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Research Article

Random Fixed Point Theorems of Random Comparable Operators and an Application

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We introduce the new concept of random comparable operators as a generalization of random monotone operators and prove several random fixed point theorems for such a class of operators in partially ordered Banach spaces. Part of the presented results generalize and extend some known results of random monotone operators. Finally, as an application, we consider the existence of the solution of a random Hammerstein integral equation.

1. Introduction and Preliminaries

In 1950s, Špaček [1] and Hanš [2] initiated the study of random fixed point theories. From then on, to study random fixed point theories had been a central topic of random theories. Moreover, the random theories played a main role in the developing theories of random differential equations and random integral equations and attracted much attention. For example, Sehgal and Waters [3] proved the random Rothe fixed point theorem in 1984 and Mukherjea [4] proved the random Schauder fixed point theorem in 1996. In recent years, random fixed point theories and their applications developed very rapidly (see Lin [5]; Xu [6]; Li and Debnath [7]; Shahzad [8]; Li and Duan [9]; Zhu and Yin [10]; and Kumam [11–26]). In particular in 2005, Li and Duan [9] proved the existence of fixed points for random monotone operators.

In this work, as a generalization of the concept of random monotone operators given by Li and Duan [9], we introduce the concept of random comparable operators and under different contractive conditions, we prove several random fixed point theorems for such operators in partially ordered Banach spaces. Some of our results generalize and extend the main results of Li and Duan [9].

Let E be a separable real Banach space, (Ω, Σ, μ) a complete measure space, and (E, β) a measurable space, where β denotes the σ-algebra of all Borel subsets generated by all open subsets of E. Suppose that D is a nonempty subset of E, and P is a cone in E. Cone P defines a partial order $\leq$ as follows: for $x, y \in E$, $x \leq y \Leftrightarrow y - x \in P$. P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. Let $u_0, v_0 \in E$; write $u_0 < v_0$ if $u_0 \leq v_0$ and $u_0 \neq v_0$. If $u_0 < v_0$, we call the set $[u_0, v_0] = \{u \in E \mid u_0 \leq u \leq v_0\}$ an order interval in E.

$A : \Omega \to E$ is called measurable if $A^{-1}(B) \in \Sigma$ for each Borel subset B of E.

$A : \Omega \times D \to E$ is said to be a random operator if for each fixed $x \in D$, $A(\cdot, x) : \Omega \to E$ is measurable.

A random operator $A : \Omega \times D \to E$ is said to be continuous if for any $\omega \in \Omega$, $A(\omega, \cdot) : D \to E$ is continuous.

A measurable map $\xi : \Omega \to E$ is called a random fixed point of a random operator $A : \Omega \times D \to E$ if $A(\omega, \xi(\omega)) = \xi(\omega)$ for almost every $\omega \in \Omega$.

Definition 1. Suppose that $u(\omega), v(\omega) : \Omega \to E$ are measurable. $u(\omega)$ and $v(\omega)$ are said to be random comparable if for any $\omega \in \Omega$, $u(\omega) \leq v(\omega)$ or $v(\omega) \leq u(\omega)$ holds.

Assume that $u(\omega), v(\omega) : \Omega \to E$ are random comparable. If for any $\omega \in \Omega$, $v(\omega) \leq u(\omega)$, then we write $u(\omega) = u(\omega) \vee v(\omega)$; if $u(\omega) \leq v(\omega)$, then we write $v(\omega) = u(\omega) \vee v(\omega)$.

Definition 2 (see [3]). A mapping $\alpha(\omega) : \Omega \to \mathscr{L}(E)$ is said to be a random endomorphism of E if $\alpha(\omega)$ is an $\mathscr{L}(E)$-valued random variable, where $\mathscr{L}(E)$ denotes the linear bounded operator space of E.

Definition 3. A random operator $A : \Omega \times D \to E$ is said to be random comparable if $A(\omega, u(\omega))$ and $A(\omega, v(\omega))$ are random comparable for any random comparable pair $u(\omega)$, $v(\omega) : \Omega \to D$.

Remark 4. The concept of random comparable operators generalizes the concept of random increasing (decreasing) operators given by Li and Duan [9].

Definition 5. A random comparable operator $A : \Omega \times D \to E$ is said to be random $\alpha(\omega)$-ordered contractive if there exists a random endomorphism $\alpha(\omega) : \Omega \to \mathscr{L}(E)$ such that for each $\omega \in \Omega$ and any measurable mappings $u(\omega), v(\omega) : \Omega \to D$, if $u(\omega)$ and $v(\omega)$ are random comparable, then

$$\begin{aligned} &(A(\omega, u(\omega)) - A(\omega, v(\omega))) \\ &\quad \vee (A(\omega, v(\omega)) - A(\omega, u(\omega))) \\ &\leq \alpha(\omega)((u(\omega) - v(\omega)) \vee (v(\omega) - u(\omega))). \end{aligned} \tag{1}$$

By the definition of random comparable operators, the following lemmas are easy, so we omit their proofs (wherein, we assume that $u(\omega)$, $v(\omega)$, $w(\omega)$, $u_n(\omega)$, $v_n(\omega) : \Omega \to E$ are measurable, $n \geq 1$).

Lemma 6. *If for each $\omega \in \Omega$, $u(\omega)$, $v(\omega)$ are random comparable, then $u(\omega) - v(\omega)$ and $v(\omega) - u(\omega)$ are random comparable and*

$$\theta \leq (u(\omega) - v(\omega)) \vee (v(\omega) - u(\omega)). \tag{2}$$

Lemma 7. *If for each $\omega \in \Omega$, $u(\omega)$ and $v(\omega)$, $u(\omega)$ and $w(\omega)$, and $v(\omega)$ and $w(\omega)$ are random comparable, then*

$$\begin{aligned} &(u(\omega) - v(\omega)) \vee (v(\omega) - u(\omega)) \\ &\quad \leq ((u(\omega) - w(\omega)) \vee (w(\omega) - u(\omega))) \\ &\quad\quad + ((w(\omega) - v(\omega)) \vee (v(\omega) - w(\omega))). \end{aligned} \tag{3}$$

Lemma 8. *If for each $\omega \in \Omega$ and any positive integer n, $u(\omega)$ and $v_n(\omega)$ are random comparable and $v_n(\omega) \to v_0(\omega)$ $(n \to \infty)$, then $u(\omega)$ and $v_0(\omega)$ are random comparable.*

Lemma 9. *If for each $\omega \in \Omega$ and any positive integer n, $u_n(\omega)$ and $v_n(\omega)$ are random comparable, and $u_n(\omega) \to u_0(\omega)$, $v_n(\omega) \to v_0(\omega)$ $(n \to \infty)$, then $u_0(\omega)$ and $v_0(\omega)$ are random comparable.*

2. Main Results

Theorem 10. *Let E be a real Banach space and P a normal cone in E with the normal constant N. Let $A : \Omega \times E \to E$ be a continuous random operator satisfying the following:*

(i) *A is a random $\alpha(\omega)$-ordered contractive operator and $0 < \|\alpha(\omega)\| < 1$, $\omega \in \Omega$;*

(ii) *there exists $x_0 \in E$ such that for any $\omega \in \Omega$, x_0 and $A(\omega, x_0)$ are random comparable.*

Then A has a random fixed point $x^*(\omega)$. Furthermore, the iterative sequence $\{A^n(\omega, x_0)\}$ converges to $x^*(\omega)$ and $\|x^*(\omega) - x_0\| \leq (1 + N\|\alpha(\omega)\|/(1 - \|\alpha(\omega)\|))\|A(\omega, x_0) - x_0\|$.

Proof. For any fixed $\omega \in \Omega$, set

$$x_1(\omega) = A(\omega, x_0), \ldots, x_n(\omega) = A(\omega, x_{n-1}(\omega)), \ldots, n \geq 1. \tag{4}$$

Since x_0 and $A(\omega, x_0)$ are random comparable, by the given condition (i), for any $n \geq 1$, $x_n(\omega)$ and $x_{n+1}(\omega)$ are random comparable and

$$\begin{aligned} \theta &\leq (x_{n+1}(\omega) - x_n(\omega)) \vee (x_n(\omega) - x_{n+1}(\omega)) \\ &= (A(\omega, x_n(\omega)) - A(\omega, x_{n-1}(\omega))) \\ &\quad \vee (A(\omega, x_{n-1}(\omega)) - A(\omega, x_n(\omega))) \\ &\leq \alpha(\omega)((x_n(\omega) - x_{n-1}(\omega)) \vee (x_{n-1}(\omega) - x_n(\omega))) \\ &\leq \cdots \leq \alpha(\omega)^n((x_1(\omega) - x_0) \vee (x_0 - x_1(\omega))). \end{aligned} \tag{5}$$

From the normality of P, we have $\|x_{n+1}(\omega) - x_n(\omega)\| \leq N\|\alpha(\omega)\|^n\|x_1(\omega) - x_0\|$. As for each $\omega \in \Omega$, $0 < \|\alpha(\omega)\| < 1$, then $\{x_n(\omega)\}$ is a Cauchy sequence in E. Hence there exists $x^*(\omega) \in E$ such that $x_n(\omega) \to x^*(\omega)$ $(n \to \infty)$. Since $A(\omega, \cdot)$ is continuous,

$$A(\omega, x^*(\omega)) = \lim_{n\to\infty} A(\omega, x_n(\omega)) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega). \tag{6}$$

Now, we prove that $x^*(\omega) : \Omega \to E$ is measurable. Since $A(\omega, x_0)$ is measurable, that is, $x_1(\omega) = A(\omega, x_0)$ is measurable, from the measurable theorem of complex operators, it is easy to prove that $x_n(\omega)$ is measurable for all $n \geq 1$. Hence $x^*(\omega) : \Omega \to E$, being the limit of a sequence of measurable mappings, is also measurable. So $x^*(\omega) : \Omega \to E$ is a random fixed point of A. Furthermore,

$$\begin{aligned} \|x^*(\omega) - x_0\| &= \lim_{n\to\infty} \|x_n(\omega) - x_0\| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i(\omega) - x_{i-1}(\omega)\| \\ &\leq \lim_{n\to\infty} \sum_{i=2}^{n} N\|\alpha(\omega)\|^{i-1}\|x_1(\omega) - x_0\| \\ &\quad + \|x_1(\omega) - x_0\| \\ &\leq \sum_{n=2}^{\infty} N\|\alpha(\omega)\|^{n-1}\|x_1(\omega) - x_0\| \\ &\quad + \|x_1(\omega) - x_0\| \\ &= \left(1 + \frac{N\|\alpha(\omega)\|}{1 - \|\alpha(\omega)\|}\right)\|A(\omega, x_0) - x_0\|. \end{aligned} \tag{7}$$

□

Theorem 11. *Let E be a real Banach space and P a normal cone in E with the normal constant N, $u_0, v_0 \in E$ with $u_0 < v_0$ and $[u_0, v_0]$ an order interval in E. Suppose that $A : \Omega \times [u_0, v_0] \to [u_0, v_0]$ is a continuous random $\beta(\omega)$-ordered contractive operator, where $0 < \|\beta(\omega)\| < 1/N$. Then A has a unique random fixed point $x^*(\omega)$.*

Proof. Define iterative sequences as follows:

$$u_1(\omega) = A(\omega, u_0), \ldots, u_{n+1}(\omega) = A(\omega, u_n(\omega)), \ldots, n \geq 1,$$
$$v_1(\omega) = A(\omega, v_0), \ldots, v_{n+1}(\omega) = A(\omega, v_n(\omega)), \ldots, n \geq 1; \tag{8}$$

then $\{u_n(\omega)\}, \{v_n(\omega)\} \subset [u_0, v_0]$. Since $u_0 < v_0$ and A is a continuous $\beta(\omega)$-random ordered contractive operator, $u_n(\omega)$, $v_n(\omega)$ are random comparable for each $n \geq 1$ and

$$\begin{aligned}\theta &\leq (u_n(\omega) - v_n(\omega)) \vee (v_n(\omega) - u_n(\omega)) \\ &= (A(\omega, u_{n-1}(\omega)) - A(\omega, v_{n-1}(\omega))) \\ &\quad \vee (A(\omega, v_{n-1}(\omega)) - A(\omega, u_{n-1}(\omega))) \\ &\leq \beta(\omega)((u_{n-1}(\omega) - v_{n-1}(\omega)) \vee (v_{n-1}(\omega) - u_{n-1}(\omega))) \\ &\leq \beta(\omega)^2((u_{n-2}(\omega) - v_{n-2}(\omega)) \vee (v_{n-2}(\omega) - u_{n-2}(\omega))) \\ &\leq \cdots \\ &\leq \beta(\omega)^n((u_0 - v_0) \vee (v_0 - u_0)).\end{aligned} \tag{9}$$

From the normality of P, we have

$$\|u_n(\omega) - v_n(\omega)\| \leq N\|\beta(\omega)\|^n \|u_0 - v_0\|. \tag{10}$$

Since $u_0 \leq u_1(\omega)$, $u_n(\omega)$ and $u_{n+1}(\omega)$ are random comparable for any $n \geq 1$ and

$$\begin{aligned}\theta &\leq (u_n(\omega) - u_{n+1}(\omega)) \vee (u_{n+1}(\omega) - u_n(\omega)) \\ &= (A(\omega, u_{n-1}(\omega)) - A(\omega, u_n(\omega))) \\ &\quad \vee (A(\omega, u_n(\omega)) - A(\omega, u_{n-1}(\omega))) \\ &\leq \beta(\omega)((u_{n-1}(\omega) - u_n(\omega)) \vee (u_n(\omega) - u_{n-1}(\omega))) \\ &\leq \beta(\omega)^2((u_{n-2}(\omega) - u_{n-1}(\omega)) \vee (u_{n-1}(\omega) - u_{n-2}(\omega))) \\ &\leq \cdots \\ &\leq \beta(\omega)^n((u_0 - u_1(\omega)) \vee (u_1(\omega) - u_0)).\end{aligned} \tag{11}$$

By the normality of P again, we get

$$\|u_n(\omega) - u_{n+1}(\omega)\| \leq N\|\beta(\omega)\|^n \|u_0 - u_1(\omega)\|. \tag{12}$$

As $0 < \|\beta(\omega)\| < 1/N \leq 1$, it is seen easily that $\{u_n(\omega)\}$ is a Cauchy sequence in E. Hence there exists $u^*(\omega) \in [u_0, v_0]$ such that $u_n(\omega) \to u^*(\omega)$ $(n \to \infty)$. Similarly, we can prove that $\{v_n(\omega)\}$ is also a Cauchy sequence in E and there exists $v^*(\omega) \in [u_0, v_0]$ such that $v_n(\omega) \to v^*(\omega)$ $(n \to \infty)$. It follows from (10) that

$$\begin{aligned}\|u^*(\omega) - v^*(\omega)\| &= \lim_{n\to\infty} \|u_n(\omega) - v_n(\omega)\| \\ &\leq \lim_{n\to\infty} N\|\beta(\omega)\|^n \|u_0 - v_0\| = 0.\end{aligned} \tag{13}$$

So $u^*(\omega) = v^*(\omega)$. Since $A(\omega, \cdot)$ is continuous, we have

$$A(\omega, u^*(\omega)) = \lim_{n\to\infty} A(\omega, u_n(\omega)) = \lim_{n\to\infty} u_{n+1}(\omega) = u^*(\omega), \tag{14}$$

$$A(\omega, v^*(\omega)) = \lim_{n\to\infty} A(\omega, v_n(\omega)) = \lim_{n\to\infty} v_{n+1}(\omega) = v^*(\omega). \tag{15}$$

Let $x^*(\omega) = u^*(\omega) = v^*(\omega)$. Equation (14) together with (15) implies that

$$A(\omega, x^*(\omega)) = x^*(\omega). \tag{16}$$

In addition, by a proof similar to that of Theorem 10, we get that $x^*(\omega) : \Omega \to E$ is a random fixed point of A.

Next we prove that $x^*(\omega)$ is the unique random fixed point of A. Suppose that $y^*(\omega) \in [u_0, v_0]$ is another random fixed point of A. By induction, one can prove that, for any $n \geq 0$, $u_n(\omega)$ and $y^*(\omega)$ are random comparable. Since $u_n(\omega) \to x^*(\omega)$ $(n \to \infty)$, by Lemma 8, $y^*(\omega)$ and $x^*(\omega)$ are also random comparable. Because A is continuous and $\beta(\omega)$-random ordered contractive,

$$\begin{aligned}\theta &\leq (x^*(\omega) - y^*(\omega)) \vee (y^*(\omega) - x^*(\omega)) \\ &= (A(\omega, x^*(\omega)) - A(\omega, y^*(\omega))) \\ &\quad \vee (A(\omega, y^*(\omega)) - A(\omega, x^*(\omega))) \\ &\leq \beta(\omega)\{(x^*(\omega) - y^*(\omega)) \vee (y^*(\omega) - x^*(\omega))\}.\end{aligned} \tag{17}$$

By the normality of P, we have

$$\|x^*(\omega) - y^*(\omega)\| \leq N\|\beta(\omega)\| \|x^*(\omega) - y^*(\omega)\|, \tag{18}$$

which implies that $\|x^*(\omega) - y^*(\omega)\| = 0$ as $0 < \|\beta(\omega)\| < 1/N$. That is $x^*(\omega) = y^*(\omega)$. □

Remark 12. In the work of Li and Duan [9], the random operator A in Theorems 2.3-2.4 needs to be random increasing and random decreasing, respectively. Hence, Theorems 10-11 in this work generalize and extend the results of Theorems 2.3-2.4 in [9], respectively.

Theorem 13. *Let E be a real Banach space and P a normal cone in E with the normal constant N. Let $A : \Omega \times E \to E$ be a continuous random comparable operator and satisfy the following:*

(i) *there exists* $0 < \lambda < 1/2$ *such that if* $u(\omega)$ *and* $v(\omega)$, $u(\omega)$ *and* $A(\omega, u(\omega))$, *and* $v(\omega)$ *and* $A(\omega, v(\omega))$ *are random comparable, then*

$$\begin{aligned}(A(\omega, v(\omega)) - A(\omega, u(\omega)))\\ \vee (A(\omega, u(\omega)) - A(\omega, v(\omega)))\\ \leq \lambda(((A(\omega, u(\omega)) - u(\omega)) \vee (u(\omega) - A(\omega, u(\omega))))\\ + ((A(\omega, v(\omega)) - v(\omega)) \vee (v(\omega) - A(\omega, v(\omega))))); \end{aligned} \quad (19)$$

(ii) *there exists* $x_0 \in E$ *such that* x_0 *and* $A(\omega, x_0)$ *are random comparable.*

Then A has a random fixed point $x^*(\omega)$. *Furthermore, the iterative sequence* $\{A^n(\omega, x_0)\}$ *converges to* $x^*(\omega)$ *and* $\|x^*(\omega) - x_0\| \leq (1 + N\lambda/(1 - 2\lambda))\|A(\omega, x_0) - x_0\|$.

Proof. For any fixed $\omega \in \Omega$, set

$$x_1(\omega) = A(\omega, x_0), \ldots, x_n(\omega) = A(\omega, x_{n-1}(\omega)), \ldots, n \geq 1. \quad (20)$$

By a similar approach as in the proof of Theorem 10, we obtain that $x_n(\omega)$ and $x_{n+1}(\omega)$ are random comparable and

$$\begin{aligned}\theta &\leq (x_n(\omega) - x_{n+1}(\omega)) \vee (x_{n+1}(\omega) - x_n(\omega))\\ &= (A(\omega, x_{n-1}(\omega)) - A(\omega, x_n(\omega)))\\ &\quad \vee (A(\omega, x_n(\omega)) - A(\omega, x_{n-1}(\omega)))\\ &\leq \lambda((A(\omega, x_n(\omega)) - x_n(\omega))\\ &\quad \vee (x_n(\omega) - A(\omega, x_n(\omega)))\\ &\quad + (A(\omega, x_{n-1}(\omega)) - x_{n-1}(\omega))\\ &\quad \vee (x_{n-1}(\omega) - A(\omega, x_{n-1}(\omega))))\\ &= \lambda((x_{n+1}(\omega) - x_n(\omega)) \vee (x_n(\omega) - x_{n+1}(\omega))\\ &\quad + (x_n(\omega) - x_{n-1}(\omega)) \vee (x_{n-1}(\omega) - x_n(\omega))).\end{aligned} \quad (21)$$

So

$$\begin{aligned}\theta &\leq (x_{n+1}(\omega) - x_n(\omega)) \vee (x_n(\omega) - x_{n+1}(\omega))\\ &\leq \frac{\lambda}{1-\lambda}((x_n(\omega) - x_{n-1}(\omega)) \vee (x_{n-1}(\omega) - x_n(\omega)))\\ &\leq \cdots\\ &\leq \left(\frac{\lambda}{1-\lambda}\right)^n ((x_1(\omega) - x_0) \vee (x_0 - x_1(\omega))).\end{aligned} \quad (22)$$

From the normality of P, we get $\|x_{n+1}(\omega) - x_n(\omega)\| \leq N(\lambda/(1-\lambda))^n\|x_1(\omega) - x_0\|$. Since $0 < \lambda < 1/2$, $\{x_n(\omega)\}$ is a Cauchy sequence in E. Hence there exists $x^*(\omega) \in E$ such that $x_n(\omega) \to x^*(\omega)\ (n \to \infty)$. The continuity of $A(\omega, \cdot)$ implies that

$$A(\omega, x^*(\omega)) = \lim_{n\to\infty} A(\omega, x_n(\omega)) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega). \quad (23)$$

By a proof similar to that of Theorem 10, we can easily prove that $x^*(\omega) : \Omega \to E$ is measurable, so $x^*(\omega)$ is a random fixed point A. Furthermore,

$$\begin{aligned}\|x^*(\omega) - x_0\| &= \lim_{n\to\infty}\|x_n(\omega) - x_0\|\\ &\leq \lim_{n\to\infty}\sum_{i=1}^{n}\|x_i(\omega) - x_{i-1}(\omega)\|\\ &\leq \lim_{n\to\infty}\sum_{i=2}^{n}\left(\frac{\lambda}{1-\lambda}\right)^{i-1} N\|x_1(\omega) - x_0\|\\ &\quad + \|x_1(\omega) - x_0\|\\ &\leq \sum_{n=2}^{\infty}\left(\frac{\lambda}{1-\lambda}\right)^{n-1} N\|x_1(\omega) - x_0\|\\ &\quad + \|x_1(\omega) - x_0\|\\ &= \left(1 + \frac{N\lambda}{1-2\lambda}\right)\|A(\omega, x_0) - x_0\|.\end{aligned} \quad (24)$$

□

Theorem 14. *Let E be a real Banach space and P a normal cone in E with the normal constant N. Suppose that* $A : \Omega \times E \to E$ *is a continuous random comparable operator satisfying the following:*

(i) *if* $u(\omega)$, $v(\omega)$ *are random comparable, then* $v(\omega)$ *and* $A(\omega, u(\omega))$, $u(\omega)$ *and* $A(\omega, v(\omega))$ *are random comparable and there exists* $0 < \lambda < 1/2$ *such that*

$$\begin{aligned}(A(\omega, v(\omega)) - A(\omega, u(\omega)))\\ \vee (A(\omega, u(\omega)) - A(\omega, v(\omega)))\\ \leq \lambda(((A(\omega, u(\omega)) - v(\omega)) \vee (v(\omega) - A(\omega, u(\omega))))\\ + ((A(\omega, v(\omega)) - u(\omega)) \vee (u(\omega) - A(\omega, v(\omega))))); \end{aligned} \quad (25)$$

(ii) *there exists* $x_0 \in E$ *such that for any* $\omega \in \Omega$, x_0, $A(\omega, x_0)$ *are random comparable and* x_0, $A^2(\omega, x_0)$ *are random comparable.*

Then A has a random fixed point $x^*(\omega)$. *Furthermore, the iterative sequence* $\{A^n(\omega, x_0)\}$ *converges to* $x^*(\omega)$, *where* $A^n(\omega, x_0) = A(\omega, A^{n-1}(\omega, x_0))$ *and*

$$\|x^*(\omega) - x_0\| \leq \left(1 + \frac{N\lambda}{1-2\lambda}\right)\|A(\omega, x_0) - x_0\|. \quad (26)$$

Proof. For any fixed $\omega \in \Omega$, put

$$x_1(\omega) = A(\omega, x_0), \ldots, x_n(\omega) = A(\omega, x_{n-1}(\omega)), \ldots, n \geq 1. \quad (27)$$

Since x_0 and $A(\omega, x_0)$, x_0 and $A^2(\omega, x_0)$ are random comparable, then according to (i), for any $n \geq 1$, $x_n(\omega)$ and $A(\omega, x_n(\omega))$, $x_n(\omega)$ and $A^2(\omega, x_n(\omega))$ are random comparable;

that is, for any $n \geq 1$, $x_n(\omega)$ and $x_{n+1}(\omega)$, $x_n(\omega)$ and $x_{n+2}(\omega)$ are random comparable, and

$$\begin{aligned}
\theta &\leq (x_n(\omega) - x_{n+1}(\omega)) \vee (x_{n+1}(\omega) - x_n(\omega)) \\
&= (A(\omega, x_{n-1}(\omega)) - A(\omega, x_n(\omega))) \\
&\quad \vee (A(\omega, x_n(\omega)) - A(\omega, x_{n-1}(\omega))) \\
&\leq \lambda ((A(\omega, x_n(\omega)) - x_{n-1}(\omega)) \\
&\qquad \vee (x_{n-1}(\omega) - A(\omega, x_n(\omega))) \\
&\qquad + (A(\omega, x_{n-1}(\omega)) - x_n(\omega)) \\
&\qquad \vee (x_n(\omega) - A(\omega, x_{n-1}(\omega)))) \\
&= \lambda ((x_{n+1}(\omega) - x_{n-1}(\omega)) \vee (x_{n-1}(\omega) - x_{n+1}(\omega))) \\
&\leq \lambda ((x_{n+1}(\omega) - x_n(\omega)) \vee (x_n(\omega) - x_{n+1}(\omega)) \\
&\qquad + (x_n(\omega) - x_{n-1}(\omega)) \\
&\qquad \vee (x_{n-1}(\omega) - x_n(\omega))) \quad (\text{by Lemma } 7).
\end{aligned} \tag{28}$$

So

$$\begin{aligned}
\theta &\leq (x_{n+1}(\omega) - x_n(\omega)) \vee (x_n(\omega) - x_{n+1}(\omega)) \\
&\leq \left(\frac{\lambda}{1-\lambda}\right) ((x_n(\omega) - x_{n-1}(\omega)) \vee (x_{n-1}(\omega) - x_n(\omega))) \\
&\leq \cdots \\
&\leq \left(\frac{\lambda}{1-\lambda}\right)^n ((x_1(\omega) - x_0) \vee (x_0 - x_1(\omega))).
\end{aligned} \tag{29}$$

The normality of P implies that

$$\|x_{n+1}(\omega) - x_n(\omega)\| \leq N \left(\frac{\lambda}{1-\lambda}\right)^n \|x_1(\omega) - x_0\|. \tag{30}$$

Since $0 < \lambda < 1/2$, $\{x_n(\omega)\}$ is a Cauchy sequence in E. Hence there exists $x^*(\omega) \in E$ such that $x_n(\omega) \to x^*(\omega)$ $(n \to \infty)$. By the continuity of $A(\omega, \cdot)$, it is easy to see that

$$A(\omega, x^*(\omega)) = \lim_{n\to\infty} A(\omega, x_n(\omega)) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega). \tag{31}$$

Using a proof similar to that of Theorem 10, it is not difficult to prove that $x^*(\omega) : \Omega \to E$ is a random fixed point of A and

$$\begin{aligned}
\|x^*(\omega) - x_0\| &= \lim_{n\to\infty} \|x_n(\omega) - x_0\| \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i(\omega) - x_{i-1}(\omega)\| \\
&\leq \lim_{n\to\infty} \sum_{i=2}^{n} \left(\frac{\lambda}{1-\lambda}\right)^{i-1} N \|x_1(\omega) - x_0\| \\
&\quad + \|x_1(\omega) - x_0\| \\
&\leq \sum_{n=2}^{\infty} \left(\frac{\lambda}{1-\lambda}\right)^{n-1} N \|x_1(\omega) - x_0\| \\
&\quad + \|x_1(\omega) - x_0\| \\
&= \left(1 + \frac{N\lambda}{1-2\lambda}\right) \|A(\omega, x_0) - x_0\|.
\end{aligned} \tag{32}$$

□

3. Applications

We consider the following random Hammerstein integral equation:

$$x(\omega, t) = Ax(\omega, t) = \int_{-\infty}^{+\infty} k(\omega, s, t) \left(1 + \sqrt{x(\omega, s)}\right) ds. \tag{33}$$

Suppose that

(i) the kernel $k(\omega, s, t)$ is nonnegative random continuous on $\Omega \times R^1 \times R^1$ satisfying

$$\frac{1}{12} \leq \int_{-\infty}^{+\infty} k(\omega, s, t)\, ds \leq \frac{1}{2}; \tag{34}$$

(ii) for any bounded continuous functions $u(t)$, $v(t)$ satisfying the following condition,

$$\frac{1}{9} \leq u(t), v(t) \leq 1, \tag{35}$$

there exists $\beta \in (0, 1)$ such that for any $\omega \in \Omega$,

$$\int_{-\infty}^{+\infty} k(\omega, s, t) \left|\sqrt{v(s)} - \sqrt{u(s)}\right| \leq \beta |v(s) - u(s)|. \tag{36}$$

Then (33) has a unique random solution $x(\omega)$.

Proof. Since the kernel $k(\omega, s, t)$ is nonnegative random continuous on $\Omega \times R^1 \times R^1$, $A : \Omega \times R^1 \to R^1$ is a random operator. Set $u_0 = 1/9$ and $v_0 = 1$; from (34), we get that $A : \Omega \times [u_0, v_0] \to [u_0, v_0]$. For any $\omega \in \Omega$, put $\beta(\omega) = \beta$; from (36), we obtain that A is a random comparable operator. Thus we prove that (33) has a unique random solution $x(\omega)$ by Theorem 11. □

Remark 15. The operator A defined by (33) is a random increasing operator (of course, it is random comparable), but just from Theorem 2.3 of Li and Duan [9], we cannot get the conclusion because A does not satisfy the condition (ii) of Theorem 2.3 of Li and Duan [9]. However, by Theorem 11 of this work, we can easily get the conclusion. Thus, from this application, it is shown that some of the results in this work generalize and extend the corresponding results in [9] again.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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