

Research Article Existence and Convergence of the Positive Solutions of a Discrete Epidemic Model

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We consider a class of system of nonlinear difference equations arising from mathematical models describing a discrete epidemic model. Sufficient conditions are established that guarantee the existence of positive solutions, the existence of a unique nonnegative equilibrium, and the convergence of the positive solutions to the nonnegative equilibrium of the system of difference equations. The obtained results are new and they complement previously known results.

1. Introduction

Over the past decades, there has been an increasing interest in studying nonlinear and rational difference equations arising in applied sciences, and hence there are many papers and books concerning theory and applications of difference equations; see, for example, [1–19] and the references therein. The existence of positive solutions for difference equations has also received a great deal of attention in the literature. To identify a few, we refer the reader to Baštinec et al. [1, 2], Diblík and Hlavičková [6], and the references therein. In [11], the authors posed some research projects about the behavior of solutions of the following deterministic epidemic models:

$$\begin{aligned} x_{n+1} &= (1 - x_n) \left(1 - e^{-Ax_n} \right), \quad n = 0, 1, \dots, \\ x_{n+1} &= (1 - x_n - x_{n-1}) \left(1 - e^{-Ax_n} \right), \quad n = 0, 1, \dots, \\ x_{n+1} &= (1 - x_n - x_{n-1} - x_{n-2}) \left(1 - e^{-Ax_n} \right), \end{aligned}$$
(1)
$$n = 0, 1, \dots$$

and, generally,

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right) \left(1 - e^{-Ax_n}\right), \quad k \in \{1, 2, 3, \ldots\}.$$
(2)

The above equations are special cases of epidemic models which were derived in [3, 4]. For some other biological

models, see, for example, [7, 10, 11] and the references therein. Recently, the dynamics of (2) and some related models have been discussed in [13, 15, 19].

Our aim in this paper is to study the existence of positive solutions, the existence of a unique nonnegative equilibrium, and the convergence of the positive solutions to the nonnegative equilibrium of the following system of nonlinear difference equations:

$$\begin{aligned} x_{n+1} &= \left(1 - \sum_{j=0}^{2} y_{n-j}\right) \left(1 - e^{-Ay_{n}}\right), \\ y_{n+1} &= \left(1 - \sum_{j=0}^{2} x_{n-j}\right) \left(1 - e^{-Bx_{n}}\right), \end{aligned}$$
(3)

where the constants $A, B \in (0, \infty)$ and the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are positive numbers which satisfy the relations

$$\begin{aligned} x_{-2} + x_{-1} + x_0 &< 1, \\ y_{-2} + y_{-1} + y_0 &< 1, \\ 1 - y_0 - y_{-1} &> \left(1 - \sum_{j=0}^2 x_{n-j}\right) \left(1 - e^{-Bx_0}\right), \end{aligned}$$

$$1 - x_{0} - x_{-1} > \left(1 - \sum_{j=0}^{2} y_{n-j}\right) \left(1 - e^{-Ay_{0}}\right),$$

$$1 - x_{0} > \left(1 - \sum_{j=0}^{2} y_{n-j}\right) \left(1 - e^{-Ay_{0}}\right)$$

$$+ \left(1 - y_{0} - y_{-1}\right) \left(1 - e^{-Ay_{0}}\right),$$

$$1 - y_{0} > \left(1 - \sum_{j=0}^{2} x_{n-j}\right) \left(1 - e^{-Bx_{0}}\right)$$

$$+ \left(1 - x_{0} - x_{-1}\right) \left(1 - e^{-Bx_{0}}\right),$$

$$y_{0} > \left(1 - \sum_{j=0}^{2} x_{n-j}\right) \left(1 - e^{-Bx_{0}}\right),$$

$$x_{0} > \left(1 - \sum_{j=0}^{2} y_{n-j}\right) \left(1 - e^{-Ay_{0}}\right).$$
(4)

System (3) can be viewed as an extension to two dimensions of (2). In fact, if A = B, $x_{-2} = y_{-2}$, $x_{-1} = y_{-1}$, and $x_0 = y_0$, then system (3) reduces to (2) for k = 3.

The rest of the paper is organized as follows. In Section 2, we establish the existence of positive solutions and the convergence to the unique zero equilibrium for the positive solutions of (3). In Section 3, we present conditions that imply the convergence of the positive solutions to the unique positive equilibrium for (3).

2. Existence and Convergence to the Unique Zero Equilibrium for the Positive Solutions of (3)

This section deals with the existence of positive solutions and the convergence to the unique zero equilibrium for the positive solutions of (3). We begin with the next lemma which furnishes some useful inequalities.

Lemma 1. Consider the following functions:

$$g(x) = (1 - 3x) (1 - e^{-Bx}),$$

$$h(x) = (1 - 3x) (1 - e^{-4x}),$$

$$G(x) = -3 (1 - e^{-Ag(x)}) + Ae^{-Ag(x)} (1 - 3g(x)),$$

$$H(x) = -3 (1 - e^{-4h(x)}) + 4e^{-4h(x)} (1 - 3h(x)),$$

(5)

where

$$1 < A \le 4,$$

$$1 < B \le 4.$$
(6)

Then the following statements hold:

(i)

$$g'(x) > h'(x)$$
, for 0.14 < x < 0.273, (7)

(ii)

$$G(x) > H(x), \quad for \ 0.14 < x < 0.273,$$
 (8)

(iii) equation h'(x) = 0 (resp., H(x) = 0) has a unique solution c (resp., d), such that

$$c \in (0.14, 0.273), d \in (c, 0.273),$$
(9)

(iv)

$$h'(x) H(x) < 1, \quad for \ c < x < d.$$
 (10)

Proof. (i) From (5), we have

$$g'(x) = -3(1 - e^{-Bx}) + Be^{-Bx}(1 - 3x),$$

$$h'(x) = -3(1 - e^{-4x}) + 4e^{-4x}(1 - 3x).$$
(11)

Set

$$F(B) = -3(1 - e^{-Bx}) + Be^{-Bx}(1 - 3x), \quad 1 < B \le 4.$$
(12)

Then

$$\frac{\mathrm{d}F}{\mathrm{d}B} = e^{-Bx} \left[1 - 6x + Bx \left(3x - 1 \right) \right]. \tag{13}$$

If dF/dB = 0, then

$$B = \frac{1 - 6x}{x(1 - 3x)}.$$
 (14)

Set

$$\Phi(x) = \frac{1 - 6x}{x(1 - 3x)}.$$
(15)

It is easy to prove that $\Phi(x)$ is a decreasing function for 0 < x < 1/3 and hence

$$\Phi(x) < \Phi(0.14) = \frac{400}{203}, \quad 0.14 \le x \le 0.273.$$
 (16)

In view of 0.14 < x < 0.273, *F*(*B*) is a decreasing function if $400/203 < B \le 4$, which implies that

$$F\left(\frac{400}{203}\right) > F(B) > F(4).$$
 (17)

Now, suppose that

$$1 < B \le \frac{400}{203}.$$
 (18)

Since $\Phi(x)$ is a decreasing function for 0 < x < 1/3, then

$$\Phi(x) \le \Phi\left(\frac{7-\sqrt{37}}{6}\right) = 1, \quad \frac{7-\sqrt{37}}{6} \le x \le 0.273.$$
 (19)

Hence, by (13), we have that F(B) is a decreasing function and therefore, by (17), we get

$$F(1) > F(B) \ge F\left(\frac{400}{203}\right) > F(4),$$
 (20)

and obviously

$$F(B) > F(4)$$
,

for
$$1 < B \le \frac{400}{203}$$
, $\frac{7 - \sqrt{37}}{6} \le x \le 0.273$. (21)

Finally, suppose that

$$1 < B \le \frac{400}{203}, \quad 0.14 \le x < \frac{7 - \sqrt{37}}{6}.$$
 (22)

Since $\Phi(x)$ is a decreasing function for 0 < x < 1/3, we have

$$1 < \Phi(x) \le \frac{400}{203}, \quad 0.14 \le x < \frac{7 - \sqrt{37}}{6}.$$
 (23)

Now by combining (13), (15), and (22) with (23) it follows that for every *x* there exists a $B_0(x)$ such that

$$\frac{dF}{dB} > 0, \quad \text{for } 1 < B < B_0(x),$$

$$\frac{dF}{dB} < 0, \quad \text{for } B_0(x) < B \le \frac{400}{203}.$$
(24)

We claim that

$$F(1) > F(4)$$
 for $0.14 \le x < \frac{7 - \sqrt{37}}{6}$. (25)

After some calculations, in order to prove (25) it is sufficient to prove that

$$e^{3x} + \frac{12x - 7}{4 - 3x} > 0$$
 for $0.14 \le x < \frac{7 - \sqrt{37}}{6}$, (26)

which is true, since if we set

$$\omega(x) = e^{3x} + \frac{12x - 7}{4 - 3x},$$
(27)

it is easy to prove that $\omega(x)$ is an increasing function for every x, such that (23) holds, and $\omega(0.14) = 0.0359 > 0$. Relations (24) and (25) imply that

$$F(B) > \min\left\{F(1), F\left(\frac{400}{203}\right)\right\} > F(4),$$

for $1 < B \le \frac{400}{203}, \ 0.14 \le x < \frac{7 - \sqrt{37}}{6}.$ (28)

Therefore, by (17), (20), and (28) we have that relation (7) is true. This completes the proof of statement (i). (ii) Since $(1-3x)(1-e^{-Bx})$ is an increasing function with

(ii) Since $(1 - 3x)(1 - e^{-bx})$ is an increasing function with respect to *B*, for $1 < B \le 4$ and $0.14 \le x \le 0.273$, then

$$0 < g(x) \le h(x), \quad 1 < B \le 4, \ 0.14 \le x \le 0.273.$$
 (29)

We claim that

$$0 < h(x) < \frac{1}{3}, \quad 0.14 \le x \le 0.273.$$
 (30)

Indeed, by (11) we have

$$h'(0.12) = 0.0388,$$

$$h'(0.273) = -1.7504,$$

$$h''(x) = -24e^{-4x} - 16e^{-4x} (1 - 3x) < 0,$$

$$0.14 \le x \le 0.273.$$

(31)

Hence, equation h'(x) = 0 has a unique solution in the interval (0.14, 0.273). Using Newton's method we can see that this solution is c = 0.1421. So after some calculations we have

$$0 < 0.12 \le \min \{h(0.14), h(0.273)\} \le h(x) \le 0.273$$

$$< \frac{1}{3},$$
 (32)

which implies that our claim (30) is true.

In addition, consider the function

$$R(x) = -3(1 - e^{-Ax}) + Ae^{-Ax}(1 - 3x).$$
(33)

It is easy to prove that R(x) is a decreasing function for $1 < A \le 4$ and 0 < x < 1/3 and hence, by (5), (29), (30), and (33), we have that

$$G(x) = R(g(x)) \ge R(h(x)),$$

$$1 < A \le 4, \ 0.14 \le x \le 0.273,$$

$$R(h(x)) = -3(1 - e^{-Ah(x)}) + Ae^{-Ah(x)}(1 - 3h(x)) \qquad (34)$$

$$> -3(1 - e^{-4h(x)}) + 4e^{-4h(x)}(1 - 3h(x))$$

$$= H(x), \quad 1 < A \le 4, \ 0.14 \le x \le 0.273,$$

which imply that (8) is true. This completes the proof of statement (ii).

(iii) By the above statement (ii), we get that equation h'(x) = 0 has a unique solution c = 0.1421 and obviously, $c \in (0.14, 0.273)$. From (5) we have

$$H'(x) = -24h'(x) e^{-4h(x)}$$

$$-16e^{-4h(x)}h'(x) (1 - 3h(x)),$$

$$H''(x) = 144e^{-4h(x)} (h'(x))^{2} - 24h''(x) e^{-4h(x)}$$
(35)

$$+ 64e^{-4h(x)} (h'(x))^{2} (1 - 3x)$$

$$- 16e^{-4h(x)}h''(x) (1 - 3h(x)).$$

Since *c* is the unique solution of equation h'(x) = 0, it follows from (31) that

$$h'(x) < h'(c) = 0, \quad c < x \le 0.273.$$
 (36)

In addition, from (35) and (36) it follows that

$$H'(x) > 0, \quad c < x \le 0.273.$$
 (37)

From (5) we get

$$H(c) = H(0.1421) = -1.154,$$

 $H(0.273) = 0.4348$ (38)

and hence, by (37), equation H(x) = 0 has a unique solution in the interval (*c*, 0.273). Using Newton's method, we can prove that this solution is d = 0.2697. Obviously, $d \in$ (*c*, 0.273). This completes the proof of statement (iii).

(iv) Consider the function

$$Q(x) = h'(x) H(x), \quad c \le x \le d,$$
 (39)

and then

$$Q'(x) = h''(x) H(x) + h'(x) H'(x),$$

$$Q''(x) = h'''(x) H(x) + 2h''(x) H'(x)$$
(40)

$$+ h'(x) H''(x),$$

where from (9) and (31)

$$h'''(x) = 144e^{-4x} + 64e^{-4x}(1-3x) > 0, \quad c \le x \le d.$$
 (41)

From (30), (31), and (35) we have

$$H''(x) > 0, \quad 0.14 \le x \le 0.273.$$
 (42)

Since *d* is the unique solution of equation H(x) = 0, it follows from (9) and (37) that

$$H(x) < H(d) = 0, \quad c \le x \le d.$$
 (43)

Since Q''(x) < 0, $c \le x \le d$, and Q(c) = Q(d) = 0, we get that equation Q'(x) = 0 has a unique solution in the interval (c, d). Using Newton's method, we can prove that this solution is x = 0.2342. Since Q(0.2342) = 0.8925 < 1, then h'(x)H(x) < 1. This completes the proof of statement (iv).

Theorem 2. Consider system (3) with the constants A, B satisfying

$$0 < A \le 4,$$

$$0 < B \le 4.$$
 (44)

Let $\{(x_n, y_n)\}$ be a solution of (3) with initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ satisfying (4). Then

$$x_n > 0,$$

 $y_n > 0,$ (45)
 $n = 1, 2, 3,$

Proof. From (3), we have

$$x_{1} = (1 - y_{0} - y_{-1} - y_{-2}) (1 - e^{-Ay_{0}}),$$

$$y_{1} = (1 - x_{0} - x_{-1} - x_{-2}) (1 - e^{-Bx_{0}}).$$
(46)

It follows from (4), (44), and (46) that

$$x_1 > 0,$$

 $y_1 > 0.$
(47)

Then from (3) we get

$$x_{2} = (1 - y_{1} - y_{0} - y_{-1}) (1 - e^{-Ay_{1}}),$$

$$y_{2} = (1 - x_{1} - x_{0} - x_{-1}) (1 - e^{-Bx_{1}}).$$
(48)

By (4) and (46), we take

$$1 - y_0 - y_{-1} > y_1,$$

$$1 - x_0 - x_{-1} > x_1,$$
(49)

which, together with (5) and (48), imply that

$$x_2 > 0,$$
 (50)
 $y_2 > 0.$

From (4) we get $y_0 > y_1$, and then

$$(1 - y_0 - y_{-1}) (1 - e^{-Ay_0}) > (1 - y_1 - y_0 - y_{-1}) (1 - e^{-Ay_1}),$$
(51)

which together with (3) and (4) yields $1 - x_0 > x_1 + x_2$, and hence, by (3) and (44), we have $x_3 > 0$. Similarly we can prove that $y_3 > 0$. We next prove that

$$x_n + x_{n-1} + x_{n-2} < 1,$$

$$y_n + y_{n-1} + y_{n-2} < 1.$$
(52)

Now, by (3), (4), and (44), we have

$$y_{3} + y_{2} + y_{1} < \left(1 - \sum_{j=0}^{2} x_{j}\right) \left(1 - e^{-Bx_{2}}\right) + \left(1 - x_{1} - x_{0}\right) \left(1 - e^{-Bx_{1}}\right) + \left(1 - x_{0}\right) \left(1 - e^{-Bx_{0}}\right).$$
(53)

We consider the function f, for x, y, z > 0, x + y + z < 1, $0 < B \le 4$, as follows:

$$f(x, y, z, B) = (1 - x - y - z) (1 - e^{-Bx}) + (1 - y - z) (1 - e^{-By}) + (1 - z) (1 - e^{-Bz}).$$
(54)

Since f is an increasing function with respect to B, we obtain

$$f(x, y, z, B) \le (1 - x - y - z) (1 - e^{-4x}) + (1 - y - z) (1 - e^{-4y}) + (1 - z) (1 - e^{-4z}),$$
(55)

where x, y, z > 0, x + y + z < 1. Set the function *h* as follows:

$$h(x, y, z) = (1 - x - y - z) (1 - e^{-4x}) + (1 - y - z) (1 - e^{-4y})$$
(56)
+ (1 - z) (1 - e^{-4z}),

where x, y, z > 0, x + y + z < 1. Then, we take the system of equations

$$\frac{\partial h}{\partial x} = -1 + e^{-4x} + 4e^{-4x} \left(1 - x - y - z\right) = 0,$$

$$\frac{\partial h}{\partial y} = -2 + e^{-4x} + e^{-4y} + 4e^{-4y} \left(1 - y - z\right) = 0,$$

$$\frac{\partial h}{\partial z} = -3 + e^{-4x} + e^{-4y} + e^{-4z} + 4e^{-4z} \left(1 - z\right) = 0,$$
(57)

where x, y, z > 0, x + y + z < 1. System (57) is equivalent to system

$$y = -\frac{1}{4} \ln \frac{2 - e^{-4x}}{e^{4x} + 4x},$$

$$z = \frac{-e^{4x} - 4x + 5 + \ln\left(\left(2 - e^{-4x}\right) / \left(e^{4x} + 4x\right)\right)\right)}{4},$$

$$-3 + e^{-4x} + \frac{2 - e^{-4x}}{e^{4x} + 4x}$$

$$+ e^{e^{4x} + 4x - 5 - \ln\left((2 - e^{-4x}) / (e^{4x} + 4x)\right)} \left(e^{4x} + 4x - \ln \frac{2 - e^{-4x}}{e^{4x} + 4x}\right) = 0.$$
(58)

We now consider the function

$$g(x) = -3 + e^{-4x} + \frac{2 - e^{-4x}}{e^{4x} + 4x} + e^{e^{4x} + 4x - 5 - \ln((2 - e^{-4x})/(e^{4x} + 4x))} \left(e^{4x} + 4x - \ln\frac{2 - e^{-4x}}{e^{4x} + 4x}\right),$$
(59)

where $0 \le x \le 1$. From (58) we get

$$g(0) = -1 + e^{-4} < 0,$$

$$g(1) = 1.333 \times 10^{6} > 0,$$
 (60)

$$g''(x) > 0, \quad 0 < x < 1.$$

From (59) and (60) we deduce that equation g(x) = 0 has a unique solution \overline{x} , such that $0 < \overline{x} < 1$. Using Newton's method we have that

$$\overline{x} = 0.2368,$$

 $\overline{y} = 0.1956,$ (61)
 $\overline{z} = 0.1732.$

In addition, by (57) we obtain

$$\frac{\partial^2 h}{\partial x^2} = -8e^{-4x} - 16e^{-4x} (1 - x - y - z) < 0,$$

$$\frac{\partial^2 h}{\partial y^2} = -8e^{-4y} - 16e^{-4y} (1 - y - z) < 0,$$

$$\frac{\partial^2 h}{\partial z^2} = -8e^{-4z} - 16e^{-4z} (1 - z) < 0,$$

$$\frac{\partial^2 h}{\partial x \partial y} = -4e^{-4x},$$

$$\frac{\partial^2 h}{\partial x \partial z} = -4e^{-4x},$$

$$\frac{\partial^2 h}{\partial y \partial z} = -4e^{-4y}.$$

(62)

The derivatives of *h* at the point $(\overline{x}, \overline{y}, \overline{z})$ provide

$$A = \frac{\partial^2 h}{\partial x^2} = -5.5499,$$

$$B = \frac{\partial^2 h}{\partial y^2} = -8.2769,$$

$$C = \frac{\partial^2 h}{\partial z^2} = -10.6186,$$

$$D = \frac{\partial^2 h}{\partial x \partial y} = -1.5513,$$

$$E = \frac{\partial^2 h}{\partial x \partial z} = -1.5513,$$

$$F = \frac{\partial^2 h}{\partial y \partial z} = -1.8292.$$
have $A < 0$

From (63), we have A < 0,

$$\begin{vmatrix} A & D \\ D & B \end{vmatrix} = 43.5294 > 0,$$

$$\begin{vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{vmatrix} = -432.5155 < 0.$$
(64)

By (63) and (64), we can get that $(\overline{x}, \overline{y}, \overline{z})$ is the maximum point of h(x, y, z) and the unique solution of system (57), and then

$$h(x, y, z) \le h(\overline{x}, \overline{y}, \overline{z}) = 0.9972 < 1.$$
(65)

$$y_3 + y_2 + y_1 < 1. (66)$$

Similarly, we can prove that

$$x_3 + x_2 + x_1 < 1. \tag{67}$$

Thus, from (66) and (67) it can be shown that (52) are satisfied for n = 3. Following this fashion, we can prove that (52) are true for all n = 3, 4, 5, ... Then it is obvious that (45) are satisfied and thus the proof is complete.

Theorem 3. Let $\{(x_n, y_n)\}$ be a solution of (3) such that (4) are satisfied. Consider the system of algebraic equations:

$$x = (1 - 3y) (1 - e^{-Ay}),$$

$$y = (1 - 3x) (1 - e^{-Bx}),$$

$$x, y \in \left(0, \frac{1}{3}\right).$$
(68)

If $0 < A \le 1$ and $0 < B \le 1$, then the following statements hold.

- (i) System (68) has a unique zero solution $(\overline{x}, \overline{y}) = (0, 0)$.
- (ii) $\{(x_n, y_n)\}$ tends to the zero equilibrium (0,0) of (3) as $n \to \infty$.

Proof. (i) Consider the functions

$$E(y) = (1 - 3y)(1 - e^{-Ay}) - y,$$

$$K(x) = (1 - 3x)(1 - e^{-Bx}) - x,$$

$$(x, y) \in \left(0, \frac{1}{3}\right).$$
(69)

Then

$$E'(y) = -3(1 - e^{-Ay}) + Ae^{-Ay}(1 - 3y) - 1 < 0,$$

$$K'(x) = -3(1 - e^{-Bx}) + Be^{-Bx}(1 - 3x) - 1 < 0,$$
(70)

which imply that

$$E(y) \le E(0) = 0,$$

 $K(x) \le K(0) = 0,$
(71)

and hence $x \le y, y \le x$. Therefore, $(\overline{x}, \overline{y}) = (0, 0)$ is the unique zero solution for system (68).

(ii) By (3), (4), (44), (45), and (52), we have

$$x_{n+1} = (1 - y_n - y_{n-1} - y_{n-2}) (1 - e^{-Ay_n})$$

< $(1 - y_n) (1 - e^{-Ay_n}) < Ay_n < y_n,$ (72)

$$n=0,1,\ldots$$

Similarly, $y_{n+1} < x_n$, which imply that $x_{2n}, x_{2n+1}, y_{2n}, y_{2n+1}$ are decreasing sequences. Therefore, there exist constants l_0, l_1, m_0, m_1 such that

$$\lim_{n \to \infty} x_{2n} = l_0,$$

$$\lim_{n \to \infty} x_{2n+1} = l_1,$$

$$\lim_{n \to \infty} y_{2n} = m_0,$$

$$\lim_{n \to \infty} y_{2n+1} = m_1.$$
(73)

Using (45), (52), and (73), we get $0 \le l_0, l_1, m_0, m_1 < 1$. Relations (72) and (73) imply that

$$l_{1} \le m_{0},$$

 $l_{0} \le m_{1},$
 $m_{0} \le l_{1},$
 $m_{1} \le l_{0},$
(74)

and, hence,

$$l_1 = m_0,$$
 (75)
 $l_0 = m_1.$

Further, from (3), (73), and (75) we have

$$l_{1} = (1 - 2l_{1} - l_{0}) (1 - e^{-Al_{1}}),$$

$$l_{0} = (1 - 2l_{0} - l_{1}) (1 - e^{-Al_{0}}).$$
(76)

First, suppose that $l_0 = 0$, then from (76) we get $l_1 = (1 - 2l_1)(1 - e^{-Al_1})$, and, hence, by using $0 < A \le 1$, $0 < B \le 1$, and Lemma 2.1 of [19] we get $l_1 = 0$. Similarly, when $l_1 = 0$, we get $l_0 = 0$. Now, suppose that

$$l_0 \neq 0,$$

$$l_1 \neq 0,$$
(77)

and then from (76) we get that

$$\frac{l_0}{1 - e^{-Al_0}} + l_0 = \frac{l_1}{1 - e^{-Al_1}} + l_1.$$
(78)

If

$$f(x) = \frac{x}{1 - e^{-cx}} + x, \quad A > 0, \ x > 0,$$
(79)

then since

$$1 - e^{-cx} - cxe^{-cx} > cx - cxe^{-cx} = cx(1 - e^{-cx}) > 0, \quad (80)$$

we deduce that

$$f'(x) = \frac{1 - e^{-cx} - cxe^{-cx}}{(1 - e^{-cx})^2} + 1 > 0,$$
(81)

and thus *f* is an increasing function. Hence, by $0 < A \le 1, 0 < B \le 1, (73)$, and (78), we have $l_1 = l_0$. From (75), we get $m_1 = m_0$. Therefore, there exist the $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$. By statement (i) of Theorem 3, we have that $l_1 = l_0 = 0$, which contradicts the suppose. Hence, $l_1 = l_0 = m_1 = m_0 = 0$ and so $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. This completes the proof of the theorem.

3. Convergence to the Unique Positive Equilibrium for the Positive Solutions of (3)

The next theorem obtains sufficient conditions that imply the convergence of the positive solutions to the unique positive equilibrium for (3).

Theorem 4. Let $\{(x_n, y_n)\}$ be a solution of (3) such that (4) are satisfied. Consider the system of algebraic equations:

$$x = (1 - 3y) (1 - e^{-Ay}),$$

$$y = (1 - 3x) (1 - e^{-Bx}),$$

$$x, y \in \left(0, \frac{1}{3}\right).$$
(82)

If $1 < A \le 4$ and $1 < B \le 4$, then the following statements hold.

- (i) System (82) has a unique positive solution $(\overline{x}, \overline{y}), \overline{x}, \overline{y} \in (0, 1/3).$
- (ii) Suppose there exists an $m \in N$ such that for $n \ge m$ either

$$x_n < \overline{x},$$

$$y_n < \overline{y}$$
(83)

or

$$x_n \ge \overline{x},$$

$$y_n \ge \overline{y}$$
(84)

hold. Then $\{(x_n, y_n)\}$ tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ of (3) as $n \to \infty$.

Proof. (i) Set

$$R(x) = (1 - 3g(x))(1 - e^{-Ag(x)}) - x, \quad x \in \left(0, \frac{1}{3}\right), \quad (85)$$

where g(x) was defined in (5). From $1 < A \le 4$ and $1 < B \le 4$, we have that

$$0 < g(x) < (1 - 3x) Bx \le \frac{B}{12} \le \frac{1}{3}, \quad x \in \left(0, \frac{1}{3}\right).$$
(86)

Further,

$$g'(x) = -3(1 - e^{-Bx}) + Be^{-Bx}(1 - 3x),$$

$$g''(x) = -6Be^{-Bx} - B^2 e^{-Bx}(1 - 3x) < 0,$$

$$x \in \left(0, \frac{1}{3}\right),$$
(87)

$$g'(0) = B > 0,$$

 $g'\left(\frac{1}{3}\right) = -3\left(1 - e^{-B/3}\right) < 0,$ (88)

and so g'(x) has a unique solution $x_0 \in (0, 1/3)$ such that

$$g'(x) > 0$$
, for $0 < x < x_0$,
 $g'(x) < 0$, for $x_0 < x < \frac{1}{3}$. (89)

Moreover, we have

$$g(0) = 0,$$

$$g\left(\frac{1}{3}\right) = 0,$$

$$R(0) = 0,$$

$$R\left(\frac{1}{3}\right) = -\frac{1}{3},$$

$$R'(x)$$

$$= -3g'(x)\left(1 - e^{-Ag(x)}\right)$$

$$+ Ag'(x) e^{-Ag(x)}\left(1 - 3g(x)\right) - 1 \qquad (91)$$

$$= g'(x)\left(-3\left(1 - e^{-Ag(x)}\right) + Ae^{-Ag(x)}\left(1 - 3g(x)\right)\right)$$

$$- 1 = g'(x)G(x) - 1.$$

Further,

$$R'(0) = g'(0) G(0) - 1 = AB - 1 > 0,$$

$$R'\left(\frac{1}{3}\right) = g'\left(\frac{1}{3}\right) G\left(\frac{1}{3}\right) - 1 = -3A\left(1 - e^{-B/3}\right) - 1 \quad (92)$$

$$< 0.$$

Relations (91) and (92) imply that equation R(x) = 0 has a solution $\overline{x} \in (0, 1/3)$. It follows from (86) that

$$0 < g\left(\overline{x}\right) < \frac{1}{3},\tag{93}$$

and so $(\overline{x}, \overline{y})$ is a solution of system (82) such that $\overline{x}, \overline{y} \in (0, 1/3)$.

We prove now that $(\overline{x}, \overline{y})$ is the unique solution of system (82). Indeed, by (5) we have

$$G'(x) = -Ag'(x)e^{-Ag(x)}(6 + A(1 - 3g(x))), \qquad (94)$$

and hence, by (6), (86), and (89), we get

$$G'(x) < 0, \quad \text{for } 0 < x < x_0,$$
 (95)

$$G'(x) > 0$$
, for $x_0 < x < \frac{1}{3}$. (96)

Firstly, assume that

$$G(x) \ge 0$$
, for $0 < x < \frac{1}{3}$. (97)

In view of (87), (89), (96), and (97), g'(x) and G(x) are positive and decreasing functions for $0 < x < x_0$. And by

(91) we get that R''(x) = g''(x)G(x) + g'(x)G'(x). Therefore, R'(x) is a decreasing function for $0 < x < x_0$. Further, by combining (89), (91) with (97) it follows that

$$R'(x) < 0$$
, for $x_0 < x < \frac{1}{3}$. (98)

Hence, by (92) we have that there exists a unique $x' \in (0, x_0)$ such that

$$R'(x) > 0, \text{ for } 0 < x < x',$$

 $R'(x) < 0, \text{ for } x' < x < \frac{1}{3}.$ (99)

Thus, \overline{x} is the unique solution of equation R(x) = 0, such that $\overline{x} \in (0, 1/3)$ and so $(\overline{x}, \overline{y})$ is the unique solution of system (82), such that $(\overline{x}, \overline{y}) \in (0, 1/3)$.

Secondly, assume that there exists an $x \in (0, 1/3)$ such that G(x) < 0. In view of (96) and G(0) = G(1/3) = A > 0, there exist exactly two real numbers x_1, x_2 such that

$$G(x_1) = G(x_2) = 0, \quad 0 < x_1 < x_0 < x_2 < \frac{1}{3}.$$
 (100)

Hence, by (96) we have that

$$G(x) > 0$$
, for $0 < x < x_1$ or $x_2 < x < \frac{1}{3}$,
 $G(x) < 0$, for $x_1 < x < x_2$.
(101)

Since $x_1 < x_0$, by (87), (89), and (101), g'(x) and G(x) are positive and decreasing functions for $0 < x < x_1$. Then

$$R''(x) = g'(x)G'(x) + g''(x)G(x) < 0,$$
(102)

and it follows that R'(x) is a decreasing function. Since

$$R'(0) = AB - 1 > 0,$$

$$R'(x_1) = g'(x_1)G(x_1) - 1 = -1 < 0,$$
(103)

we deduce that there exists an $x'' \in (0, x_1)$ such that

$$R'(x) > 0$$
, for $0 < x < x''$,
 $R'(x) < 0$, for $x'' < x < x_1$. (104)

In addition, since $x_1 < x_0$, it follows from (89), (91), and (101) that

$$R'(x) < 0, \quad \text{for } x_1 < x < x_0.$$
 (105)

We also claim that

$$R'(x) < 0, \quad \text{for } x_0 < x < x_2.$$
 (106)

First, we prove that $c \le x_0$ and $x_2 \le d$, where c, d are defined in statement (iii) of Lemma 1. Since x_0 is the unique solution of equation g'(x) = 0 for 0 < x < 1/3, and (87) holds, in order to prove $c \le x_0$ it is sufficient to prove that g'(c) > 0 for any $1 < B \le 4$. From statements (i) and (iii) of Lemma 1 we have g'(c) > h'(c) = 0, and hence, $c \le x_0$.

Moreover, from (89) and (101), in order to prove $x_2 \le d$, it is sufficient to prove that $x_0 \le d$ and G(d) > 0 for any $1 < A \le 4$ and $1 < B \le 4$. From (12) and (20), we get that

$$g'(x) < -3(1 - e^{-x}) + e^{-x}(1 - 3x),$$

$$1 < B \le 4, \ \frac{7 - \sqrt{37}}{6} \le x \le 0.273,$$
(107)

and since, from statement (iii) of Lemma 1, d = 0.2697, it follows that

$$g'(d) < -3(1 - e^{-d}) + e^{-d}(1 - 3d) = -0.5634 < 0 \quad (108)$$

and $g'(x_0) = 0$, g''(x) < 0, which imply that $x_0 \le d$.

In addition, from statements (ii) and (iii) of Lemma 1 we have

$$G(d) > H(d) = 0, \quad 1 < A \le 4, \ 1 < B \le 4,$$
 (109)

and, thus, G(d) > 0. Now, by combining relations (89), (101) and statements (i) and (ii) of Lemma 1 with $0.14 < x_1 < x_0 < x_2 < 0.273$, it follows that

$$g'(x) G(x) < h'(x) H(x) < 1, \quad x_0 < x < x_2.$$
 (110)

Hence,

$$R'(x) = g'(x)G(x) - 1 < 0, \quad x_0 < x < x_2,$$
(111)

which proves our claim.

Finally, since $x_0 < x_2$, by (89), (104), and (105), we have

$$R'(x) < 0, \quad \text{for } x_2 < x < \frac{1}{3}.$$
 (112)

In summary, since

$$R'(x) > 0 \quad \text{for } 0 < x < x'',$$

$$R'(x) < 0 \quad \text{for } x'' < x < \frac{1}{3},$$
(113)

and G(0) = 0, G(1/3) = -1/3, we deduce that \overline{x} is the unique solution of equation R(x) = 0 such that $\overline{x} \in (0, 1/3)$, and $(\overline{x}, \overline{y})$ is the unique solution of system (82) such that $\overline{x}, \overline{y} \in (0, 1/3)$.

(ii) Assuming that there exists an $m \in N$ such that, for $n \ge m$,

$$\begin{aligned} x_n < \overline{x}, \\ y_n < \overline{y}, \end{aligned} \tag{114}$$

then

$$x_{n+1} \ge (1 - 3\overline{y}) \left(1 - e^{-Ay_n} \right),$$

$$y_{n+1} \ge (1 - 3\overline{x}) \left(1 - e^{-Bx_n} \right),$$

$$n = m + 1, m + 2, \dots.$$
(115)

Using Lemma 2.7 of [19], we have

$$\begin{aligned} x_{n+1} &\geq \left(1 - 3\overline{y}\right) \left(1 - e^{-A\overline{y}}\right) \frac{1 - e^{-Ay_n}}{1 - e^{-A\overline{y}}} = \overline{x} \frac{1 - e^{-Ay_n}}{1 - e^{-A\overline{y}}} \\ &\geq \frac{\overline{x}}{\overline{y}} y_n, \\ y_{n+1} &\geq \left(1 - 3\overline{x}\right) \left(1 - e^{-B\overline{x}}\right) \frac{1 - e^{-Bx_n}}{1 - e^{-B\overline{x}}} = \overline{y} \frac{1 - e^{-Bx_n}}{1 - e^{-B\overline{x}}} \\ &\geq \frac{\overline{y}}{\overline{x}} x_n, \end{aligned}$$
(116)

which lead to

$$x_{n+1} \ge x_{n-1},$$

 $y_{n+1} \ge y_{n-1},$ (117)
 $n = 1, 2, 3, \dots$

Since $x_{n+1} = (1 - y_n - y_{n-1} - y_{n-2})(1 - e^{-Ay_n}) < Ay_n < y_n$ and, similarly, $y_{n+1} < x_n$, then $x_{2n}, x_{2n+1}, y_{2n}, y_{2n+1}$ are decreasing sequences. Therefore, there exist constants l_0, l_1, m_0, m_1 such that

$$\lim_{n \to \infty} x_{2n} = l_0,$$

$$\lim_{n \to \infty} x_{2n+1} = l_1,$$

$$\lim_{n \to \infty} y_{2n} = m_0,$$

$$\lim_{n \to \infty} y_{2n+1} = m_1.$$
(118)

Since $\{(x_n, y_n)\}$ are the positive solutions of (3), we get

$$0 \le l_0, l_1, m_0, m_1 < 1.$$
(119)

Relations (116) imply that

$$l_{0} \geq \frac{\overline{x}}{\overline{y}} m_{1},$$

$$l_{1} \geq \frac{\overline{x}}{\overline{y}} m_{0},$$

$$m_{0} \geq \frac{\overline{y}}{\overline{x}} l_{1},$$

$$m_{1} \geq \frac{\overline{y}}{\overline{x}} l_{0},$$
(120)

and, hence,

$$l_0 = \frac{\overline{x}}{\overline{y}} m_1,$$

$$l_1 = \frac{\overline{x}}{\overline{y}} m_0.$$
(121)

Further, by using (3) and (118), we have

$$m_{1} = (1 - 2l_{0} - l_{1}) (1 - e^{-Bl_{0}}),$$

$$m_{0} = (1 - 2l_{1} - l_{0}) (1 - e^{-Bl_{1}}).$$
(122)

Thus, relations (121) and (122) imply that

$$m_{1} = \left(1 - \left(m_{0} + 2m_{1}\right)\frac{\overline{x}}{\overline{y}}\right)\left(1 - e^{-Bm_{1}(\overline{x}/\overline{y})}\right),$$

$$m_{0} = \left(1 - \left(m_{1} + 2m_{0}\right)\frac{\overline{x}}{\overline{y}}\right)\left(1 - e^{-Bm_{0}(\overline{x}/\overline{y})}\right).$$
(123)

Then we get

$$\frac{m_1}{1 - e^{-Bm_1(\overline{x}/\overline{y})}} + m_1\frac{\overline{x}}{\overline{y}} = \frac{m_0}{1 - e^{-Bm_0(\overline{x}/\overline{y})}} + m_0\frac{\overline{x}}{\overline{y}}.$$
 (124)

If

$$f(x) = \frac{(\overline{y}/\overline{x})x}{1 - e^{-Bx}} + x, \quad x > 0, \ \overline{x} > 0, \ \overline{y} > 0,$$
(125)

then since

$$1 - e^{-Bx} - Bxe^{-Bx} > Bx - Bxe^{-Bx} = Bx\left(1 - e^{-Bx}\right)$$

> 0, (126)

we get

$$f'(x) = \frac{(\overline{y}/\overline{x})(1 - e^{-Bx} - Bxe^{-Bx})}{(1 - e^{-Bx})^2} + 1 > 0, \qquad (127)$$

which implies that f is an increasing function, and we take $m_1 = m_0$. Then from (121) it is obvious that $l_1 = l_0$. Since from statement (i) of Theorem 4 $(\overline{x}, \overline{y})$ is the unique positive equilibrium of (3), and thus $(x_n, y_n) \to (\overline{x}, \overline{y})$. This completes the proof of the theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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