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Research Article

Fixed Point Theorems of Binary Contraction Comparable Operators and an Application

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The aim of this paper is to present the concept of binary comparable operators in partially ordered Banach spaces and prove several fixed point theorems under some contractive conditions. The results of this paper can be used to investigate a large class of nonlinear problems. As an application, we study the existence of solution of a nonlinear integral equation.

1. Introduction

The Banach contraction principle [1] as a popular tool for solving problems in nonlinear analysis was invented in 1922. Since then, the study of fixed points of mappings with contractive property has been at the center of various research activities. For example, Liu and Zhu [2] studied the solvability of a binary operator equation satisfying certain contractive conditions; Romaguera [3] obtained several fixed point theorems of mappings satisfying some generalized contractive conditions. For more details on fixed point results of contractive type mappings and applications, we refer to Yan et al. [4], Mukherjea [5], Ran and Reurings [6], Hussain et al. [7], Amini-Harandi [8], Sintunavarat and Kumam [9], Nieto and Rodríguez-López [10, 11], and O'Regan and Saadati [12].

One of the common properties of the above results is that the involved operators must satisfy the monotone or mixed monotone conditions. In 2005, Zhang [13] studied an ordinal L-ordering symmetric contraction operator without the mixed monotone property and proved some coupled fixed point theorems. On the other hand, Qiao [14] investigated the fixed point theorems of the ordered contractive operators with the comparable property.

In the present paper, we introduce the concept of binary comparable operators, which can be seen as a generalization of the concept of mixed monotone operators and the concept of antimixed monotone operators. Using the iterative techniques ([15, 16]), we obtain several fixed point theorems for such operators under some contractive conditions. The results of this paper generalize several classical results in the literature. As an application, the existence of solution of an integral equation is presented.

For the sake of convenience, let us recall the following definitions and lemmas (see [14, 17, 18] for more details and recent results).

Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(i) P is closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in R$, $a, b \geq 0$, and $x, y \in P$ implies $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.

Let E be a normed Banach space, which is partially ordered by a cone P. The cone P is said to be normal if there exists a constant $N > 0$, such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E.

Definition 1 (see [14]). For some $x, y \in E$, if $x \leq y$ or $y \leq x$ holds, x and y are said to be comparable. Moreover, one will write $x \vee y = y$ to indicate that $x \leq y$, while $x \vee y = x$ stand for $y \leq x$.

Lemma 2 (see [14]). *If $x, y \in E$ are comparable, then $x - y$ and $y - x$ are comparable, and*

$$\theta \leq (x - y) \vee (y - x). \tag{1}$$

Lemma 3 (see [14]). *For some $x, y, z \in E$, if any two of them are comparable, then*

$$\begin{aligned}(x - y) \vee (y - x) \leq {} & [(x - z) \vee (z - x)] \\ & + [(z - y) \vee (y - z)].\end{aligned} \tag{2}$$

Lemma 4 (see [14]). *If, for each positive integer n, x and y_n are comparable and $\lim_{n\to\infty} y_n \to y_0$, then x and y_0 are comparable.*

Lemma 5 (see [14]). *If, for each positive integer n, x_n and y_n are comparable and $\lim_{n\to\infty} x_n \to x_0$, $\lim_{n\to\infty} y_n \to y_0$, then x_0 and y_0 are comparable.*

Definition 6. A binary operator $A : E \times E \to E$ is said to be comparable if, for all comparable pairs $x_1, x_2 \in E$ and $y_1, y_2 \in E$, $A(x_1, y_1)$ and $A(x_2, y_2)$ are comparable.

Definition 7. A comparable operator $A : E \times E \to E$ is said to be α-contractive, if, for all comparable pairs $x_1, x_2 \in E$ and $y_1, y_2 \in E$, there exists a constant $\alpha \in (0, 1/2)$ such that

$$\begin{aligned}&[A(x_1, y_1) - A(x_2, y_2)] \vee [A(x_2, y_2) - A(x_1, y_1)] \\ &\quad \leq \alpha \{[(x_1 - x_2) \vee (x_2 - x_1)] \\ &\quad + [(y_1 - y_2) \vee (y_2 - y_1)]\}.\end{aligned} \tag{3}$$

2. Main Results

Theorem 8. *Let E be a real Banach space, P a normal cone of E with the normal constant N, and $\leq$ a partial order with respect to P. Let $A : E \times E \to E$ be demicontinuous. Suppose that the following two conditions are satisfied:*

(i) *A is α-contractive and comparable, where $\alpha \in (0, 1/2)$;*

(ii) *there exists a comparable pair (x_0, y_0) in E, such that x_0 and $A(x_0, y_0)$, y_0 and $A(y_0, x_0)$ are comparable.*

Then A has a fixed point x^* in E; that is, $A(x^*, x^*) = x^*$. Moreover, the iterative sequences $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ converge to x^*, and

$$\begin{aligned}&\|x_0 - x^*\| \\ &\quad \leq \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) - x_0\| + \|A(y_0, x_0) - y_0\|), \\ &\|y_0 - x^*\| \\ &\quad \leq \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) - x_0\| + \|A(y_0, x_0) - y_0\|).\end{aligned} \tag{4}$$

Proof. Consider the iterative sequences

$$\begin{aligned}x_1 &= A(x_0, y_0), \\ &\vdots \\ x_n &= A(x_{n-1}, y_{n-1}); \\ y_1 &= A(y_0, x_0), \\ &\vdots \\ y_n &= A(y_{n-1}, x_{n-1}); \\ & \qquad n = 1, 2, \ldots.\end{aligned} \tag{5}$$

Since x_0 and y_0 are comparable and $A : E \times E \to E$ is a comparable operator, it is easy to verify that $x_1 = A(x_0, y_0)$ and $y_1 = A(y_0, x_0)$ are comparable. By inductions, we can prove that x_n and y_n are comparable for any positive integer n.

Since $A : E \times E \to E$ is a α-contractive comparable operator, we have

$$\begin{aligned}&(x_n - y_n) \vee (y_n - x_n) \\ &\quad = [A(x_{n-1}, y_{n-1}) - A(y_{n-1}, x_{n-1})] \\ &\qquad \vee [A(y_{n-1}, x_{n-1}) - A(x_{n-1}, y_{n-1})] \\ &\quad \leq 2\alpha [(x_{n-1} - y_{n-1}) \vee (y_{n-1} - x_{n-1})].\end{aligned} \tag{6}$$

If we continue in the same way, for each $n \in N$, we have

$$(x_n - y_n) \vee (y_n - x_n) \leq 2^n \alpha^n [(x_0 - y_0) \vee (y_0 - x_0)]. \tag{7}$$

By the normality of P, we have

$$\|x_n - y_n\| \leq N 2^n \alpha^n \|x_0 - y_0\|. \tag{8}$$

Since (x_0, y_0) is a comparable pair in E and, moreover, x_0 and $A(x_0, y_0)$, y_0 and $A(y_0, x_0)$ are comparable, we know that $A(x_0, y_0)$ and $A[A(x_0, y_0), A(y_0, x_0)]$, $A(y_0, x_0)$ and $A[A(y_0, x_0), A(x_0, y_0)]$ are comparable, which implies that x_1 and x_2, y_1 and y_2 are comparable. By induction, we know that (x_n, x_{n+1}) and (y_n, y_{n+1}) are all comparable pairs for each $n \in N$.

Furthermore, for each $n \in N$, we have

$$\begin{aligned}&(x_n - x_{n+1}) \vee (x_{n+1} - x_n) = [A(x_{n-1}, y_{n-1}) \\ &\quad - A(x_n, y_n)] \vee [A(x_n, y_n) - A(x_{n-1}, y_{n-1})] \\ &\quad \leq \alpha \{[(x_n - x_{n-1}) \vee (x_{n-1} - x_n)] \\ &\quad + [(y_n - y_{n-1}) \vee (y_{n-1} - y_n)]\}, \\ &(y_n - y_{n+1}) \vee (y_{n+1} - y_n) = [A(y_{n-1}, x_{n-1}) \\ &\quad - A(y_n, x_n)] \vee [A(y_n, x_n) - A(y_{n-1}, x_{n-1})] \\ &\quad \leq \alpha \{[(x_n - x_{n-1}) \vee (x_{n-1} - x_n)] \\ &\quad + [(y_n - y_{n-1}) \vee (y_{n-1} - y_n)]\}.\end{aligned} \tag{9}$$

By induction we can prove that

$$\begin{aligned}
&(x_n - x_{n+1}) \vee (x_{n+1} - x_n) \\
&\quad \le 2^{n-1}\alpha^n \{[(A(x_0, y_0) - x_0) \vee (x_0 - A(x_0, y_0))] \\
&\quad + [(A(y_0, x_0) - y_0) \vee (y_0 - A(y_0, x_0))]\}, \\
&(y_n - y_{n+1}) \vee (y_{n+1} - y_n) \\
&\quad \le 2^{n-1}\alpha^n \{[(A(x_0, y_0) - x_0) \vee (x_0 - A(x_0, y_0))] \\
&\quad + [(A(y_0, x_0) - y_0) \vee (y_0 - A(y_0, x_0))]\}.
\end{aligned} \tag{10}$$

Since $\alpha \in (0, 1/2)$, we can prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in E. Then there exist two points $x^*, y^* \in E$, such that

$$\begin{aligned}
x_n &\longrightarrow x^*, \\
y_n &\longrightarrow y^* \\
&(n \longrightarrow \infty).
\end{aligned} \tag{11}$$

Since A is demicontinuous, $x_n = A(x_{n-1}, y_{n-1})$ converges to $A(x^*, y^*)$ weakly and $y_n = A(y_{n-1}, x_{n-1})$ converges to $A(y^*, x^*)$ weakly and thus $A(x^*, y^*) = x^*$, $A(y^*, x^*) = y^*$; that is, (x^*, y^*) is a coupled fixed point of A.

Taking limit in (8) as $n \to \infty$, we get

$$x^* = y^*. \tag{12}$$

This means that

$$A(x^*, x^*) = x^*. \tag{13}$$

Hence x^* is a fixed point of A.

Moreover,

$$\begin{aligned}
\|x_0 - x^*\| &= \lim_{n\to\infty} \|x_n - x_0\| \le \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i - x_{i-1}\| \\
&\le \sum_{i=1}^{\infty} \frac{N}{2} 2^{i-1} \alpha^{i-1} (\|A(x_0, y_0) - x_0\| \\
&\quad + \|A(y_0, x_0) - y_0\|) = \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) \\
&\quad - x_0\| + \|A(y_0, x_0) - y_0\|).
\end{aligned} \tag{14}$$

By a similar method, we can prove

$$\begin{aligned}
&\|y_0 - x^*\| \\
&\le \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) - x_0\| + \|A(y_0, x_0) - y_0\|).
\end{aligned} \tag{15}$$

Then we complete the proof of Theorem 8. □

Theorem 9. *Let E be a real Banach space, P a normal cone of E with the normal constant N, and $\le$ the partial order with respect to P. Suppose that the following two conditions are satisfied:*

(i) *$A : E \times E \to E$ is α-contractive comparable one, where $\alpha \in (0, 1/2)$;*

(ii) *there exists a comparable pair (x_0, y_0) in E, such that x_0 and $x_n = A(x_{n-1}, y_{n-1})$, y_0 and $y_n = A(y_{n-1}, x_{n-1})$ are comparable for each $n \in N$.*

Then A has a fixed point x^* in E; that is, $A(x^*, x^*) = x^*$. Moreover, the iterative sequences $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$ converge to x^*, and

$$\begin{aligned}
&\|x_0 - x^*\| \\
&\le \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) - x_0\| + \|A(y_0, x_0) - y_0\|), \\
&\|y_0 - x^*\| \\
&\le \frac{N}{2(1-2\alpha)} (\|A(x_0, y_0) - x_0\| + \|A(y_0, x_0) - y_0\|).
\end{aligned} \tag{16}$$

Proof. By a similar approach as in the proof of Theorem 8, we can prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in E, and there exist two points $x^*, y^* \in E$, such that

$$\begin{aligned}
x_n &\longrightarrow x^*, \\
y_n &\longrightarrow y^* \\
&(n \longrightarrow \infty).
\end{aligned} \tag{17}$$

Furthermore

$$x^* = y^*. \tag{18}$$

We now prove that x^* is a fixed point of A. For some $m, n \in N$, assume that $n < m$; then by condition (ii), we know that x_0 and $x_{m-n} = A(x_{m-n-1}, y_{m-n-1})$, y_0 and $y_{m-n} = A(y_{m-n-1}, x_{m-n-1})$ are comparable; hence $A(x_0, y_0)$ and $A(x_{m-n}, y_{m-n})$, $A(y_0, x_0)$ and $A(y_{m-n}, x_{m-n})$ are comparable. If we continue in the same way, we can prove that x_n and x_m, y_n and y_m are comparable. As $m \to \infty$, by Lemma 4, we know that, for each $n \in N$, x_n and x^*, y_n and y^* are comparable.

Hence $A(x_n, y_n)$ and $A(x^*, y^*)$, $A(y_n, x_n)$ and $A(y^*, x^*)$ are comparable; furthermore

$$\begin{aligned}
&[A(x_n, y_n) - A(x^*, y^*)] \vee [A(x^*, y^*) - A(x_n, y_n)] \\
&\quad \le \alpha \{[(x_n - x^*) \vee (x^* - x_n)] \\
&\quad + [(y_n - y^*) \vee (y^* - y_n)]\} \\
&[A(y_n, x_n) - A(y^*, x^*)] \vee [A(y^*, x^*) - A(y_n, x_n)] \\
&\quad \le \alpha \{[(x_n - x^*) \vee (x^* - x_n)] \\
&\quad + [(y_n - y^*) \vee (y^* - y_n)]\}.
\end{aligned} \tag{19}$$

From the normality of P, we have

$$\begin{aligned}&\|A(x_n,y_n)-A(x^*,y^*)\|\\&\quad\le\alpha\{\|x_n-x^*\|+\|y_n-y^*\|\}\\&\|A(y_n,x_n)-A(y^*,x^*)\|\\&\quad\le\alpha\{\|x_n-x^*\|+\|y_n-y^*\|\}.\end{aligned}\tag{20}$$

Those imply that

$$\begin{aligned}&\|x_{n+1}-A(x^*,y^*)\|\le\alpha\{\|x_n-x^*\|+\|y_n-y^*\|\}\\&\|y_{n+1}-A(y^*,x^*)\|\le\alpha\{\|x_n-x^*\|+\|y_n-y^*\|\}.\end{aligned}\tag{21}$$

Taking limit in (21) as $n\to\infty$, we get $A(x^*,y^*)=x^*$, $A(y^*,x^*)=y^*$; that is, (x^*,y^*) is a coupled fixed point of A.

By (18), we can prove that x^* is a fixed point of A.

Using the same argument as that in the proof of Theorem 8, we can obtain

$$\begin{aligned}&\|x_0-x^*\|\le\frac{N}{2(1-2\alpha)}(\|A(x_0,y_0)-x_0\|\\&\quad+\|A(y_0,x_0)-y_0\|),\\&\|y_0-x^*\|\le\frac{N}{2(1-2\alpha)}(\|A(x_0,y_0)-x_0\|\\&\quad+\|A(y_0,x_0)-y_0\|).\end{aligned}\tag{22}$$

Then we complete the proof of Theorem 9. □

Theorem 10. *Let E be a Banach space and P a normal cone in E with the normal constant N, $u_0,v_0\in E$ with $u_0\le v_0$, and $[u_0,v_0]$ an order interval. Suppose that $A:[u_0,v_0]\times[u_0,v_0]\to[u_0,v_0]$ is α-contractive and comparable, where $\alpha\in(0,1/2)$; then A has a unique fixed point u^* in E. Moreover, for any initial $(x_0,y_0)\in[u_0,v_0]\times[u_0,v_0]$, the iterative sequences $x_n=A(x_{n-1},y_{n-1})$ and $y_n=A(y_{n-1},x_{n-1})$ converge to u^*.*

Proof. Consider the iterative sequences

$$\begin{aligned}u_1&=A(u_0,v_0),\\&\vdots\\u_n&=A(u_{n-1},v_{n-1});\\v_1&=A(v_0,u_0),\\&\vdots\\v_n&=A(v_{n-1},u_{n-1});\\&\qquad n=1,2,\ldots.\end{aligned}\tag{23}$$

Since u_0 and v_0 are comparable and $A:E\times E\to E$ is a comparable operator, it is easy to verify that $u_1=A(u_0,v_0)$ and $v_1=A(v_0,u_0)$ are comparable. By inductions, we can prove that u_n and v_n are comparable for any positive integer n.

Since $A:E\times E\to E$ is a α-contractive comparable operator, we have

$$\begin{aligned}&(u_n-v_n)\vee(v_n-u_n)\\&\quad=[A(u_{n-1},v_{n-1})-A(u_{n-1},v_{n-1})]\\&\qquad\vee[A(v_{n-1},u_{n-1})-A(u_{n-1},v_{n-1})]\\&\quad\le2\alpha[(u_{n-1}-v_{n-1})\vee(v_{n-1}-u_{n-1})].\end{aligned}\tag{24}$$

If we continue in the same way, for each $n\in N$, we have

$$(u_n-v_n)\vee(v_n-u_n)\le2^n\alpha^n[(u_0-v_0)\vee(v_0-u_0)].\tag{25}$$

By the normality of P, we have

$$\|u_n-v_n\|\le N2^n\alpha^n\|u_0-v_0\|.\tag{26}$$

Since $u_0\le v_0$, we know that u_0 and $u_1=A(u_0,v_0)$, v_0 and $v_1=A(v_0,u_0)$ are comparable; thus $A(u_0,v_0)$ and $A(u_1,v_1)$, $A(v_0,u_0)$ and $A(v_1,u_1)$ are comparable, which implies that u_1 and u_2, v_1 and v_2 are comparable. By induction, we know that (u_n,u_{n+1}) and (v_n,v_{n+1}) are all comparable pairs for each $n\in N$.

Furthermore, for each $n\in N$, we have

$$\begin{aligned}&(u_n-u_{n+1})\vee(u_{n+1}-u_n)=[A(u_{n-1},v_{n-1})\\&\quad-A(u_n,v_n)]\vee[A(u_n,y_n)-A(u_{n-1},v_{n-1})]\\&\quad\le\alpha\{[(u_n-u_{n-1})\vee(u_{n-1}-u_n)]\\&\quad+[(v_n-v_{n-1})\vee(v_{n-1}-v_n)]\}\\&\quad\le\alpha\{[A(u_{n-1},v_{n-1})-A(u_{n-2},v_{n-2})]\\&\quad\vee[A(u_{n-2},y_{n-2})-A(u_{n-1},v_{n-1})]\\&\quad+[A(v_{n-1},u_{n-1})-A(v_{n-2},u_{n-2})]\\&\quad\vee[A(v_{n-2},u_{n-2})-A(v_{n-1},u_{n-1})]\}\\&\quad\le2\alpha^2\{[(u_{n-1}-u_{n-2})\vee(u_{n-2}-u_{n-1})]\\&\quad+[(v_{n-1}-v_{n-2})\vee(v_{n-2}-v_{n-1})]\}\le\cdots\\&\quad\le2^{n-1}\alpha^n\{[(u_1-u_0)\vee(u_0-u_1)]\\&\quad+[(v_1-v_0)\vee(v_0-v_1)]\},\\&(v_n-v_{n+1})\vee(v_{n+1}-v_n)\\&\quad\le2^{n-1}\alpha^n\{[(u_1-u_0)\vee(u_0-u_1)]\\&\quad+[(v_1-v_0)\vee(v_0-v_1)]\}.\end{aligned}\tag{27}$$

By the normality of P, we have

$$\begin{aligned}\|u_{n+1}-u_n\|&\le N2^{n-1}\alpha^n\|u_1-u_0\|\\\|v_{n+1}-v_n\|&\le N2^{n-1}\alpha^n\|v_1-v_0\|.\end{aligned}\tag{28}$$

For $\alpha \in (0, 1/2)$, we can prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences.

Since $\{u_n\}, \{v_n\} \subset [u_0, v_0]$, there exist two points u^*, v^* in $[u_0, v_0]$, such that

$$\begin{aligned} u_n &\longrightarrow u^*, \\ v_n &\longrightarrow v^* \\ &(n \longrightarrow \infty). \end{aligned} \tag{29}$$

Taking limit in (26) as $n \to \infty$, we get

$$u^* = v^*. \tag{30}$$

We now prove that u^* is a fixed point of A. For some $m, n \in N$, assume that $n < m$; since $\{u_n\}, \{v_n\} \subset [u_0, v_0]$, we know that u_0 and $u_{m-n} = A(u_{m-n-1}, v_{m-n-1})$, v_0 and $v_{m-n} = A(v_{m-n-1}, u_{m-n-1})$ are comparable; hence $A(u_0, v_0)$ and $A(u_{m-n}, v_{m-n})$, $A(v_0, u_0)$ and $A(v_{m-n}, u_{m-n})$ are comparable. If we continue in the same way, we can prove that u_n and u_m, v_n and v_m are comparable. As $m \to \infty$, by Lemma 4, we know that, for each $n \in N$, u_n and u^*, v_n and v^* are comparable.

Hence $A(u_n, v_n)$ and $A(u^*, v^*)$, $A(v_n, u_n)$ and $A(u^*, v^*)$ are comparable; furthermore

$$\begin{aligned} &[A(u_n, v_n) - A(u^*, v^*)] \vee [A(u^*, v^*) - A(u_n, v_n)] \\ &\quad \le \alpha \{[(u_n - u^*) \vee (u^* - u_n)] \\ &\quad + [(v_n - v^*) \vee (v^* - v_n)]\} \\ &[A(v_n, u_n) - A(v^*, u^*)] \vee [A(v^*, u^*) - A(v_n, u_n)] \\ &\quad \le \alpha \{[(u_n - u^*) \vee (u^* - u_n)] \\ &\quad + [(v_n - v^*) \vee (v^* - v_n)]\}. \end{aligned} \tag{31}$$

From the normality of P, we have

$$\begin{aligned} &\|A(u_n, v_n) - A(u^*, v^*)\| \\ &\quad \le N\alpha \{\|u_n - u^*\| + \|v_n - v^*\|\} \\ &\|A(v_n, u_n) - A(v^*, u^*)\| \\ &\quad \le N\alpha \{\|u_n - u^*\| + \|v_n - v^*\|\}. \end{aligned} \tag{32}$$

This implies that

$$\begin{aligned} \|u_{n+1} - A(u^*, v^*)\| &\le N\alpha \{\|u_n - u^*\| + \|v_n - v^*\|\} \\ \|v_{n+1} - A(v^*, u^*)\| &\le N\alpha \{\|u_n - u^*\| + \|v_n - v^*\|\}. \end{aligned} \tag{33}$$

Taking limit in (33) as $n \to \infty$, we get $A(u^*, v^*) = u^*$, $A(v^*, u^*) = v^*$; that is, (u^*, v^*) is a coupled fixed point of A.

By (30), we can prove that u^* is a fixed point of A.

For any initial $(x_0, y_0) \in [u_0, v_0] \times [u_0, v_0]$, we construct iterative sequences $x_n = A(x_{n-1}, y_{n-1})$ and $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$. Since $u_0 \le x_0$, $y_0 \le v_0$, that is, x_0 and u_0, y_0 and v_0 are comparable, we know that $A(x_0, y_0)$ and $A(u_0, v_0)$, $A(y_0, x_0)$ and $A(v_0, u_0)$ are comparable, which mean that x_1 and u_1, y_1 and v_1 are comparable. By inductions, we know that x_n and u_n, y_n and v_n are comparable for each $n \in N$. Moreover,

$$\begin{aligned} &(x_n - u_n) \vee (u_n - x_n) = [A(x_n, y_n) - A(u_n, v_n)] \\ &\quad \vee [A(u_n, v_n) - A(x_n, y_n)] \\ &\quad \le \alpha \{[(u_n - x_n) \vee (x_n - u_n)] \\ &\quad + [(v_n - y_n) \vee (y_n - v_n)]\} \\ &\quad \le \alpha \{[A(u_{n-1}, v_{n-1}) - A(x_{n-1}, y_{n-1})] \\ &\quad \vee [A(x_{n-1}, y_{n-1}) - A(u_{n-1}, v_{n-1})] \\ &\quad + [A(v_{n-1}, u_{n-1}) - A(y_{n-1}, x_{n-1})] \\ &\quad \vee [A(y_{n-1}, x_{n-1}) - A(v_{n-1} - u_{n-1})]\} \\ &\quad \le 2\alpha^2 \{[(u_{n-1} - x_{n-1}) \vee (x_{n-1} - u_{n-1})] \\ &\quad + [(v_{n-1} - y_{n-1}) \vee (y_{n-1} - v_{n-1})]\} \le \cdots \\ &\quad \le 2^n \alpha^{n+1} \{[(u_0 - x_0) \vee (u_0 - x_0)] \\ &\quad + [(y_0 - v_0) \vee (v_0 - y_0)]\}, \\ &(y_n - v_n) \vee (v_n - y_n) \\ &\quad \le 2^n \alpha^{n+1} \{[(u_0 - x_0) \vee (u_0 - x_0)] \\ &\quad + [(y_0 - v_0) \vee (v_0 - y_0)]\}. \end{aligned} \tag{34}$$

From the normality of P, we have

$$\begin{aligned} \|u_n - x_n\| &\le N2^n \alpha^{n+1} \{\|u_0 - x_0\| + \|y_0 - v_0\|\} \\ \|v_n - y_n\| &\le N2^n \alpha^{n+1} \{\|u_0 - x_0\| + \|y_0 - v_0\|\}. \end{aligned} \tag{35}$$

This implies that

$$\begin{aligned} x_n &\longrightarrow u^*, \\ y_n &\longrightarrow v^* (u^*) \\ &(n \longrightarrow \infty). \end{aligned} \tag{36}$$

For the uniqueness, we assume that there exists a point $\omega^* \in [u_0, v_0]$, such that $A(\omega^*, \omega^*) = \omega^*$. Construct the iterative sequence as $\omega_1 = A(\omega^*, \omega^*)$, $\omega_2 = A(\omega_1, \omega_1), \ldots$, and $\omega_n = A(\omega_{n-1}, \omega_{n-1}), \ldots$, and we know that $\omega_n \to u^*$ $(n \to \infty)$. On the other hand, for $\omega^* = A(\omega^*, \omega^*)$, $\omega^* = \omega_1 = \cdots = \omega_n$, and hence $\omega^* = u^*$.

Then we complete the proof of Theorem 10. □

Remark 11. It is easy to verify that mixed monotone operators and antimixed monotone operators are precisely comparable operators. However, a comparable operator is not necessarily a mixed monotone operator or an antimixed monotone operator. Thus, the fixed point theorems in this work generalize and extend the fixed point theorems of mixed monotone operators and antimixed monotone operators.

3. Applications

As an application, we consider the nonlinear Hammerstein integral equation of the following form.

Let I be the closed unit interval $[0,1]$ in R. Consider the following integral equation:

$$\begin{aligned} x(s) &= A(x(s), y(s)) \\ &= \int_0^s \frac{k(s,t)}{s^2-t^2} \cdot \frac{x(t)+x(t)\,y(t)-1}{1+y(t)} dt, \quad s \in I. \end{aligned} \tag{37}$$

Suppose that $k(s,t) : I \times I \to R^+$ is continuous about s and bounded measurable about t and, moreover, nonnegative on $[0,1]\times[0,1]$ and $\alpha = \sup_{t\in I} \int_0^1 (k(s,t)/(s^2-t^2))dt < 1/2$; then (37) has a unique solution $x^*(s)$.

Proof. Let $E = C[0,1]$ denote the Banach space of all real valued continuous functions φ on $[0,1]$, with $\|\varphi\| = \sup_{s\in[0,1]}|\varphi(s)|$, and let $P = \{\varphi \in E \mid \varphi(s) \geq 0,\ s \in [0,1]\}$. Clearly, P is a normal cone of E with the normal constant $N = 1$.

We transform integral equation (37) to the form

$$\begin{aligned} x(s) &= A(x(s), y(s)) \\ &= \int_0^1 \frac{k(s,t)}{s^2-t^2} \left[x(t) - \frac{1}{1+y(t)} \right] dt. \end{aligned} \tag{38}$$

Since $k(s,t)/(s^2-t^2)$ is nonnegative on $[0,1]\times[0,1]$, we get that A is a binary operator from $[0,1]\times[0,1]$ to $[0,1]$, and apparently A is comparable.

For $0 \leq x_1(s), x_2(s), y_1(s), y_2(s) \leq 1$, we get

$$\begin{aligned} &[A(x_1,y_1) - A(x_2,y_2)] \vee [A(x_2,y) - A(x_1,y_1)] \\ &\quad = |A(x_1,y_1) - A(x_2,y_2)| \\ &\quad = \int_0^1 \frac{R(s,t)}{s^2-t^2} |x_1(t) - x_2(t)| + \frac{|y_1(t) - y_2(t)|}{(1+y_1(t))} dt \\ &\quad \leq \alpha (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|). \end{aligned} \tag{39}$$

Then we obtain that A is α-contractive. According to Theorem 10, we can prove that the integral equation (37) has a unique solution. □

Remark 12. The operator $A(x,y)$ defined by (37) is a comparable operator, but A does not satisfy the mixed monotone or antimixed monotone condition. However, by Theorem 10 of this work, we can easily get the conclusion. Thus, from this application, it is shown that some of the results in this work generalize and extend the corresponding results of mixed monotone operators and antimixed monotone operators again.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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