\Delta u - u + \kappa(x) |u|^{p-2} u + f(u) = 0 \quad \text{in } \mathbb{R}^n. \tag{83}$$

We consider the critical values of the energy functional $E(u)$ defined by (70). In order to obtain infinite critical values by Lemma 8, we verify that the functional $E(u)$ is bounded from below and satisfies the $(P.S.)$ condition.

Lemma 16. *The functional $E(u)$ defined by (70) is bounded from below.*

Proof. The functional is

$$\begin{aligned} E(u) = {} & \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \, dx \\ & + \frac{1}{p} \int_{\mathbb{R}^n} \kappa(x) |u|^p \, dx + \int_{\mathbb{R}^n} F(u) \, dx. \end{aligned} \tag{84}$$

By estimate (57), it is easy to verify that the functional $E(u)$ is bounded from below. □

Lemma 17. *Let $E(u)$ be a functional defined by (70) and $d \in \mathbb{R}$ be a constant, then any sequence $\{u_n\} \subset H^1_\kappa(\mathbb{R}^n)$ such that*

$$\begin{aligned} & E(u_n) \leq d \quad \forall n \in \mathbb{N}, \\ & E'(u_n) \longrightarrow 0 \end{aligned} \tag{85}$$

contains a convergent subsequence.

Proof. Since $E(u_n) \leq d$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx + \frac{1}{p} \int_{\mathbb{R}^n} \kappa(x) |u_n|^p \, dx + \int_{\mathbb{R}^n} F(u) \, dx \\ & \quad \leq d + \frac{1}{2} \int_{\mathbb{R}^n} |u_n|^2 \, dx. \end{aligned} \tag{86}$$

Similarly to estimate (19), we have

$$\int_{\mathbb{R}^n} |u_n|^2 \, dx \le C(\kappa) + \frac{1}{p} \int_{\mathbb{R}^n} \kappa(x) |u_n|^p \, dx, \tag{87}$$

and combining the above estimates and $F(u) \ge 0$, we have

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx + \frac{1}{2p} \int_{\mathbb{R}^n} \kappa(x) |u_n|^p \, dx \le d + \frac{C(\kappa)}{2}. \tag{88}$$

It follows that $\|u_n\|_{H^1_\kappa}$ is bounded.

Going if necessary to a subsequence, we assume that

$$u_n \rightharpoonup u \quad \text{in } H^1_\kappa(\mathbb{R}^n). \tag{89}$$

By Lemma 15, we have $u_n \to u$ in $L^2(\mathbb{R}^n)$. Observe that

$$\begin{aligned} &\langle E'(u_n) - E'(u), u_n - u \rangle \\ &\quad - \int_{\mathbb{R}^n} (f(u_n) - f(u))(u_n - u) \, dx \\ &= \int_{\mathbb{R}^n} |\nabla(u_n - u)|^2 \, dx - \int_{\mathbb{R}^n} |u_n - u|^2 \, dx \\ &\quad + \int_{\mathbb{R}^n} \left(\kappa |u_n|^{p-2} u_n - \kappa |u|^{p-2} u\right)(u_n - u) \, dx \\ &\ge \int_{\mathbb{R}^n} |\nabla(u_n - u)|^2 \, dx - \int_{\mathbb{R}^n} |u_n - u|^2 \, dx \\ &\quad + \alpha \int_{\mathbb{R}^n} \kappa |u_n - u|^p \, dx. \end{aligned} \tag{90}$$

Since $E'(u_n) \to 0$ and $u_n \to u$ in $L^2(\mathbb{R}^n)$, we can obtain that

$$\begin{aligned} &\langle E'(u_n) - E'(u), u_n - u \rangle \longrightarrow 0, \quad n \longrightarrow \infty, \\ &\int_{\mathbb{R}^n} |u_n - u|^2 \, dx \longrightarrow 0, \quad n \longrightarrow \infty, \\ &- \int_{\mathbb{R}^n} (f(u_n) - f(u))(u_n - u) \, dx \\ &\quad = - \int_{\mathbb{R}^n} \left(\int_u^{u_n} f'(s) \, ds \right)(u_n - u) \, dx \\ &\quad \le l \int_{\mathbb{R}^n} |u_n - u|^2 \, dx \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{91}$$

Thus we have proved that $\|u_n - u\|_{H^1_\kappa} \to 0, n \to \infty$. □

Now, we prove that the global attractor contains infinite distinct pairs of equilibrium points.

Proof of Theorem 4. By the proof of Theorem 2, we obtain that, for arbitrary $r > 0$, there exists a subspace $V_r \subset V$ with $\dim V_r = r$ and $\rho > 0$, such that

$$\sup_{u \in V_r \cap S_\rho} E(u) < E(0) = 0. \tag{92}$$

The above two lemmas show that E also satisfies conditions (B_2) and (B_3) of Lemma 8; thus we obtain infinite pairs of critical points, which implies the conclusion. □

Remark 18. In this paper, we suppose that the nonlinear term f is continuous. If $f(u)$ is a weak continuous function in space $H^1_\kappa(\mathbb{R}^n)$ or $f(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping of C^1 in u, all conclusions in this paper are still valid.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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