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Research Article

Almost Periodic Solution of a Multispecies Discrete Mutualism System with Feedback Controls

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We consider an almost periodic multispecies discrete Lotka-Volterra mutualism system with feedback controls. We firstly obtain the permanence of the system by utilizing the theory of difference equation. By means of constructing a suitable Lyapunov function, sufficient conditions are obtained for the existence of a unique positive almost periodic solution which is uniformly asymptotically stable. An example together with numerical simulation indicates the feasibility of the main result.

1. Introduction

The mutualism system [1] has been studied by more and more scholars. Topics such as permanence, global attractivity, and global stability of continuous differential mutualism system were extensively investigated (see [2–7] and the references cited therein). In addition, some recent attention was on the permanence and global stability of discrete mutualism system, and many excellent results have been derived (see [3, 8–13] and the references cited therein).

Recently, the multispecies descrete Lotka-Volterra ecosystem is increasingly concerned (see [12–21] and the references cited therein). Yang and Li [19] studied a discrete nonlinear N-species cooperation system with time delays and feedback controls. Sufficient conditions which ensure the permanence of the system are obtained. Li and Zhang [21] studied a discrete n-species cooperation system with time-varying delays and feedback controls. Sufficient conditions are obtained for the permanence of the system.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, and harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important, and more general when we consider the effects of the environmental factors. In fact, there have been many nice works on the positive almost periodic solutions of continuous and discrete dynamics model with almost periodic coefficients (see [7, 12, 13, 22–28] and the references cited therein).

As we all known, investigating the almost periodic solutions of discrete population dynamics model with feedback control has more extensively practical application value (see [11, 22, 23, 29–34] and the references cited therein). Wang [22] considered a nonlinear single species discrete with feedback control and obtained some sufficient conditions which assure the unique existence and global attractivity of almost positive periodic solution. Niu and Chen [30] studied a discrete Lotka-Volterra competitive system with feedback control and obtain the existence and uniqueness of the almost periodic solution which is uniformly asymptotically stable.

Motivated by above, in this paper, we are concerned with the following multispecies discrete Lotka-Volterra mutualism system with feedback controls

$$\begin{aligned} x_i(k+1) &= x_i(k)\exp\Big\{a_i(k) - b_i(k)x_i(k) \end{aligned}$$

$$+\sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{x_j(k)}{d_{ij}(k)+x_j(k)} - e_i(k)u_i(k)\Bigg\},$$

$$\Delta u_i(k) = -f_i(k)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)x_j(k), \quad i=1,2,\ldots,n, \tag{1}$$

where $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$, $\{d_{ij}(k)\}$, $\{e_i(k)\}$, $\{f_i(k)\}$ and $\{g_{ij}(k)\}$ are bounded nonnegative almost periodic sequences such that

$$\begin{aligned} &0 < a_i^l \le a_i(k) \le a_i^u, \qquad 0 < b_i^l \le b_i(k) \le b_i^u,\\ &0 < c_{ij}^l \le c_{ij}(k) \le c_{ij}^u, \qquad 0 < d_{ij}^l \le d_{ij}(k) \le d_{ij}^u,\\ &0 < e_i^l \le e_i(k) \le e_i^u, \qquad 0 < f_i^l \le f_i(k) \le f_i^u < 1,\\ &0 < g_{ij}^l \le g_{ij}(k) \le g_{ij}^u, \end{aligned} \tag{2}$$

$i,j = 1,2,\ldots,n$, $k \in \mathbf{Z}$. For any bounded sequence $\{f(k)\}$ defined on $\mathbf{Z}$, $f^u = \sup_{k\in\mathbf{Z}} f(k)$ and $f^l = \inf_{k\in\mathbf{Z}} f(k)$.

By the biological meaning, we will focus our discussion on the positive solutions of system (1). So it is assumed that the initial conditions of system (1) are the form:

$$x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1,2,\ldots,n. \tag{3}$$

One can easily show that the solutions of system (1) with the initial condition (3) are defined and remain positive for all $k \in N^+ = \{0,1,2,3,\ldots\}$.

To the best of our knowledge, this is the first paper to investigate the uniformly asymptotical stability of positive almost periodic solution of multispecies discrete Lotka-Volterra mutualism system with feedback controls. The aim of this paper is to obtain sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution of system (1) with initial condition (3), by utilizing the theory of difference equation and constructing a suitable Lyapunov function and applying the analysis technique of papers [11, 22, 29, 31, 32].

The remaining part of this paper is organized as follows. In Section 2, we will introduce some definitions and several useful lemmas. In the next section, we establish the permanence of system (1). Then, in Section 4, we establish sufficient conditions to ensure the existence of a unique positive almost periodic solution, which is uniformly asymptotically stable. The main result is illustrated by an example with a numerical simulation in the last section.

2. Preliminaries

First, we give the definitions of the terminologies involved.

Definition 1 (see [35, 36]). A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : |x(n+\tau) - x(n)| < \varepsilon, \forall n \in Z\} \tag{4}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$|x(n+\tau) - x(n)| < \varepsilon, \quad \forall n \in Z. \tag{5}$$

τ is called an ε-translation number of $x(n)$.

Definition 2 (see [37]). Let D be an open subset of R^m, $f : Z\times D \to R^m$. $f(n,x)$ is said to be almost periodic in n uniformly for $x \in D$ if, for any $\varepsilon > 0$ and any compact set $S \subset D$, there exists a positive integer $l = l(\varepsilon, S)$ such that any interval of length l contains an integer τ for which

$$\left|f(n+\tau, x) - f(n,x)\right| < \varepsilon, \quad \forall (n,x) \in Z\times S. \tag{6}$$

τ is called an ε-translation number of $f(n,x)$.

Definition 3 (see [38]). The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \{g(n,x) : \lim_{k\to\infty} f(n+\tau_k, x) = g(n,x) \text{ uniformly on } Z\times S\}, \tag{7}$$

for some sequence $\{\tau_k\}$, where S is any compact set in D.

Now, we state several lemmas which will be useful in proving our main result.

Lemma 4 (see [39]). *$\{x(n)\}$ is an almost periodic sequence if and only if, for any integer sequence $\{k_i'\}$, there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that the sequence $\{x(n+k_i)\}$ converges uniformly for all $n \in Z$ as $i \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Lemma 5 (see [9]). *Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and*

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\} \tag{8}$$

for $n \in \mathbf{N}$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp\{a^u - 1\}. \tag{9}$$

Lemma 6 (see [9]). *Assume that $\{x(n)\}$ satisfies*

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0, \qquad \limsup_{n\to+\infty} x(n) \le x^*, \tag{10}$$

and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \geq \min\left\{\frac{a^l}{b^u}\exp\left\{a^l - b^u x^*\right\}, \frac{a^l}{b^u}\right\}. \quad (11)$$

Lemma 7 (see [40]). *Assume that $A > 0$ and $y(0) > 1$, and further suppose that*

$$y(n+1) \leq Ay(n) + B(n), \quad n = 1, 2, 3, \ldots. \quad (12)$$

Then, for any integer $k \leq n$,

$$y(n) \leq A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \quad (13)$$

Specifically, if $A < 1$ and B is bounded above with respect to M, then

$$\limsup_{n\to\infty} y(n) \leq \frac{M}{1-A}. \quad (14)$$

Lemma 8 (see [40]). *Assume that $A > 0$ and $y(0) > 1$, and further suppose that*

$$y(n+1) \geq Ay(n) + B(n), \quad n = 1, 2, 3, \ldots. \quad (15)$$

Then, for any integer $k \leq n$,

$$y(n) \geq A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \quad (16)$$

Specifically, if $A < 1$ and B is bounded below with respect to m, then

$$\liminf_{n\to\infty} y(n) \geq \frac{m}{1-A}. \quad (17)$$

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \quad (18)$$

where $f : Z^+ \times S_B \to R^K$, $S_B = \{x \in R^k : \|x\| < B\}$, and $f(n,x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x. The product system of (18) is the following system:

$$x(n+1) = f(n, x(n)), \qquad y(n+1) = f(n, y(n)), \quad (19)$$

and Zhang [38] obtained the following theorem.

Theorem 9 (see [38]). *Suppose that there exists a Lyapunov function $V(n,x,y)$ defined for $n \in Z^+$, $\|x\| < B$, and $\|y\| < B$ satisfying the following conditions:*

(i) *$a(\|x-y\|) \leq V(n,x,y) \leq b(\|x-y\|)$, where $a, b \in K$ with $K = \{a \in C(R^+, R^+) : a(0) = 0$ and a is increasing$\}$;*

(ii) *$\|V(n,x_1,y_1) - V(n,x_2,y_2)\| \leq L[\|x_1 - x_2\| + \|y_1 - y_2\|]$, where $L > 0$ is a constant;*

(iii) *$\Delta V_{(19)}(n,x,y) \leq -\alpha V(n,x,y)$, where $0 < \alpha < 1$ is a constant, and*

$$\Delta V_{(19)}(n,x,y) \equiv V(n+1, f(n,x), f(n,y)) - V(n,x,y). \quad (20)$$

Moreover, if there exists a solution $\varphi(n)$ of (18) such that $\|\varphi(n)\| \leq B^* < B$ for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of (18) which is bounded by B^*. In particular, if $f(n,x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of (18) of period ω.

3. Permanence

In this section, we establish the permanence result for system (1).

Theorem 10. *Assume that the conditions (2) and (3) hold; furthermore,*

$$a_i^l - e_i^u N_i > 0, \quad (21)$$

and then system (1) is permanent; that is, there exist positive constants m_i, M_i, n_i, and N_i ($i = 1, 2, \ldots, n$) which are independent of the solutions of system (1), such that, for any positive solution $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of system (1), one has

$$m_i \leq \liminf_{k\to+\infty} x_i(k) \leq \limsup_{k\to+\infty} x_i(k) \leq M_i,$$
$$n_i \leq \liminf_{k\to+\infty} u_i(k) \leq \limsup_{k\to+\infty} u_i(k) \leq N_i, \quad (22)$$
$$i = 1, 2, \ldots, n.$$

Proof. Let $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ be any positive solution of system (1). From the first equation of system (1), it follows that

$$x_i(k+1) \leq x_i(k)\exp\left\{a_i(k) + \sum_{j=1, j\neq i}^{n} c_{ij}(k) - b_i(k)x_i(k)\right\}. \quad (23)$$

Thus, as a direct corollary of Lemma 5, according to (23), one has

$$\limsup_{k\to+\infty} x_i(k) \leq \frac{1}{b_i^l}\exp\left\{a_i^u + \sum_{j=1, j\neq i}^{n} c_{ij}^u - 1\right\} \triangleq M_i > 0. \quad (24)$$

For any small positive constant $\varepsilon > 0$, from (24), it follows that there exists a positive constant $K_1 > 0$ such that, for all $k > K_1$ and $i = 1, 2, \ldots, n$,

$$x_i(k) \leq M_i + \varepsilon. \quad (25)$$

For $k \geq K_1$, from (25) and system (1), we have

$$u_i(k+1) \leq \left(1 - f_i^l\right)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)\left(M_j + \varepsilon\right). \quad (26)$$

Then, as a direct corollary of Lemma 7, according to (26), one has

$$\limsup_{k\to+\infty} u_i(k) \le \frac{1}{f_i^l}\sum_{j=1}^{n} g_{ij}^u \left(M_j+\varepsilon\right). \tag{27}$$

Letting $\varepsilon \to 0$, it follows that

$$\limsup_{k\to+\infty} u_i(k) \le \frac{1}{f_i^l}\sum_{j=1}^{n} g_{ij}^u M_j \triangleq N_i. \tag{28}$$

Thus, there exists a positive integer $K_2 > K_1$, and we have, for $k > K_2$,

$$u_i(k) \le N_i + \varepsilon. \tag{29}$$

For $k \ge K_2$, from (29) and system (1), we have

$$x_i(k+1) \ge x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k) - e_i(k)\left(N_i+\varepsilon\right)\right\}. \tag{30}$$

Assuming that $a_i^l - e_i^u N_i > 0$, for any $\varepsilon > 0$, there exists a positive integer $K_3 > K_2$ such that $a_i(k) - e_i(k)(N_i+\varepsilon) > 0$ for $k > K_3$. Thus, as a direct corollary of Lemma 6, according to (30), one has

$$\liminf_{k\to+\infty} x_i(k) \ge \min\left\{m_{i_1\varepsilon}, m_{i_2\varepsilon}\right\}, \tag{31}$$

where

$$\begin{gathered} m_{i_1\varepsilon} = \frac{a_i^l - e_i^u\left(N_i+\varepsilon\right)}{b_i^u}, \\ m_{i_2\varepsilon} = m_{i_1\varepsilon}\exp\left\{a_i^l - e_i^u\left(N_i+\varepsilon\right) - b_i^u M_i\right\}. \end{gathered} \tag{32}$$

Letting $\varepsilon \to 0$, it follows that

$$\liminf_{k\to+\infty} x_i(k) \ge \frac{1}{2}\min\left\{m_{i_1}, m_{i_2}\right\} \triangleq m_i > 0, \tag{33}$$

where

$$\begin{gathered} m_{i_1} = \frac{a_i^l - e_i^u N_i}{b_i^u}, \\ m_{i_2} = m_{i_1\varepsilon}\exp\left\{a_i^l - e_i^u N_i - b_i^u M_i\right\}. \end{gathered} \tag{34}$$

From (33), for any $\varepsilon > 0$, there exists a positive integer $K_4 > K_3$ such that

$$x_i(k) \ge m_i - \varepsilon \tag{35}$$

for $k > K_4$.

From (35) and system (1), we have

$$u_i(k+1) \ge \left(1 - f_i^u\right)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)\left(m_j - \varepsilon\right). \tag{36}$$

Then, as a direct corollary of Lemma 8, according to (36), one has

$$\liminf_{k\to+\infty} u_i(k) \ge \frac{1}{f_i^u}\sum_{j=1}^{n} g_{ij}^l \left(m_j-\varepsilon\right). \tag{37}$$

Letting $\varepsilon \to 0$, it follows that

$$\liminf_{k\to+\infty} u_i(k) \ge \frac{1}{f_i^u}\sum_{j=1}^{n} g_{ij}^l m_j \triangleq n_i > 0. \tag{38}$$

Then, (24), (28), (33), and (38) show that system (1) is permanent. The proof is completed. □

According to Theorem 9, we first prove that there is a bounded solution of system (1), and then construct a suitable Lyapunov function for system (1).

We denote by Ω the set of all solutions $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of system (1) satisfying $m_i \le x_i(k) \le M_i$ and $n_i \le u_i(k) \le N_i$ $(i = 1, 2, \ldots, n)$ for all $k \in Z^+$.

Proposition 11. *Assume that the conditions* (2), (3), *and* (21) *hold. Then* $\Omega \ne \Phi$.

Proof. By the almost periodicity of $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$, $\{d_{ij}(k)\}$, $\{e_i(k)\}$, $\{f_i(k)\}$, and $\{g_{ij}(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to \infty$ as $p \to \infty$ such that

$$\begin{gathered} a_i\left(k+\delta_p\right) \longrightarrow a_i(k), \qquad b_i\left(k+\delta_p\right) \longrightarrow b_i(k), \\ c_{ij}\left(k+\delta_p\right) \longrightarrow c_{ij}(k), \qquad d_{ij}\left(k+\delta_p\right) \longrightarrow d_{ij}(k), \\ e_i\left(k+\delta_p\right) \longrightarrow e_i(k), \qquad f_i\left(k+\delta_p\right) \longrightarrow f_i(k), \\ g_{ij}\left(k+\delta_p\right) \longrightarrow g_{ij}(k) \\ \text{as } p \longrightarrow \infty. \end{gathered} \tag{39}$$

Let ε be an arbitrary small positive number. It follows from Theorem 10 that there exists a positive integer N_0 such that

$$\begin{gathered} m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon, \qquad n_i - \varepsilon \le u_i(k) \le N_i + \varepsilon, \\ k > N_0. \end{gathered} \tag{40}$$

Write $x_{ip}(k) = x_i(k+\delta_p)$ and $u_{ip}(k) = u_i(k+\delta_p)$ for $k \ge N_0 - \delta_p$ and $p = 1, 2, \ldots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p \ge q\}$ such that the sequence $x_p(k)$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z^+ as $p \to \infty$. Thus we have a sequence $\{y_i(k)\}$ such that

$$\begin{gathered} x_{ip}(k) \longrightarrow y_i(k), \qquad u_{ip}(k) \longrightarrow v_i(k) \\ \text{for } k \in Z^+ \quad \text{as } p \longrightarrow \infty. \end{gathered} \tag{41}$$

This, combined with

$$\begin{aligned} x_{ip}(k+1) &= x_{ip}(k)\exp\left\{a_i\left(k+\delta_p\right)-b_i\left(k+\delta_p\right)x_{ip}(k)\right. \\ &\quad+\sum_{j=1,j\neq i}^{n}c_{ij}\left(k+\delta_p\right) \\ &\quad\times\frac{x_{jp}(k)}{d_{ij}\left(k+\delta_p\right)+x_{jp}(k)} \\ &\quad\left.-e_i\left(k+\delta_p\right)u_{ip}(k)\right\}, \\ u_{ip}(k+1) &= \left(1-f_i\left(k+\delta_p\right)\right)u_{ip}(k) \\ &\quad+\sum_{j=1}^{n}g_{ij}\left(k+\delta_p\right)x_{jp}(k), \quad i=1,2,\ldots,n, \end{aligned} \tag{42}$$

gives us

$$\begin{aligned} y_i(k+1) &= y_i(k)\exp\left\{a_i(k)-b_i(k)y_i(k)\right. \\ &\quad+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{y_j(k)}{d_{ij}(k)+y_j(k)} \\ &\quad\left.-e_i(k)v_i(k)\right\}, \\ v_i(k+1) &= \left(1-f_i(k)\right)v_i(k) \\ &\quad+\sum_{j=1}^{n}g_{ij}(k)y_j(k), \quad i=1,2,\ldots,n. \end{aligned} \tag{43}$$

We can easily see that $(y_1(k), y_2(k), \ldots, y_n(k), v_1(k), v_2(k), \ldots, v_n(k))$ is a solution of system (1) and $m_i - \varepsilon \le y_i(k) \le M_i + \varepsilon$ and $n_i - \varepsilon \le v_i(k) \le N_i + \varepsilon$ for $k \in Z^+$. Since ε is an arbitrary small positive number, it follows that $m_i \le y_i(k) \le M_i$ and $n_i \le v_i(k) \le N_i$ and hence we complete the proof. □

4. Stability of Almost Periodic Solution

In this section, by constructing a nonnegative Lyapunov function, we will obtain sufficient conditions for uniform asymptotical stability of positive almost periodic solution of system (1).

Theorem 12. *Assume that the conditions* (2), (3), *and* (21) *hold; moreover,* $0 < \beta < 1$, *where*

$$\begin{aligned} \beta &= \min\left\{\min_{1\le i\le n}\beta_i^1, \min_{1\le i\le n}\beta_i^2\right\}, \\ \beta_i^1 &= 2b_i^l m_i - b_i^{u2}M_i^2 - \sum_{j=1}^{n}g_{ji}^{u2}M_i^2 \\ &\quad-\sum_{j=1,j\neq i}^{n}c_{ji}^{u2}-\left(1+2b_i^uM_i\right)c_{ij}^u \\ &\quad-\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}c_{li}^uc_{lj}^u-\frac{1}{2}\sum_{l=1}^{n}g_{li}^ug_{lj}^uM_iM_j \\ &\quad-e_i^u\left(b_i^uM_i+1\right)-g_{ii}^uM_i\left(1-f_i^l\right) \\ &\quad-\sum_{j=1,j\neq i}^{n}\left[\left(1+2b_j^uM_j\right)c_{ji}^u+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}c_{li}^uc_{lj}^u\right. \\ &\quad\left.+\frac{1}{2}\sum_{l=1}^{n}g_{li}^ug_{lj}^uM_iM_j+\left(1-f_j^l\right)g_{ji}^uM_i\right. \\ &\quad\left.+e_j^uc_{ji}^u\right], \\ \beta_i^2 &= f_i^l\left(2-f_i^u\right)-e_i^u\left(b_i^uM_i+1\right)-g_{ii}^uM_i\left(1-f_i^l\right) \\ &\quad-e_i^{u2}-\sum_{j=1,j\neq i}^{n}\left[\left(1-f_i^l\right)g_{ij}^uM_j+e_i^uc_{ij}^u\right], \end{aligned} \tag{44}$$

$i = 1, 2, \ldots, n$. *Then there exists a unique uniformly asymptotically stable almost periodic solution* $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ *of system* (1) *which is bounded by* Ω *for all* $k \in N^+$.

Proof. Let $p_i(k) = \ln x_i(k)$, $i = 1, 2, \ldots, n$. From system (1), we have

$$\begin{aligned} p_i(k+1) &= p_i(k)+a_i(k)-b_i(k)e^{p_i(k)} \\ &\quad+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{e^{p_j(k)}}{d_{ij}(k)+e^{p_j(k)}}-e_i(k)u_i(k), \\ \Delta u_i(k) &= -f_i(k)u_i(k)+\sum_{j=1}^{n}g_{ij}(k)e^{p_j(k)}, \\ &\quad i=1,2,\ldots,n. \end{aligned} \tag{45}$$

From Proposition 11, we know that system (45) has a bounded solution $(p_1(k), p_2(k), \ldots, p_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ satisfying

$$\begin{aligned} \ln m_i \le p_i(k) \le \ln M_i, \qquad n_i \le u_i(k) \le N_i, \\ i=1,2,\ldots,n, \quad k\in Z^+. \end{aligned} \tag{46}$$

Hence, $|p_i(k)| \le A_i$ and $|u_i(k)| \le B_i$, where $A_i = \max\{|\ln m_i|, |\ln M_i|\}$ and $B_i = \max\{n_i, N_i\}$, $i = 1, 2, \ldots, n$.

For $(X, U) \in R^{n+n}$, we define the norm $\|(X, U)\| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |u_i|$.

Consider that the product system of (45) is

$$\begin{aligned} p_i(k+1) &= p_i(k) + a_i(k) - b_i(k) e^{p_i(k)} \\ &\quad + \sum_{j=1, j\neq i}^{n} c_{ij}(k) \frac{e^{p_j(k)}}{d_{ij}(k) + e^{p_j(k)}} - e_i(k) u_i(k), \\ \Delta u_i(k) &= -f_i(k) u_i(k) + \sum_{j=1}^{n} g_{ij}(k) e^{p_j(k)}, \\ q_i(k+1) &= q_i(k) + a_i(k) - b_i(k) e^{q_i(k)} \\ &\quad + \sum_{j=1, j\neq i}^{n} c_{ij}(k) \frac{e^{q_j(k)}}{d_{ij}(k) + e^{q_j(k)}} - e_i(k) w_i(k), \\ \Delta w_i(k) &= -f_i(k) w_i(k) + \sum_{j=1}^{n} g_{ij}(k) e^{q_j(k)}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{47}$$

We assume that $Q = (p_1(k), p_2(k), \ldots, p_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ and $W = (q_1(k), q_2(k), \ldots, q_n(k), w_1(k), w_2(k), \ldots, w_n(k))$ are any two solutions of system (45) defined on $Z^+ \times S^*$; then, $\|Q\| \le B$ and $\|W\| \le B$, where $B = \sum_{i=1}^{n}\{A_i + B_i\}$ and $S^* = \{(p_1(k), p_2(k), \ldots, p_n(k), u_1(k), u_2(k), \ldots, u_n(k)) \mid \ln m_i \le p_i(n) \le \ln M_i, n_i \le u_i(k) \le N_i, i = 1, 2, \ldots, n, k \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(k, Q, W) = \sum_{i=1}^{n} \left\{ \left(p_i(k) - q_i(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2 \right\}. \tag{48}$$

It is obvious that the norm $\|Q - W\| = \sum_{i=1}^{n}\{|p_i(k) - q_i(k)| + |u_i(k) - w_i(k)|\}$ is equivalent to $\|Q - W\|_* = [\sum_{i=1}^{n}\{(p_i(k) - q_i(k))^2 + (u_i(k) - w_i(k))^2\}]^{1/2}$; that is, there are two constants $c_1 > 0$ and $c_2 > 0$, such that

$$c_1 \|Q - W\| \le \|Q - W\|_* \le c_2 \|Q - W\|, \tag{49}$$

and then,

$$\left(c_1 \|Q - W\|\right)^2 \le V(k, Q, W) \le \left(c_2 \|Q - W\|\right)^2. \tag{50}$$

Let $\psi, \varphi \in C(R^+, R^+)$, $\psi(x) = c_1^2 x^2$, $\varphi(x) = c_2^2 x^2$; then, condition (i) of Theorem 9 is satisfied.

Moreover, for any $(k, Q, W), (k, \overline{Q}, \overline{W}) \in Z^+ \times S^* \times S^*$, we have

$$\begin{aligned} &\left|V(k, Q, W) - V\left(k, \overline{Q}, \overline{W}\right)\right| \\ &= \left|\sum_{i=1}^{n} \left\{ \left(p_i(k) - q_i(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2 \right\} \right. \\ &\qquad \left. - \sum_{i=1}^{n} \left\{ \left(\overline{p_i}(k) - \overline{q_i}(k)\right)^2 + \left(\overline{u_i}(k) - \overline{w_i}(k)\right)^2 \right\} \right| \\ &\le \sum_{i=1}^{n} \left| \left(p_i(k) - q_i(k)\right)^2 - \left(\overline{p_i}(k) - \overline{q_i}(k)\right)^2 \right| \\ &\quad + \sum_{i=1}^{n} \left| \left(u_i(k) - w_i(k)\right)^2 - \left(\overline{u_i}(k) - \overline{w_i}(k)\right)^2 \right| \\ &= \sum_{i=1}^{n} \left| \left(p_i(k) - q_i(k)\right) + \left(\overline{p_i}(k) - \overline{q_i}(k)\right) \right| \\ &\quad \times \left| \left(p_i(k) - q_i(k)\right) - \left(\overline{p_i}(k) - \overline{q_i}(k)\right) \right| \\ &\quad + \sum_{i=1}^{n} \left| \left(u_i(k) - w_i(k)\right) + \left(\overline{u_i}(k) - \overline{w_i}(k)\right) \right| \\ &\quad \times \left| \left(u_i(k) - w_i(k)\right) - \left(\overline{u_i}(k) - \overline{w_i}(k)\right) \right| \\ &\le \sum_{i=1}^{n} \left(\left|p_i(k)\right| + \left|q_i(k)\right| + \left|\overline{p_i}(k)\right| + \left|\overline{q_i}(k)\right| \right) \\ &\quad \times \left(\left|p_i(k) - \overline{p_i}(k)\right| + \left|q_i(k) - \overline{q_i}(k)\right| \right) \\ &\quad + \sum_{i=1}^{n} \left(\left|u_i(k)\right| + \left|w_i(k)\right| + \left|\overline{u_i}(k)\right| + \left|\overline{w_i}(k)\right| \right) \\ &\quad \times \left(\left|u_i(k) - \overline{u_i}(k)\right| + \left|w_i(k) - \overline{w_i}(k)\right| \right) \\ &\le L \left[\sum_{i=1}^{n} \left\{ \left|p_i(k) - \overline{p_i}(k)\right| + \left|u_i(k) - \overline{u_i}(k)\right| \right\} \right. \\ &\qquad \left. + \sum_{i=1}^{n} \left\{ \left|q_i(k) - \overline{q_i}(k)\right| + \left|w_i(k) - \overline{w_i}(k)\right| \right\} \right] \\ &= L \left(\left\|Q - \overline{Q}\right\| + \left\|W - \overline{W}\right\| \right), \end{aligned} \tag{51}$$

where $\overline{Q} = (\overline{p_1}(k), \overline{p_2}(k), \ldots, \overline{p_n}(k), \overline{u_1}(k), \overline{u_2}(k), \ldots, \overline{u_n}(k))$, $\overline{W} = (\overline{q_1}(k), \overline{q_2}(k), \ldots, \overline{q_n}(k), \overline{w_1}(k), \overline{w_2}(k), \ldots, \overline{w_n}(k))$, and $L = 4\max\{\max_{1\le i\le n}\{A_i\}, \max_{1\le i\le n}\{B_i\}\}$. Thus, condition (ii) of Theorem 9 is satisfied.

Finally, calculating the $\Delta V(k)$ of $V(k)$ along the solutions of system (47), we have

$$\begin{aligned} &\Delta V_{(47)}(k) \\ &= V(k+1) - V(k) \\ &= \sum_{i=1}^{n} \left\{ \left(p_i(k+1) - q_i(k+1)\right)^2 + \left(u_i(k+1) - w_i(k+1)\right)^2 \right\} \\ &\quad - \sum_{i=1}^{n} \left\{ \left(p_i(k) - q_i(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2 \right\} \\ &= \sum_{i=1}^{n} \left\{ \left(p_i(k+1) - q_i(k+1)\right)^2 - \left(p_i(k) - q_i(k)\right)^2 \right. \end{aligned}$$

$$
\begin{aligned}
&+\left(u_i(k+1)-w_i(k+1)\right)^2-\left(u_i(k)-w_i(k)\right)^2\Big\}\\
&=\sum_{i=1}^{n}\Bigg\{\Bigg[\left(p_i(k)-q_i(k)\right)-b_i(k)\left(e^{p_i(k)}-e^{q_i(k)}\right)\\
&\quad+\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad-e_i(k)(u_i(k)-w_i(k))\Bigg]^2-\left(p_i(k)-q_i(k)\right)^2\\
&\quad+\Bigg[\left(1-f_i(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\qquad+\sum_{j=1}^{n}g_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)\Bigg]^2\\
&\quad-\left(u_i(k)-w_i(k)\right)^2\Bigg\}\\
&=\sum_{i=1}^{n}\Bigg\{-2b_i(k)\left(p_i(k)-q_i(k)\right)\left(e^{p_i(k)}-e^{q_i(k)}\right)\\
&\quad+b_i^2(k)\left(e^{p_i(k)}-e^{q_i(k)}\right)^2-2b_i(k)\left(e^{p_i(k)}-e^{q_i(k)}\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad+2b_i(k)\,e_i(k)\left(e^{p_i(k)}-e^{q_i(k)}\right)\left(u_i(k)-w_i(k)\right)\\
&\quad-2e_i(k)\left(u_i(k)-w_i(k)\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad+\left(\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\right)^2\\
&\quad+2\left(p_i(k)-q_i(k)\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad-2e_i(k)\left(p_i(k)-q_i(k)\right)\\
&\quad\times(u_i(k)-w_i(k))+e_i^2(k)\left(u_i(k)-w_i(k)\right)^2\\
&\quad+f_i(k)\left(f_i(k)-2\right)\left(u_i(k)-w_i(k)\right)^2\\
&\quad+2\left(1-f_i(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\quad\times\sum_{j=1}^{n}g_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)\\
&\quad+\left(\sum_{j=1}^{n}g_{ij}(k)\left(e^{p_j(k)}-e^{q_j(k)}\right)\right)^2\Bigg\}.
\end{aligned}
\tag{52}
$$

By the mean value theorem, it derives that

$$
e^{p_i(k)}-e^{q_i(k)}=\xi_i(k)\left(p_i(k)-q_i(k)\right),\quad i=1,2,\ldots,n,
\tag{53}
$$

where $\xi_i(k)$ lies between $e^{p_i(k)}$ and $e^{q_i(k)}$. Then, we have

$$
\begin{aligned}
&\Delta V_{(47)}(k)\\
&=\sum_{i=1}^{n}\Bigg\{-2b_i(k)\,\xi_i(k)\left(p_i(k)-q_i(k)\right)^2\\
&\quad+b_i^2(k)\,\xi_i^2(k)\left(p_i(k)-q_i(k)\right)^2\\
&\quad-2b_i(k)\,\xi_i(k)\left(p_i(k)-q_i(k)\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad+2b_i(k)\,e_i(k)\,\xi_i(k)\left(p_i(k)-q_i(k)\right)\\
&\quad\times\left(u_i(k)-w_i(k)\right)-2e_i(k)\left(u_i(k)-w_i(k)\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad+\left(\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\right)^2\\
&\quad+2\left(p_i(k)-q_i(k)\right)\\
&\quad\times\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\quad-2e_i(k)\left(p_i(k)-q_i(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\quad+e_i^2(k)\left(u_i(k)-w_i(k)\right)^2\\
&\quad+f_i(k)\left(f_i(k)-2\right)\left(u_i(k)-w_i(k)\right)^2\\
&\quad+2\left(1-f_i(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\quad\times\sum_{j=1}^{n}g_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{j=1}^{n} g_{ij}(k)\,\xi_j(k)\left(p_j(k)-q_j(k)\right)\right)^2\Bigg\}\\
&=\sum_{i=1}^{n}\Bigg\{\Bigg(-2b_i(k)\,\xi_i(k)+b_i^2(k)\,\xi_i^2(k)+\sum_{j=1}^{n} g_{ji}^2(k)\,\xi_i^2(k)\\
&\quad+\sum_{j=1,j\neq i}^{n}\frac{c_{ji}^2(k)\,d_{ji}^2(k)\,\xi_i^2(k)}{\left(d_{ji}(k)+e^{p_i(k)}\right)^2\left(d_{ji}(k)+e^{q_i(k)}\right)^2}\Bigg)\\
&\quad\times\left(p_i(k)-q_i(k)\right)^2\\
&\quad+2\sum_{j=1,j\neq i}^{n}\Bigg(\frac{\left(1-2b_i(k)\,\xi_i(k)\right)c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\\
&\qquad+\frac{1}{2}\\
&\qquad\times\sum_{l=1,l\neq i,j}^{n}\frac{c_{li}(k)\,d_{li}(k)\,\xi_i(k)}{\left(d_{li}(k)+e^{p_i(k)}\right)\left(d_{li}(k)+e^{q_i(k)}\right)}\\
&\qquad\times\frac{c_{lj}(k)\,d_{lj}(k)\,\xi_j(k)}{\left(d_{lj}(k)+e^{p_j(k)}\right)\left(d_{lj}(k)+e^{q_j(k)}\right)}\\
&\qquad+\frac{1}{2}\sum_{l=1}^{n} g_{li}(k)\,g_{lj}(k)\,\xi_i(k)\,\xi_j(k)\Bigg)\\
&\quad\times\left(p_i(k)-q_i(k)\right)\left(p_j(k)-q_j(k)\right)\\
&\quad+2\left[e_i(k)\left(b_i(k)\,\xi_i(k)-1\right)\right.\\
&\qquad\left.+g_{ii}(k)\,\xi_i(k)\left(1-f_i(k)\right)\right]\\
&\quad\times\left(p_i(k)-q_i(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\quad+2\sum_{j=1,j\neq i}^{n}\Bigg[\left(1-f_i(k)\right)g_{ij}(k)\,\xi_j(k)-e_i(k)\\
&\qquad\times\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\Bigg]\\
&\quad\times\left(p_j(k)-q_j(k)\right)\left(u_i(k)-w_i(k)\right)\\
&\quad+\left[e_i^2(k)+f_i(k)\left(f_i(k)-2\right)\right]\left(u_i(k)-w_i(k)\right)^2\Bigg\}\\
&\le\sum_{i=1}^{n}\Bigg\{\Bigg(-2b_i(k)\,\xi_i(k)+b_i^2(k)\,\xi_i^2(k)\\
&\quad+\sum_{j=1}^{n} g_{ji}^2(k)\,\xi_i^2(k)\\
&\quad+\sum_{j=1,j\neq i}^{n}\frac{c_{ji}^2(k)\,d_{ji}^2(k)\,\xi_i^2(k)}{\left(d_{ji}(k)+e^{p_i(k)}\right)^2\left(d_{ji}(k)+e^{q_i(k)}\right)^2}\Bigg)\\
&\quad\times\left(p_i(k)-q_i(k)\right)^2+2\\
&\quad\times\Bigg|\sum_{j=1,j\neq i}^{n}\Bigg(\frac{\left(1-2b_i(k)\,\xi_i(k)\right)c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}+\frac{1}{2}\\
&\qquad\times\sum_{l=1,l\neq i,j}^{n}\frac{c_{li}(k)\,d_{li}(k)\,\xi_i(k)}{\left(d_{li}(k)+e^{p_i(k)}\right)\left(d_{li}(k)+e^{q_i(k)}\right)}\\
&\qquad\times\frac{c_{lj}(k)\,d_{lj}(k)\,\xi_j(k)}{\left(d_{lj}(k)+e^{p_j(k)}\right)\left(d_{lj}(k)+e^{q_j(k)}\right)}\\
&\qquad+\frac{1}{2}\sum_{l=1}^{n} g_{li}(k)\,g_{lj}(k)\,\xi_i(k)\,\xi_j(k)\Bigg)\\
&\qquad\times\left(p_i(k)-q_i(k)\right)\left(p_j(k)-q_j(k)\right)\Bigg|\\
&\quad+2\Big|\left[e_i(k)\left(b_i(k)\,\xi_i(k)-1\right)\right.\\
&\qquad\left.+g_{ii}(k)\,\xi_i(k)\left(1-f_i(k)\right)\right]\\
&\qquad\times\left(p_i(k)-q_i(k)\right)\left(u_i(k)-w_i(k)\right)\Big|\\
&\quad+2\Bigg|\sum_{j=1,j\neq i}^{n}\Bigg[\left(1-f_i(k)\right)g_{ij}(k)\,\xi_j(k)-e_i(k)\\
&\qquad\times\frac{c_{ij}(k)\,d_{ij}(k)\,\xi_j(k)}{\left(d_{ij}(k)+e^{p_j(k)}\right)\left(d_{ij}(k)+e^{q_j(k)}\right)}\Bigg]\\
&\qquad\times\left(p_j(k)-q_j(k)\right)\left(u_i(k)-w_i(k)\right)\Bigg|\\
&\quad+\left[e_i^2(k)+f_i(k)\left(f_i(k)-2\right)\right]\\
&\quad\times\left(u_i(k)-w_i(k)\right)^2\Bigg\}.
\end{aligned}
\tag{54}
$$

Then, we have

$$
\Delta V_{(47)}(k)\le\sum_{i=1}^{n}\left\{V_{i1}+V_{i2}+V_{i3}+V_{i4}+V_{i5}\right\},
\tag{55}
$$

where

$$
\begin{aligned}
V_{i1}=&\Bigg(-2b_i(k)\,\xi_i(k)+b_i^2(k)\,\xi_i^2(k)+\sum_{j=1}^{n} g_{ji}^2(k)\,\xi_i^2(k)\\
&+\sum_{j=1,j\neq i}^{n}\frac{c_{ji}^2(k)\,d_{ji}^2(k)\,\xi_i^2(k)}{\left(d_{ji}(k)+e^{p_i(k)}\right)^2\left(d_{ji}(k)+e^{q_i(k)}\right)^2}\Bigg)
\end{aligned}
$$

$$\times \left(p_i(k) - q_i(k)\right)^2$$

$$\leq \left(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1}^{n} g_{ji}^{u2} M_i^2 + \sum_{j=1, j\neq i}^{n} c_{ji}^{u2}\right)$$

$$\times \left(p_i(k) - q_i(k)\right)^2,$$

$$V_{i2} = 2\left|\sum_{j=1, j\neq i}^{n}\left(\frac{\left(1 - 2b_i(k)\xi_i(k)\right)c_{ij}(k)d_{ij}(k)\xi_j(k)}{\left(d_{ij}(k) + e^{p_j(k)}\right)\left(d_{ij}(k) + e^{q_j(k)}\right)}\right.\right.$$

$$+ \frac{1}{2}\sum_{l=1, l\neq i,j}^{n}\frac{c_{li}(k)d_{li}(k)\xi_i(k)}{\left(d_{li}(k) + e^{p_i(k)}\right)\left(d_{li}(k) + e^{q_i(k)}\right)}$$

$$\times \frac{c_{lj}(k)d_{lj}(k)\xi_j(k)}{\left(d_{lj}(k) + e^{p_j(k)}\right)\left(d_{lj}(k) + e^{q_j(k)}\right)}$$

$$\left.+\frac{1}{2}\sum_{l=1}^{n} g_{li}(k)g_{lj}(k)\xi_i(k)\xi_j(k)\right)$$

$$\left.\times \left(p_i(k) - q_i(k)\right)\left(p_j(k) - q_j(k)\right)\right|$$

$$\leq \sum_{j=1, j\neq i}^{n}\left(\left(1 + 2b_i^u M_i\right)c_{ij}^u + \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u\right.$$

$$\left.+\frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j\right)$$

$$\times \left[\left(p_i(k) - q_i(k)\right)^2 + \left(p_j(k) - q_j(k)\right)^2\right],$$

$$V_{i3} = 2\left|\left[e_i(k)\left(b_i(k)\xi_i(k) - 1\right) + g_{ii}(k)\xi_i(k)\left(1 - f_i(k)\right)\right]\right.$$

$$\left.\times \left(p_i(k) - q_i(k)\right)\left(u_i(k) - w_i(k)\right)\right|$$

$$\leq \left[e_i^u\left(b_i^u M_i + 1\right) + g_{ii}^u M_i\left(1 - f_i^l\right)\right]$$

$$\times \left[\left(p_i(k) - q_i(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2\right],$$

$$V_{i4} = 2\left|\sum_{j=1, j\neq i}^{n}\left[\left(1 - f_i(k)\right)g_{ij}(k)\xi_j(k)\right.\right.$$

$$\left.-e_i(k)\frac{c_{ij}(k)d_{ij}(k)\xi_j(k)}{\left(d_{ij}(k) + e^{p_j(k)}\right)\left(d_{ij}(k) + e^{q_j(k)}\right)}\right]$$

$$\left.\times \left(p_j(k) - q_j(k)\right)\left(u_i(k) - w_i(k)\right)\right|$$

$$\leq \sum_{j=1, j\neq i}^{n}\left[\left(1 - f_i^l\right)g_{ij}^u M_j + e_i^u c_{ij}^u\right]$$

$$\times \left[\left(p_j(k) - q_j(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2\right],$$

$$V_{i5} = \left[e_i^2(k) + f_i(k)\left(f_i(k) - 2\right)\right]\left(u_i(k) - w_i(k)\right)^2$$

$$\leq \left[e_i^{u2} + f_i^l\left(f_i^u - 2\right)\right]\left(u_i(k) - w_i(k)\right)^2. \tag{56}$$

Hence, we have

$$\Delta V_{(47)}(k)$$

$$\leq \sum_{i=1}^{n}\left\{\left(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1}^{n} g_{ji}^{u2} M_i^2\right.\right.$$

$$+ \sum_{j=1, j\neq i}^{n} c_{ji}^{u2} + \left(1 + 2b_i^u M_i\right)c_{ij}^u + \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u$$

$$+ \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j + e_i^u\left(b_i^u M_i + 1\right)$$

$$\left.+ g_{ii}^u M_i\left(1 - f_i^l\right)\right)\left(p_i(k) - q_i(k)\right)^2$$

$$+ \left(\left[e_i^u\left(b_i^u M_i + 1\right) + g_{ii}^u M_i\left(1 - f_i^l\right)\right]\right.$$

$$+ \left[e_i^{u2} + f_i^l\left(f_i^u - 2\right)\right]$$

$$\left.+ \sum_{j=1, j\neq i}^{n}\left[\left(1 - f_i^l\right)g_{ij}^u M_j + e_i^u c_{ij}^u\right]\right)$$

$$\times \left(u_i(k) - w_i(k)\right)^2$$

$$+ \sum_{j=1, j\neq i}^{n}\left(\left(1 + 2b_i^u M_i\right)c_{ij}^u + \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u\right.$$

$$+ \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j$$

$$\left.+ \left(1 - f_i^l\right)g_{ij}^u M_j + e_i^u c_{ij}^u\right)$$

$$\left.\times \left(p_j(k) - q_j(k)\right)^2\right\}$$

$$= \sum_{i=1}^{n}\left\{\left(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1}^{n} g_{ji}^{u2} M_i^2\right.\right.$$

$$+ \sum_{j=1, j\neq i}^{n} c_{ji}^{u2} + \left(1 + 2b_i^u M_i\right)c_{ij}^u$$

$$+ \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u + \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j$$

$$
\begin{aligned}
&+ e_i^u\left(b_i^u M_i + 1\right) + g_{ii}^u M_i\left(1 - f_i^l\right)\Bigg) \\
&\times \left(p_i(k) - q_i(k)\right)^2 \\
&+ \Bigg(e_i^u\left(b_i^u M_i + 1\right) + g_{ii}^u M_i\left(1 - f_i^l\right) \\
&\quad + e_i^{u2} + f_i^l\left(f_i^u - 2\right) \\
&\quad + \sum_{j=1, j\neq i}^{n}\left[\left(1 - f_i^l\right) g_{ij}^u M_j + e_i^u c_{ij}^u\right]\Bigg) \\
&\times \left(u_i(k) - w_i(k)\right)^2 \\
&+ \sum_{j=1, j\neq i}^{n}\Bigg(\left(1 + 2b_j^u M_j\right) c_{ji}^u + \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \\
&\quad + \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j \\
&\quad + \left(1 - f_j^l\right) g_{ji}^u M_i + e_j^u c_{ji}^u\Bigg) \\
&\times \left(p_i(k) - q_i(k)\right)^2\Bigg\} \\
= &-\sum_{i=1}^{n}\Bigg\{\Bigg(2b_i^l m_i - b_i^{u2} M_i^2 - \sum_{j=1}^{n} g_{ji}^{u2} M_i^2 \\
&\quad - \sum_{j=1, j\neq i}^{n} c_{ji}^{u2} - \left(1 + 2b_i^u M_i\right) c_{ij}^u \\
&\quad - \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u - \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j \\
&\quad - e_i^u\left(b_i^u M_i + 1\right) - g_{ii}^u M_i\left(1 - f_i^l\right) \\
&\quad - \sum_{j=1, j\neq i}^{n}\Bigg[\left(1 + 2b_j^u M_j\right) c_{ji}^u + \frac{1}{2}\sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \\
&\qquad + \frac{1}{2}\sum_{l=1}^{n} g_{li}^u g_{lj}^u M_i M_j \\
&\qquad + \left(1 - f_j^l\right) g_{ji}^u M_i + e_j^u c_{ji}^u\Bigg]\Bigg) \\
&\quad \times \left(p_i(k) - q_i(k)\right)^2 \\
&\quad + \Bigg(f_i^l\left(2 - f_i^u\right) - e_i^u\left(b_i^u M_i + 1\right) \\
&\qquad - g_{ii}^u M_i\left(1 - f_i^l\right) - e_i^{u2} \\
&\qquad - \sum_{j=1, j\neq i}^{n}\left[\left(1 - f_i^l\right) g_{ij}^u M_j + e_i^u c_{ij}^u\right]\Bigg) \\
&\qquad \times \left(u_i(k) - w_i(k)\right)^2\Bigg\} \\
\le &-\sum_{i=1}^{n}\left[\beta_{i1}\left(p_i(k) - q_i(k)\right)^2 + \beta_{i2}\left(u_i(k) - w_i(k)\right)^2\right] \\
\le &-\beta\sum_{i=1}^{n}\left[\left(p_i(k) - q_i(k)\right)^2 + \left(u_i(k) - w_i(k)\right)^2\right] \\
= &-\beta V(k, Q, W),
\end{aligned}
\tag{57}
$$

where $\beta = \min\{\min_{1\le i\le n}\beta_i^1, \min_{1\le i\le n}\beta_i^2\}$. That is, there exists a positive constant $0 < \beta < 1$ such that

$$\Delta V_{(47)}(k, Q, W) \le -\beta V(k, Q, W). \tag{58}$$

From $0 < \beta < 1$, the condition (iii) of Theorem 9 is satisfied. Then, according to Theorem 9, there exists a unique uniformly asymptotically stable almost periodic solution $(p_1(k), p_2(k), \ldots, p_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of (45) which is bounded by S^* for all $k \in Z^+$. It means that there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_n(k))$ of system (1) which is bounded by Ω for all $k \in Z^+$. This completed the proof. □

Remark 13. If $n = 2$, the conditions of Theorem 12 can be simplified. Therefore, we have the following result.

Corollary 14. *Let $n = 2$, and assume further that $0 < \beta < 1$, where*

$$
\begin{aligned}
\beta &= \min\left\{\beta_{12}^1, \beta_{21}^1, \beta_{12}^2, \beta_{21}^2\right\}, \\
\beta_{ij}^1 &= 2b_i^l m_i - b_i^{u2} M_i^2 - \sum_{l=1}^{2} g_{li}^{u2} M_i^2 - c_{ji}^{u2} \\
&\quad - \left(1 + 2b_i^u M_i\right) c_{ij}^u - \sum_{l=1}^{2} g_{li}^u g_{lj}^u M_i M_j \\
&\quad - e_i^u\left(b_i^u M_i + 1\right) - g_{ii}^u M_i\left(1 - f_i^l\right) \\
&\quad - \left(1 + 2b_j^u M_j\right) c_{ji}^u - \left(1 - f_j^l\right) g_{ji}^u M_i - e_j^u c_{ji}^u, \\
\beta_{ij}^2 &= f_i^l\left(2 - f_i^u\right) - e_i^u\left(b_i^u M_i + 1\right) \\
&\quad - g_{ii}^u M_i\left(1 - f_i^l\right) - e_i^{u2} - \left[\left(1 - f_i^l\right) g_{ij}^u M_j + e_i^u c_{ij}^u\right],
\end{aligned}
\tag{59}
$$

$i, j = 1, 2$, $j \neq i$. Then system (1) admits a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), u_1(k), u_2(k))$ which is bounded by Ω for all $k \in Z^+$.

5. Example and Numerical Simulation

In this section, we give the following example to check the feasibility of our result.

Example 15. Consider the following almost periodic discrete Lotka-Volterra mutualism system with feedback controls:

$$\begin{aligned}
x_1(k+1) &= x_1(k) \times \exp\Bigg\{1.2 - 0.02\sin\left(\sqrt{2}k\right) \\
&\quad - \left(1.05 + 0.01\sin\left(\sqrt{2}k\right)\right)x_1(k) \\
&\quad + \frac{\left(0.025 + 0.002\cos\left(\sqrt{2}k\right)\right)x_2(k)}{0.2 + 0.003\cos\left(\sqrt{2}k\right) + x_2(k)} \\
&\quad + \frac{\left(0.02 + 0.001\cos\left(\sqrt{2}k\right)\right)x_3(k)}{0.4 + 0.03\cos\left(\sqrt{2}k\right) + x_3(k)} \\
&\quad - \left(0.025 + 0.002\cos\left(\sqrt{3}k\right)\right)u_1(k)\Bigg\}, \\
x_2(k+1) &= x_2(k) \times \exp\Bigg\{1.1 - 0.025\sin\left(\sqrt{2}k\right) \\
&\quad - \left(1.08 + 0.015\sin\left(\sqrt{2}k\right)\right)x_2(k) \\
&\quad + \frac{\left(0.02 + 0.003\cos\left(\sqrt{2}k\right)\right)x_1(k)}{0.3 + 0.02\cos\left(\sqrt{3}k\right) + x_1(k)} \\
&\quad + \frac{\left(0.025 + 0.002\cos\left(\sqrt{2}k\right)\right)x_3(k)}{0.2 + 0.07\sin\left(\sqrt{2}k\right) + x_3(k)} \\
&\quad - \left(0.025 + 0.002\sin\left(\sqrt{2}k\right)\right)u_2(k)\Bigg\}, \\
x_3(k+1) &= x_3(k) \times \exp\Bigg\{1.15 - 0.03\sin\left(\sqrt{2}k\right) \\
&\quad - \left(1.1 + 0.002\sin\left(\sqrt{2}k\right)\right)x_3(k) \\
&\quad + \frac{\left(0.03 + 0.0025\cos\left(\sqrt{2}k\right)\right)x_1(k)}{0.2 + 0.004\sin\left(\sqrt{2}k\right) + x_1(k)} \\
&\quad + \frac{\left(0.028 + 0.0015\cos\left(\sqrt{2}k\right)\right)x_2(k)}{0.2 + 0.004\cos\left(\sqrt{3}k\right) + x_2(k)} \\
&\quad - \left(0.025 + 0.002\cos\left(\sqrt{2}k\right)\right)u_3(k)\Bigg\}, \\
\Delta u_1(k) &= -\left(0.925 - 0.03\sin\left(\sqrt{2}k\right)\right)u_1(k) \\
&\quad + \left(0.015 - 0.005\sin\left(\sqrt{2}k\right)\right)x_1(k) \\
&\quad + \left(0.013 - 0.004\sin\left(\sqrt{3}k\right)\right)x_2(k) \\
&\quad + \left(0.02 - 0.006\cos\left(\sqrt{2}k\right)\right)x_3(k), \\
\Delta u_2(k) &= -\left(0.925 - 0.04\sin\left(\sqrt{3}k\right)\right)u_2(k) \\
&\quad + \left(0.016 - 0.004\sin\left(\sqrt{3}k\right)\right)x_1(k) \\
&\quad + \left(0.015 - 0.005\cos\left(\sqrt{2}k\right)\right)x_2(k) \\
&\quad + \left(0.014 - 0.004\sin\left(\sqrt{2}k\right)\right)x_3(k), \\
\Delta u_3(k) &= -\left(0.935 - 0.035\cos\left(\sqrt{2}k\right)\right)u_3(k) \\
&\quad + \left(0.017 - 0.006\cos\left(\sqrt{2}k\right)\right)x_1(k) \\
&\quad + \left(0.013 - 0.005\sin\left(\sqrt{3}k\right)\right)x_2(k) \\
&\quad + \left(0.016 - 0.005\cos\left(\sqrt{2}k\right)\right)x_3(k).
\end{aligned} \tag{60}$$

By simple computation, we derive

$$\begin{aligned}
&\beta_1^1 \approx 0.0092, \quad \beta_2^1 \approx 0.0042, \quad \beta_3^1 \approx 0.0014, \\
&\beta_1^2 \approx 0.0031, \quad \beta_2^2 \approx 0.0081, \quad \beta_3^2 \approx 0.0072.
\end{aligned} \tag{61}$$

Then

$$0 < \beta = \min\left\{\beta_1^1, \beta_2^1, \beta_3^1, \beta_1^2, \beta_2^2, \beta_3^2\right\} < 1. \tag{62}$$

Also it is easy to see that the conditions of Theorem 12 are verified. Therefore, system (60) has a unique positive almost periodic solution which is uniformly asymptotically stable. Our numerical simulations support our results (see Figure 1).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper, and there are no financial conflict of interests between the authors and the commercial identity.

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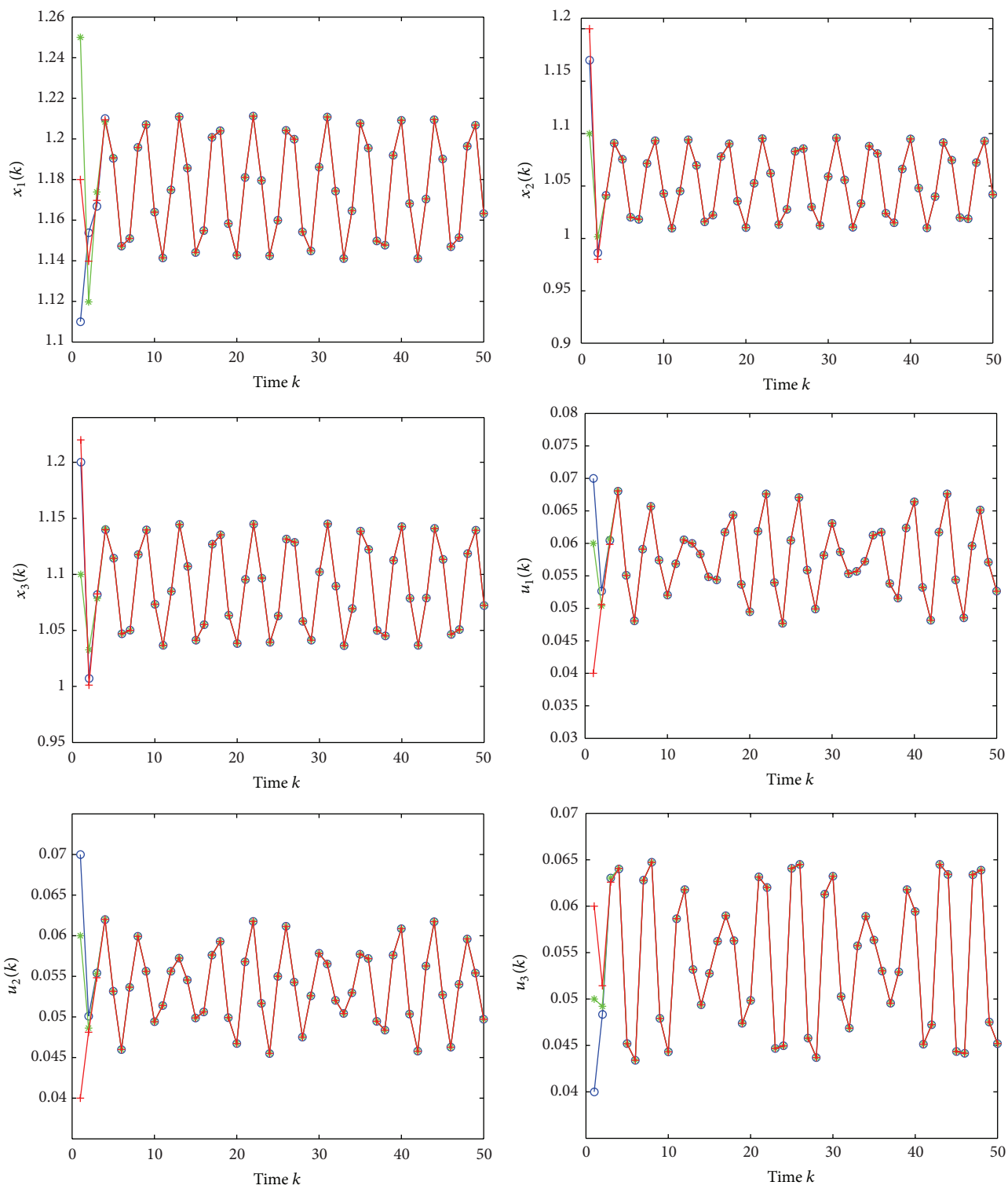

Figure 1: Dynamic behavior of positive almost periodic solution $(x_1(k), x_2(k), x_3(k), u_1(k), u_2(k), u_3(k))$ of system (60) with the three initial conditions (1.11, 1.17, 1.2, 0.07, 0.07, 0.04), (1.25, 1.1, 1.1, 0.06, 0.06, 0.05), and (1.18, 1.2, 1.22, 0.04, 0.04, 0.06) for $k \in [1, 50]$, respectively.

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