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Research Article

Asymptotic Behavior of Positive Solutions of a Competitive System Subject to Environmental Noise

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A competitive system subject to environmental noise is established. By using the theory of stochastic differential equations and Lyapunov function, sufficient conditions for the existence, uniqueness, stochastic boundedness, and global attraction of the positive solution of the above system are established, respectively. An example together with its corresponding numerical simulations is presented to confirm our analytical results.

1. Introduction

Mathematical modelling plays an important role in the mathematical ecology. In the past several years, ecological models based on determinate systems emerged in large numbers (see [1–9]). While the disturbance of environmental noise is unavoidable in the real world, more and more researchers start to pay attention to the study on nonlinear dynamic systems with environmental noise and many valuable results have been obtained (see [10–24]).

In [25], Gopalsamy introduced the following competitive system:

$$\begin{aligned} dx_1(t) &= x_1(t)\left[r_1 - a_1 x_1(t) - \frac{c_2 x_2(t)}{1 + x_2(t)}\right] dt, \\ dx_2(t) &= x_2(t)\left[r_2 - a_2 x_2(t) - \frac{c_1 x_1(t)}{1 + x_1(t)}\right] dt, \end{aligned} \quad (1)$$

where $x_i(t)$ may represent the densities of species. The coefficients r_i, a_i, b_i, and c_i are all positive constants. In the absence of interspecific interactions, each species is governed by the logistic equation; however, in the presence of interspecific interactions, each species retains the average growth rate of the other. In this contribution, we consider the influence of environmental noise and obtain the following form:

$$\begin{aligned} dx_1(t) &= x_1(t)\left[r_1 - a_1 x_1(t) - \frac{c_2 x_2(t)}{1 + x_2(t)}\right] dt \\ &\quad + \sigma_1 x_1(t)\, dw_1(t), \\ dx_2(t) &= x_2(t)\left[r_2 - a_2 x_2(t) - \frac{c_1 x_1(t)}{1 + x_1(t)}\right] dt \\ &\quad + \sigma_2 x_2(t)\, dw_2(t), \end{aligned} \quad (2)$$

where $x_i(0) > 0$, $dw_i(t)$ is independent white noise with $w_i(0) = 0$, $t \geq 0$, and σ_i^2 represents the intensity of the noise, $i = 1, 2$. $w_i(t)$ is standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets).

In this paper, we focus on the asymptotic behavior of positive solution of system (2). To the best of our knowledge, there are few published papers concerning system (2). The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. The existence, uniqueness, and stochastic boundedness of positive solution of system (2) are discussed in Section 3. The global attraction of system (2) is studied in Section 4. As an application of our main results, we present an example and its numerical simulations to support our theoretical results in Section 5.

2. Preliminaries

In this section, we introduce some definitions and lemmas which are useful for establishing our main results.

Definition 1. The solution $(x_1(t), x_2(t))$ of system (2) is stochastically bounded if, for any $\varepsilon_i \in (0,1)$, there exist positive constants $H_i = H(\varepsilon_i)$ such that

$$\lim_{t\to+\infty} \sup P\{|x_i(t)| > H_i\} < \varepsilon_i, \quad i = 1, 2. \tag{3}$$

Definition 2. Let $(x_1(t), x_2(t))$ be a positive solution of system (2). If another positive solution $(x_1^*(t), x_2^*(t))$ of system (2) satisfies

$$\lim_{t\to+\infty} E\left(\left|(x_1(t), x_2(t)) - (x_1^*(t), x_2^*(t))\right|\right) = 0, \tag{4}$$

then $(x_1(t), x_2(t))$ is global attractive.

Definition 3. Let $(x_1(t), x_2(t))$ be a positive solution of system (2). The solution $x(t) = (x_1(t), x_2(t))$ to system (2) is said to be exponentially extinct with probability one if

$$\lim_{t\to\infty}\sup \frac{\ln x_i(t)}{t} < 0, \quad \text{a.s. } i = 1, 2. \tag{5}$$

Lemma 4 (C_p inequality). *Suppose that $a_1, a_2, \ldots, a_n$ are all real numbers; then for any positive real number p we have*

$$|a_1 + a_2 + \cdots + a_n|^p \le C_p\left(|a_1|^p + |a_2|^p + \cdots + |a_n|^p\right), \tag{6}$$

where

$$C_p = \begin{cases} 1, & 0 < p \le 1, \\ n^{p-1}, & p > 1. \end{cases} \tag{7}$$

Lemma 5 (see [26]). *Let $f(t)$ be a nonnegative integrable and uniformly continuous function defined on $[0, +\infty)$ such that $f(t)$ is integrable and uniformly continuous on $[0, +\infty)$. Then $\lim_{t\to+\infty} f(t) = 0$.*

Lemma 6 (see [27, 28]). *Suppose that a stochastic process $X(t)$ on $t \ge 0$ satisfies the condition*

$$E\left|X(t) - X(s)\right|^{\alpha} \le c\left|t - s\right|^{1+\beta}, \quad 0 \le s,\ t < +\infty, \tag{8}$$

for some positive constants α, β, and c. Then there exists a continuous modification $\widetilde{X}(t)$ of $X(t)$, which has the property that, for every $\gamma \in (0, \beta/\alpha)$, there is a positive random variable $h(\omega)$ such that

$$P\left\{\omega : \sup_{0<|t-s|<h(\omega), 0\le s,t<+\infty} \frac{\left|\widetilde{X}(t,\omega) - \widetilde{X}(s,\omega)\right|}{|t-s|^{\gamma}} \ge \frac{2}{1-2^{-\gamma}}\right\} = 1. \tag{9}$$

In other words, almost every sample path of $\widetilde{X}$ is locally but uniformly Hölder continuous with exponent γ.

3. Existence, Uniqueness, Stochastic Boundedness, and Extinction

We first present the existence and uniqueness of positive solution of system (2).

Theorem 7. *System (2) has a unique positive solution, say $(x_1(t), x_2(t))$, on $t \ge 0$. Furthermore, the solution will remain in $R_+^2 = \{(x_1, x_2) \in R^2 : x_i > 0, i = 1, 2\}$ with probability one.*

Proof. The proof of this lemma is rather standard. It is obvious that the coefficients of system (2) are local Lipschitz continuous. Then, for any initial value $(x_1(0), x_2(0))$ with $x_i(0) > 0$, there exists a unique local solution $(x_1(t), x_2(t))$, $t \in [0, \tau_*)$, where τ_* is the explosion time (see [10, 19]). Therefore, to prove that the local solution is also global, we only need to show that $\tau_* = +\infty$ a.s. Let $n_0 > 0$ be sufficiently large so that every component of $(x_1(0), x_2(0))$ lies in $[1/n_0, n_0]$. For each integer $n \ge n_0$, we define the stopping time as follows:

$$\tau_n = \inf\left\{t \in [0, \tau_*) : x_1(t) \notin \left(\frac{1}{n}, n\right) \text{ or } x_2(t) \notin \left(\frac{1}{n}, n\right)\right\}. \tag{10}$$

Here we set $\inf \emptyset = +\infty$ ($\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to +\infty$. Denote $\tau_{+\infty} = \lim_{n\to+\infty}\tau_n$, whence $\tau_{+\infty} \le \tau_*$ a.s. We need to show that $\tau_{+\infty} = +\infty$ a.s. Otherwise, there exist constants $T > 0$ and $\varepsilon \in (0,1)$ such that $P\{\tau_{+\infty} \le T\} > \varepsilon$. Then, by denoting $\Omega_n = \{\tau_n \le T\}$, there exists an integer $n_1 \ge n_0$ such that for all $n \ge n_1$,

$$P(\Omega_n) \ge \varepsilon. \tag{11}$$

We now define a C^2-function V as

$$V(x_1, x_2) = [x_1 - \ln x_1 - 1] + [x_2 - \ln x_2 - 1], \tag{12}$$

where $x_1 > 0$ and $x_2 > 0$. It is obvious that $V(x_1, x_2)$ is nonnegative. By Itô's formula, one has

$$\begin{aligned} dV&(x_1(t), x_2(t)) \\ &= \Big[(r_1 + a_1)x_1 - \frac{c_2 x_2(t)}{1 + x_2(t)}(1 - x_1(t)) \\ &\quad - a_1 x_1^2(t) + \frac{1}{2}\sigma_1^2 - r_1 \\ &\quad + (r_2 + a_2)x_2 - \frac{c_1 x_1(t)}{1 + x_1(t)}(1 - x_2(t)) \\ &\quad - a_2 x_2^2(t) + \frac{1}{2}\sigma_2^2 - r_2\Big]dt \\ &\quad + \left(1 - x_1(t)^{-1}\right)\sigma_1 x_1(t)\,dw_1(t) \\ &\quad + \left(1 - x_2(t)^{-1}\right)\sigma_2 x_2(t)\,dw_2(t) \\ &= f(x_1(t), x_2(t))\,dt + \left(1 - x_1(t)^{-1}\right)\sigma_1 x_1(t)\,dw_1(t) \\ &\quad + \left(1 - x_2(t)^{-1}\right)\sigma_2 x_2(t)\,dw_2(t), \end{aligned} \tag{13}$$

where

$$f(x_1, x_2) = (r_1 + a_1)x_1 + (r_2 + a_2)x_2 - \frac{c_2 x_2}{1+x_2}(1-x_1) - \frac{c_1 x_1}{1+x_1}(1-x_2) - a_1 x_1^2 - a_2 x_2^2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - r_1 - r_2. \quad (14)$$

A calculation can show that $f(x_1, x_2)$ is upper bounded, denoted by H. Thus (13) can be rewritten as

$$dV(x_1(t), x_2(t)) \le Hdt + \left(1 - x_1(t)^{-1}\right)\sigma_1 x_1(t)\, dw_1(t) + \left(1 - x_2(t)^{-1}\right)\sigma_2 x_2(t)\, dw_2(t). \quad (15)$$

Integrating both sides from 0 to $\tau_n \wedge T$, we acquire that

$$\int_0^{\tau_n\wedge T} V(x_1(t), x_2(t)) \le \int_0^{\tau_n\wedge T}\left[\left(1 - x_1(t)^{-1}\right)\sigma_1 x_1(t)\, dw_1(t) + \left(1 - x_2(t)^{-1}\right)\sigma_2 x_2(t)\, dw_2(t)\right] + \int_0^{\tau_n\wedge T} Hdt. \quad (16)$$

As a consequence, one has

$$V(x_1(\tau_n\wedge T), x_2(\tau_n\wedge T)) - V(x_1(0), x_2(0)) \le H(\tau_n\wedge T) + \int_0^{\tau_n\wedge T}\left[\left(1 - x_1(t)^{-1}\right)\sigma_1 x_1(t)\, dw_1(t) + \left(1 - x_2(t)^{-1}\right)\sigma_2 x_2(t)\, dw_2(t)\right]. \quad (17)$$

Since $\tau_n \wedge T > 0$, taking expectations one shows that

$$E(V(x_1(\tau_n\wedge T), x_2(\tau_n\wedge T))) \le V(x_1(0), x_2(0)) + HE(\tau_n\wedge T). \quad (18)$$

Thus

$$E(V(x_1(\tau_n\wedge T), x_2(\tau_n\wedge T))) \le V(x_1(0), x_2(0)) + HT. \quad (19)$$

On the other hand, for every $\omega \in \Omega_n$, either $x_1(\tau_n, \omega)$ or $x_2(\tau_n, \omega)$ equals to either n or $1/n$. Then $V(x_1(\tau_n, \omega), x_2(\tau_n, \omega))$ is not less than either $n-1-\ln n$ or $1/n-1+\ln n$. Consequently, from (19) we have

$$V(x_1(0), x_2(0)) + HT \ge E\left[1_{\Omega_n} V(x_1(\tau_n, \omega), x_2(\tau_n, \omega))\right] \ge \varepsilon\left[(n-1-\ln n)\wedge\left(\frac{1}{n} - 1 + \ln n\right)\right], \quad (20)$$

where 1_{Ω_n} is the indicator function of Ω_n. Let $n \to +\infty$, one can show the following contradiction:

$$+\infty \le V(x_1(0), x_2(0)) + HT < +\infty. \quad (21)$$

Hence, $\tau_{+\infty} = +\infty$ a.s. and there exists a unique positive solution $(x_1(t), x_2(t))$ of system (2) on $t \ge 0$. This completes the proof. □

Next, we investigate the stochastic boundedness of the positive solutions of system (2). To this end, we first give the following Lemma 8.

Lemma 8. *If $x_i(0) < r_i/a_i$, $i = 1, 2$, then for any real number $p \ge 1$ the solution $(x_1(t), x_2(t))$ of system* (2) *satisfies*

$$E\left(x_i^p(t)\right) \le K_i(p), \quad (22)$$

where

$$K_i(p) = \left[\frac{r_i + ((p-1)/2)\sigma_i^2}{a_i}\right]^p, \quad i = 1, 2. \quad (23)$$

Proof. By *Itô*'s formula, one can show that

$$\begin{aligned} dx_i^p(t) &= px_i^{p-1}(t)\, dx_i(t) + \frac{p(p-1)}{2}x_i^{p-1}(t)\, dx_i(t)\, dx_i(t) \\ &= \left\{px_i^{p-1}(t)\, x_i(t)\left[r_i - a_i x_i(t) - \frac{c_j x_j(t)}{1+x_j(t)}\right] + \frac{p(p-1)}{2}\sigma_i^2 x_i^p(t)\right\}dt \\ &\quad + p\sigma_i x_i^p(t)\, dw_i(t). \end{aligned} \quad (24)$$

Integrating from 0 to t, we have

$$x_i^p(t) - x_i^p(0) = \int_0^t p\left\{x_i^p(s)\left[r_i - a_i x_i(s) - \frac{c_j x_j(t)}{1+x_j(t)} + \frac{(p-1)}{2}\sigma_i^2\right]\right\}ds + \int_0^t p\sigma_i x_i^p(t)\, dw_i(t). \quad (25)$$

Taking expectations, we obtain that

$$E\left(x_i^p(t)\right) - E\left(x_i^p(0)\right) = \int_0^t pE\left\{x_i^p(s)\left[r_i - a_i x_i(s) - \frac{c_j x_j(s)}{1+x_j(s)} + \frac{(p-1)}{2}\sigma_i^2\right]\right\}ds. \quad (26)$$

So

$$\frac{dE\left(x_i^p(t)\right)}{dt} = pE\left\{x_i^p(t)\left[r_i - a_i x_i(t) - \frac{c_j x_j(t)}{1+x_j(t)} + \frac{(p-1)}{2}\sigma_i^2\right]\right\} \le pr_i E\left(x_i^p(t)\right) - pa_i E\left(x_i^{p+1}(t)\right) + \frac{(p-1)}{2}\sigma_i^2 E\left(x_i^p(t)\right) = pE\left(x_i^p(t)\right)\left\{\left[r_i + \frac{(p-1)}{2}\sigma_i^2\right] - a_i\left[E\left(x_i^p(t)\right)\right]^{1/p}\right\}. \tag{27}$$

Let

$$Z_i(t) = E\left(x_i^p(t)\right); \tag{28}$$

we have

$$\frac{dZ_i(t)}{dt} \le pZ_i(t)\left[r_i + \frac{(p-1)}{2}\sigma_i^2 - a_i(Z_i(t))^{1/p}\right]. \tag{29}$$

As a consequence

$$\frac{dZ_i(t)}{Z_i(t)} \le p\left[r_i + \frac{(p-1)}{2}\sigma_i^2 - a_i(Z_i(t))^{1/p}\right]dt. \tag{30}$$

Noting that $x_i(0) < (r_i + (1/2)(p-1)\sigma_i^2)/a_i$, $i = 1, 2$, we have

$$0 < a_i(Z_i(0))^{1/p} = a_i x_i(0) < r_i + \frac{1}{2}(p-1)\sigma_i^2. \tag{31}$$

Furthermore, using the standard comparison principle, one can show that

$$\left[E(x_i(t))^p\right]^{1/p} = (Z_i(t))^{1/p} \le \frac{r_i + (1/2)(p-1)\sigma_i^2}{a_i}. \tag{32}$$

Then we can obtain that

$$E\left(x_i^p(t)\right) \le K_i(p), \tag{33}$$

where

$$K_i(p) = \left[\frac{r_i + ((p-1)/2)\sigma_i^2}{a_i}\right]^p. \tag{34}$$

This completes the proof. □

Finally, we discuss the stochastic boundedness of the positive solutions of system (2).

Theorem 9. *If $x_i(0) < r_i/a_i$, $i = 1, 2$, then the solution $(x_1(t), x_2(t))$ of system* (2) *is stochastically bounded.*

Proof. On one hand, for any positive number δ_i, $i = 1, 2$, one derives that

$$P\left\{\left|x_i(t) - E(x_i(t))\right| \ge \delta_i\right\} \ge P\left\{x_i(t) \ge E(x_i(t)) + \delta_i\right\}. \tag{35}$$

On the other hand, by the *Chebyshev* inequality and Lemma 8, we obtain that

$$P\left\{\left|x_i(t) - E(x_i(t))\right| > \delta_i\right\} \le \frac{\operatorname{Var}(x_i(t))}{\delta_i^2} \le \frac{E\left(x_i^2(t)\right)}{\delta_i^2} \le \frac{\left(r_i + (1/2)\sigma_i^2\right)^2}{a_i^2\delta_i^2}. \tag{36}$$

Let $\delta_i = (r_i + (1/2)\sigma_i^2)/\sqrt{\varepsilon_i}a_i$. Then we have

$$P\left\{x_i(t) \ge E(x_i(t)) + \frac{r_i + (1/2)\sigma_i^2}{\sqrt{\varepsilon_i}a_i}\right\} \le \varepsilon_i. \tag{37}$$

It follows from Lemma 8 that

$$E(x_i(t)) \le \frac{r_i}{a_i}, \tag{38}$$

which, together with (37), leads to

$$P\left\{x_i(t) \ge \frac{r_i}{a_i} + \frac{r_i + (1/2)\sigma_i^2}{\sqrt{\varepsilon_i}a_i}\right\} \le P\left\{x_i(t) \ge E(x_i(t)) + \frac{r_i + (1/2)\sigma_i^2}{\sqrt{\varepsilon_i}a_i}\right\} \le \varepsilon_i. \tag{39}$$

Let $H_i(\varepsilon_i) = ((\sqrt{\varepsilon_i} + 1)r_i + (1/2)\sigma_i^2)/a_i\sqrt{\varepsilon_i}$, and noting that $x_i(t) > 0$, one shows that

$$P\left\{\left|x_i(t)\right| \ge H_i(\varepsilon_i)\right\} < \varepsilon_i. \tag{40}$$

Therefore,

$$\lim_{t\to\infty} \sup P\left\{\left|x_i(t)\right| \ge H_i(\varepsilon_i)\right\} < \varepsilon_i, \tag{41}$$

which implies that the solution of system (2) is stochastically bounded. The proof is complete. □

Theorem 10. *Suppose that all coefficients of system* (2) *are positive and $r_i < \sigma_i^2/2$, $i = 1, 2$. Then the solution $(x_1(t), x_2(t))$ of system* (2) *is exponentially extinct with probability one.*

Proof. Define, respectively, Lyapunov functions $\ln x_1(t)$ and $\ln x_2(t)$. Then the following conclusions can be obtained by *Itô*'s formula

$$\begin{aligned} d(\ln x_1(t)) &= \left[r_1 - \frac{1}{2}\sigma_1^2 - a_1 x_1(t) - \frac{c_2 x_2(t)}{1 + x_2(t)}\right]dt + \sigma_1 dw_1(t), \\ d(\ln x_2(t)) &= \left[r_2 - \frac{1}{2}\sigma_2^2 - a_2 x_1(t) - \frac{c_1 x_1(t)}{1 + x_1(t)}\right]dt + \sigma_2 dw_2(t). \end{aligned} \tag{42}$$

Integrating from 0 to t, one concludes that

$$\begin{aligned} \ln x_1(t) &\le \ln x_1(0) + \left(r_1 - \frac{1}{2}\sigma_1^2\right)t + \int_0^t \sigma_1 dw_1(s), \\ \ln x_2(t) &\le \ln x_2(0) + \left(r_2 - \frac{1}{2}\sigma_2^2\right)t + \int_0^t \sigma_2 dw_2(s). \end{aligned} \tag{43}$$

Dividing t on both sides of (43), sending $t \to \infty$, and employing the strong law of large numbers for local martingales, one acquires that

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} < 0, \quad \text{a.s. } i = 1, 2. \tag{44}$$

This completes the proof. □

4. Global Attraction

In this section, we first introduce Lemma 11 before we show the global attraction of system (2).

Lemma 11. *If $c_i \le a_i$, $x_i(0) < r_i/a_i$, $i = 1, 2$, then almost every sample path of the solution $(x_1(t), x_2(t))$ of system (2) is uniformly continuous on $t \ge 0$.*

Proof. It follows from system (2) that

$$\begin{aligned} x_1(t) = x_1(0) &+ \int_0^t g_1(s, x_1(s), x_2(s))\, ds \\ &+ \int_0^t h_1(s, x_1(s), x_2(s))\, dw_1(s), \end{aligned} \tag{45}$$

where

$$\begin{aligned} g_1(s, x_1(s), x_2(s)) &= x_1(s)\left[r_1 - a_1 x_1(s) - \frac{c_2 x_2(s)}{1 + x_2(s)}\right], \\ h_1(s, x_1(s), x_2(s)) &= \sigma_1 x_1(s). \end{aligned} \tag{46}$$

Applying Lemmas 4 and 8, for any $p > 1$, one derives that

$$\begin{aligned} &E\left(\left|g_1(s, x_1(s), x_2(s))\right|^p\right) \\ &\quad = E\left(x_1^p(s)\left|r_1 - a_1 x_1(s) - \frac{c_2 x_2(s)}{1 + x_2(s)}\right|^p\right) \\ &\quad \le \frac{1}{2}E\left(x_1^{2p}(s)\right) + \frac{1}{2}E\left(\left(r_1 - a_1 x_1(s) - \frac{c_2 x_2(s)}{1 + x_2(s)}\right)^{2p}\right) \\ &\quad \le \frac{1}{2}E\left(x_1^{2p}(s)\right) + \frac{1}{2}E\left[3^{2p-1}|r_1|^{2p} + 3^{2p-1}|a_1 x_1(s)|^{2p}\right. \\ &\qquad\qquad \left. + 3^{2p-1}|c_2 x_2(s)|^{2p}\right] \\ &\quad = \frac{1}{2}E\left(x_1^{2p}(s)\right) + \frac{1}{2}3^{2p-1}|r_1|^{2p} + \frac{1}{2}3^{2p-1}|a_1|^{2p}E\left(x_1^{2p}(s)\right) \\ &\qquad + \frac{1}{2}3^{2p-1}|c_2|^{2p-1}E\left(x_2^{2p}(s)\right) \\ &\quad \le \frac{1}{2}K_1(2p) + \frac{1}{2}3^{2p-1}|r_1|^{2p} + \frac{1}{2}3^{2p-1}|a_1|^{2p}K_1(2p) \\ &\qquad + \frac{1}{2}3^{2p-1}|c_2|^{2p-1}K_2(2p) \triangleq L_1(p), \\ &E\left(\left|h_1(s, x_1(s), x_2(s))\right|^p\right) \\ &\quad = E\left(\sigma_1^p x_1^p(s)\right) \le \sigma_1^p E\left(x_1^p(s)\right) \triangleq M_1(p). \end{aligned} \tag{47}$$

Without loss of generality, we assume that $p > 2$. Using the moment inequality (see [10]) to stochastic integral (45), we can obtain that

$$\begin{aligned} &E\left|\int_{t_1}^{t_2} h_1(s, x_1(s), x_2(s))\, dw_1(s)\right|^p \\ &\quad \le \left[\frac{p(p-1)}{2}\right]^{p/2} (t_2 - t_1)^{p/2} \\ &\qquad \times \int_{t_1}^{t_2} E\left|h_1(s, x_1(s), x_2(s))\right|^p ds, \end{aligned} \tag{48}$$

where $0 \le t_1 < t_2 < +\infty$ and $p > 2$. We further let

$$t_2 - t_1 < 1, \qquad \frac{1}{p} + \frac{1}{q} = 1; \tag{49}$$

then by (47), (48), and Lemma 4, one yields that

$$\begin{aligned} &E\left|x_1(t_2) - x_1(t_1)\right|^p \\ &\quad = E\left[\int_{t_1}^{t_2} g_1(s, x_1(s), x_2(s))\, ds \right. \\ &\qquad\qquad \left. + \int_{t_1}^{t_2} h_1(s, x_1(s), x_2(s))\, dw_1(s)\right]^p \\ &\quad \le E\left[2^{p-1}\left(\int_{t_1}^{t_2} g_1(s, x_1(s), x_2(s))\, ds\right)^p \right. \\ &\qquad\qquad \left. + 2^{p-1}\left(\int_{t_1}^{t_2} h_1(s, x_1(s), x_2(s))\, dw_1(s)\right)^p\right] \\ &\quad \le 2^{p-1}E\left(\int_{t_1}^{t_2} \left|g_1(s, x_1(s), x_2(s))\right|^p ds\right) \\ &\qquad + 2^{p-1}E\left(\int_{t_1}^{t_2} \left|h_1(s, x_1(s), x_2(s))\right|^p dw_1(s)\right) \\ &\quad \le 2^{p-1}\left(\int_{t_1}^{t_2} 1^q ds\right)^{p/q} E\left(\int_{t_1}^{t_2} \left|g_1(s, x_1(s), x_2(s))\right|^p ds\right) \\ &\qquad + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2} (t_2 - t_1)^{(p-2)/2} \\ &\qquad \times \int_{t_1}^{t_2} E\left|h_1(s, x_1(s), x_2(s))\right|^p ds \\ &\quad \le 2^{p-1}\left(\int_{t_1}^{t_2} 1^q ds\right)^{p/q} E\left(\int_{t_1}^{t_2} L_1(p)\, ds\right) \\ &\qquad + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2} (t_2 - t_1)^{(p-2)/2} \int_{t_1}^{t_2} M_1(p)\, ds \\ &\quad = 2^{p-1}L_1(p)(t_2 - t_1)^{p/2} \end{aligned}$$

$$+ 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{(p-2)/2}M_1(p)$$
$$\leq 2^{p-1}(t_2-t_1)^{(p-2)/2}\left[L_1(p)+\left[\frac{p(p-1)}{2}\right]^{p/2}M_1(p)\right]. \tag{50}$$

It follows from Lemma 6 that almost every sample path of $x_1(t)$ is uniformly continuous on $t \geq 0$. Similarly, we can show that almost every sample path of $x_2(t)$ is uniformly continuous on $t \geq 0$. Therefore, $(x_1(t), x_2(t))$ is uniformly continuous on $t \geq 0$, a.s. This completes the proof. □

We can now present the result on global attraction of system (2).

Theorem 12. *If $c_i \leq a_i$, $x_i(0) < r_i/a_i$, $i = 1,2$, then system* (2) *has a unique global attractive positive solution, denoted by* $(x_1(t), x_2(t))$*, on* $t \geq 0$.

Proof. It follows from Theorem 7 that system (2) has a unique positive solution $(x_1(t), x_2(t))$. Assume that $(x_1^*(t), x_2^*(t))$ is another positive solution of system (2). Consider a Lyapunov function $V(t)$ defined by

$$V(t) = \left|\ln x_1(t) - \ln x_1^*(t)\right| + \left|\ln x_2(t) - \ln x_2^*(t)\right|, \quad t \geq 0. \tag{51}$$

Applying *Itô*'s formula, a calculation of the right differential $D^+V(t)$ of $V(t)$ along the solution, one yields that

$$\begin{aligned}
&D^+V(t)\\
&= \operatorname{sgn}(x_1(t)-x_1^*(t))\left\{\left[\frac{dx_1(t)}{x_1(t)}-\frac{(dx_1(t))^2}{2x_1^2(t)}\right]-\left[\frac{dx_1^*(t)}{x_1^*(t)}-\frac{(dx_1^*(t))^2}{2x_1^*(t)^2}\right]\right\}\\
&+ \operatorname{sgn}(x_2(t)-x_2^*(t))\left\{\left[\frac{dx_2(t)}{x_2(t)}-\frac{(dx_2(t))^2}{2x_2^2(t)}\right]-\left[\frac{dx_2^*(t)}{x_2^*(t)}-\frac{(dx_2^*(t))^2}{2x_2^*(t)^2}\right]\right\}\\
&= \operatorname{sgn}(x_1(t)-x_1^*(t))\\
&\times\left\{\left[\left(r_1-a_1x_1(t)-\frac{c_2x_2(t)}{1+x_2(t)}-\frac{\sigma_1^2}{2}\right)dt+\sigma_1dw_1(t)\right]\right.\\
&\left.-\left[\left(r_1-a_1x_1^*(t)-\frac{c_2x_2^*(t)}{1+x_2^*(t)}-\frac{\sigma_1^2}{2}\right)dt+\sigma_1dw_1(t)\right]\right\}\\
&+ \operatorname{sgn}(x_2(t)-x_2^*(t))\\
&\times\left\{\left[\left(r_2-a_2x_2(t)-\frac{c_1x_1(t)}{1+x_1(t)}-\frac{\sigma_2^2}{2}\right)dt+\sigma_2dw_2(t)\right]\right.\\
&\left.-\left[\left(r_2-a_2x_2^*(t)-\frac{c_1x_1^*(t)}{1+x_1^*(t)}-\frac{\sigma_2^2}{2}\right)dt+\sigma_2dw_2(t)\right]\right\}\\
&= \operatorname{sgn}(x_1(t)-x_1^*(t))\\
&\times\left[a_1(x_1^*(t)-x_1(t))+c_2\left(\frac{x_2^*(t)}{1+x_2^*(t)}-\frac{x_2(t)}{1+x_2(t)}\right)\right]dt\\
&+ \operatorname{sgn}(x_2(t)-x_2^*(t))\\
&\times\left[a_2(x_2^*(t)-x_2(t))+c_1\left(\frac{x_1^*(t)}{1+x_1^*(t)}-\frac{x_1(t)}{1+x_1(t)}\right)\right]dt.
\end{aligned} \tag{52}$$

Integrating from 0 to t and taking expectations one can show that

$$\begin{aligned}
&E(V(t)-V(0))\\
&= E\left\{\int_0^t \operatorname{sgn}(x_1(s)-x_1^*(s))\right.\\
&\times\left[a_1(x_1^*(s)-x_1(s))+c_2\left(\frac{x_2^*(s)}{1+x_2^*(s)}-\frac{x_2(s)}{1+x_2(s)}\right)\right]ds\\
&+\int_0^t \operatorname{sgn}(x_2(s)-x_2^*(s))\\
&\left.\times\left[a_2(x_2^*(s)-x_2(s))+c_1\left(\frac{x_1^*(s)}{1+x_1^*(s)}-\frac{x_1(s)}{1+x_1(s)}\right)\right]ds\right\}.
\end{aligned} \tag{53}$$

Thus

$$\frac{dE(V(t))}{dt} = E\left\{\operatorname{sgn}(x_1(t)-x_1^*(t))\right.$$

$$\times\left[a_1\left(x_1^*(t)-x_1(t)\right)+c_2\left(\frac{x_2^*(t)}{1+x_2^*(t)}-\frac{x_2(t)}{1+x_2(t)}\right)\right]$$
$$+\operatorname{sgn}\left(x_2(t)-x_2^*(t)\right)\times\left[a_2\left(x_2^*(t)-x_2(t)\right)+c_1\left(\frac{x_1^*(t)}{1+x_1^*(t)}-\frac{x_1(t)}{1+x_1(t)}\right)\right]\Bigg\}$$
$$\leq -a_1E\left(\left|x_1(t)-x_1^*(t)\right|\right)+c_2E\left(\left|x_2(t)-x_2^*(t)\right|\right)-a_2E\left(\left|x_2(t)-x_2^*(t)\right|\right)+c_1E\left(\left|x_1(t)-x_1^*(t)\right|\right)$$
$$=\left(c_1-a_1\right)E\left(\left|x_1(t)-x_1^*(t)\right|\right)+\left(c_2-a_2\right)E\left(\left|x_2(t)-x_2^*(t)\right|\right), \tag{54}$$

and hence integrating from 0 to t one derives that

$$E(V(t))\leq\int_0^t\left(c_1-a_1\right)E\left(\left|x_1(s)-x_1^*(s)\right|\right)ds+\int_0^t\left(c_2-a_2\right)E\left(\left|x_2(s)-x_2^*(s)\right|\right)ds+V(0), \tag{55}$$

which implies that

$$\begin{aligned}&E\left(\left|\left(x_1(t),x_2(t)\right)-\left(x_1^*(t),x_2^*(t)\right)\right|\right)\\&=E\left\{\left[\left|x_1(t)-x_1^*(t)\right|^2+\left|x_2(t)-x_2^*(t)\right|^2\right]^{1/2}\right\}\\&\leq E\left(\left|x_1(t)-x_1^*(t)\right|\right)+E\left(\left|x_2(t)-x_2^*(t)\right|\right)\in L^1[0,+\infty).\end{aligned} \tag{56}$$

So

$$\begin{aligned}&E\left(\left|\left(x_1(t),x_2(t)\right)-\left(x_1^*(t),x_2^*(t)\right)\right|\right)\\&=E\left\{\left[\left|x_1(t)-x_1^*(t)\right|^2+\left|x_2(t)-x_2^*(t)\right|^2\right]^{1/2}\right\}\\&\leq E\left(\left|x_1(t)-x_1^*(t)\right|\right)+E\left(\left|x_2(t)-x_2^*(t)\right|\right)\in L^1[0,\infty),\end{aligned} \tag{57}$$

which, together with Lemmas 5 and 11, leads to

$$\lim_{t\to\infty}E\left(\left|\left(x_1(t),x_2(t)\right)-\left(x_1^*(t),x_2^*(t)\right)\right|\right)=0 \tag{58}$$

and hence $(x_1(t),x_2(t))$ is global attractive on $t\geq 0$. □

5. An Example

In this section, we first give an example to verify the feasibilities of Theorems 9 and 12. Using the Milsten method mentioned in Higham [29], we can derive the following discrete version of system (2):

$$\begin{aligned}x_1(k+1)-x_1(k)&=x_1(k)\left[r_1-a_1x_1(k)-\frac{c_2x_2(k)}{1+x_2(k)}\right]\Delta t\\&\quad+\sigma_1x_1(k)\sqrt{\Delta t}N_1(k)\\&\quad+\frac{1}{2}\sigma_1^2x_1(k)\left(N_1^2(k)-1\right)\Delta t,\\x_2(k+1)-x_2(k)&=x_2(k)\left[r_2-a_2x_2(k)-\frac{c_1x_1(k)}{1+x_1(k)}\right]\Delta t\\&\quad+\sigma_2x_2(k)\sqrt{\Delta t}N_2(k)\\&\quad+\frac{1}{2}\sigma_2^2x_2(k)\left(N_2^2(k)-1\right)\Delta t,\end{aligned} \tag{59}$$

where $N_1(k)$ and $N_2(k)$ are Gaussian random variables which follow $N(0,1)$. Let us choose $r_1=0.5$, $a_1=0.4$, $c_1=0.1$, $\sigma_1=0.1$, $r_2=0.6$, $a_2=0.5$, $c_2=0.3$, $\sigma_2=0.1$, $\Delta t=0.001$, and $(x_1(0),x_2(0))=(0.8,1.1)$, $(x_1^*(0),x_2^*(0))=(0.3,0.4)$. A calculation shows that the conditions of Theorems 9 and 12 are satisfied. Figures 1 and 2 show that the positive solution of system (59) is stochastically bounded and global attractive on $t\geq 0$.

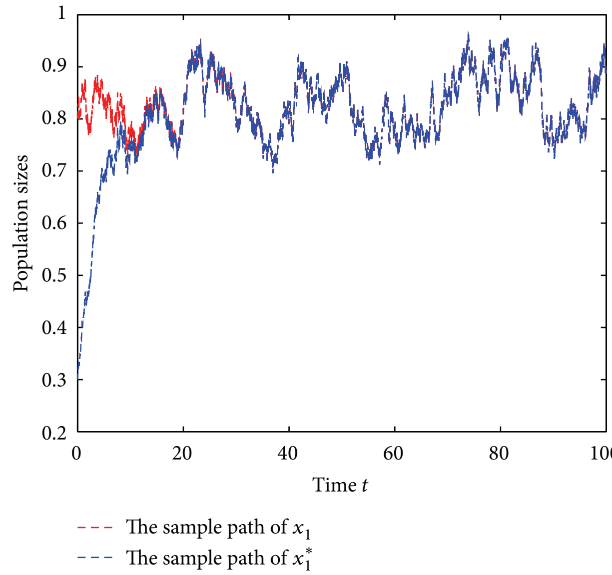

Figure 1: The sample paths of $x_1(t)$ and $x_1^*(t)$.

Recalling the whole paper, we have derived sufficient conditions for the existence, uniqueness, stochastic boundedness, and global attraction of the positive solutions of system (2). However, there are still some limitations in our work which need to be improved. We only especially consider the white noise which is an idealized situation. In fact, the effect

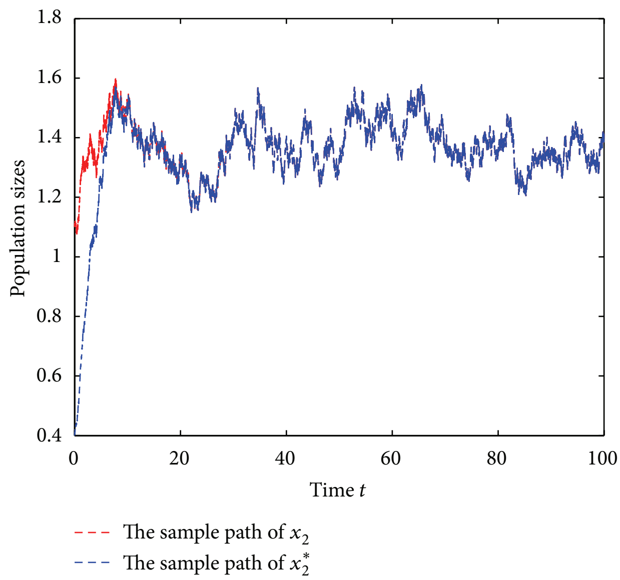

Figure 2: The sample paths of $x_2(t)$ and $x_2^*(t)$.

of colorful noise on system (2) is more general in line with the actual situation, and we leave it for our future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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