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Research Article

Bifurcation of an Orbit Homoclinic to a Hyperbolic Saddle of a Vector Field in $\mathbb{R}^4$

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We perform a bifurcation analysis of an orbit homoclinic to a hyperbolic saddle of a vector field in $\mathbb{R}^4$. We give an expression of the gap between returning points in a transverse section by renormalizing system, through which we find the existence of homoclinic-doubling bifurcation in the case $1+\alpha>\beta>\nu$. Meanwhile, after reparametrizing the parameter, a periodic-doubling bifurcation appears and may be close to a saddle-node bifurcation, if the parameter is varied. These scenarios correspond to the occurrence of chaos. Based on our analysis, bifurcation diagrams of these bifurcations are depicted.

1. Introduction and Problems

Homoclinic orbits are crucial to know dynamics of differential systems in many application fields. For example, the famous FitzHugh-Nagumo equations, given by PDEs (see [1]), describe how action potentials in neurons occur and spread

$$\begin{aligned} u_t &= u_x x - f_a(u) - w, \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{1}$$

where $f_a(u) = u(u-a)(u-1)$. Through the variable transforming $\zeta = x + ct$, system (1) is then in an ODE form:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= cv + f_a(u) + w, \\ \dot{w} &= \frac{\varepsilon}{c}(u - \gamma w). \end{aligned} \tag{2}$$

It has an orbit homoclinic to the equilibrium $(u, v, w) = 0$ which corresponds to a solitary wave $(u, w)(x, t) = (u, w)(\zeta)$ of system (1). The authors detected how homoclinic branch converted a 1-homoclinic orbit to a N-homoclinic orbit.

In [2], a reversible water wave model was studied:

$$\frac{2}{15}v^{iv} - bv'' + av + \frac{3}{2}v^2 - \frac{1}{2}(v')^2 + [vv']' = 0. \tag{3}$$

The system admits a flip orbit

$$r(t) = 3\left(b + \frac{1}{2}\right)\operatorname{sech}^2\left(t\sqrt{\frac{3}{4}(2b+1)}\right), \tag{4}$$

for $b > 2$, $a > 0$, and shows the existence of the N-homoclinic orbit in some circumstances on two sides of the flip bifurcation.

In fact, the homoclinic-doubling bifurcation, which switches a 2^{n-1}-homoclinic orbit to a 2^n-homoclinic orbit, exists extensively in systems with flips; see [3–6] and the references therein. A simple and analytic model permitting these flips was initially given by Sandstede in a three-dimensional system in [7]. From then on, more and more excellent work has been done based on the model (see, e.g., [8, 9]). Now researchers even extend these flips phenomena to heterodimensional cycles and homoclinic bellows to study periodic orbits and homoclinic orbits; see [10–12]. But none of them aimed to investigate the homoclinic-doubling bifurcations. So in this paper we focus on the homoclinic-doubling problem for a kind of homoclinic flips.

Throughout the paper, we consider the following ODE system:

$$\dot{x} = f(x, \xi), \quad (x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^l, \tag{5}$$

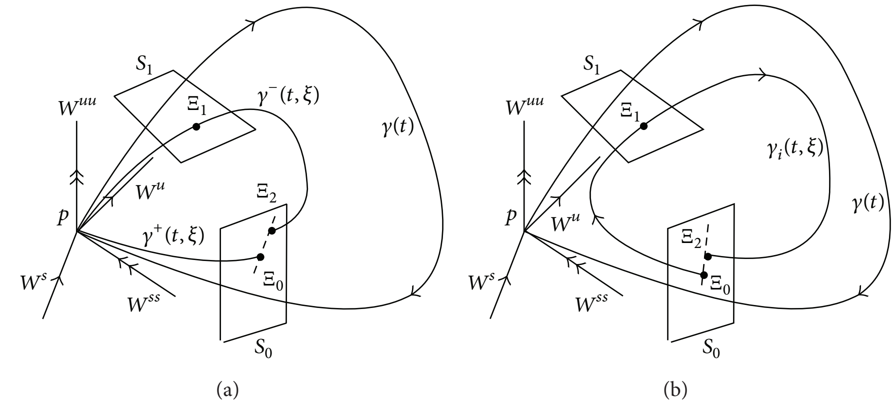

Figure 1: The gap $\|\gamma^-(T,\xi)-\gamma^+(T,\xi)\| \neq 0$ in (a); there is no homoclinic orbit in general. The gap $\|\gamma_i(T+T_i+\tau_i,\xi)-\gamma_i(T,\xi)\| \neq 0$ in (b); there is no periodic orbit in general.

where f is sufficiently smooth and $l \geq 4$. Suppose that the system (5) has an orbit $\gamma(t)$ of codimension-1 homoclinic to a saddle equilibrium p at $\xi = 0$. Let T be a certain time, such that $\gamma(T)$ and $\gamma(-T)$ are in some small neighborhood U of p. Then we can take two sections vertical to $T_{\gamma(\pm T)}$:

$$S_0 : \{x \mid x = x(T)\} \subset U; \qquad S_1 : \{x \mid x = x(-T)\} \subset U. \tag{6}$$

Generally, if the small parameter $\xi \neq 0$, the homoclinic orbit $\gamma(t)$ will not exist. But the system (5) must have solutions $\gamma^{\pm}(t,\xi)$ with the properties

$$\begin{gathered}\dot{\gamma}^{\pm} = f(\gamma^{\pm},\xi),\\ \gamma^{+}(t,\xi) \in W^{s}(p), \quad \gamma^{-}(t,\xi) \in W^{u}(p),\\ \gamma^{\pm}(t,0) = \gamma(t),\\ \gamma^{+}(T,\xi) \in S_0, \quad \gamma^{-}(-T,\xi) \in S_1,\\ \left\|\gamma^{-}(T,\xi) - \gamma^{+}(T,\xi)\right\| \ll 1,\end{gathered} \tag{7}$$

where $W^s(p)$ and $W^u(p)$ are the stable and unstable manifolds of the equilibrium p, and $\dim(T_{\gamma(t)}W^s \cap T_{\gamma(t)}W^u) = 1$. Notice that if the gap $\|\gamma^-(T,\xi)-\gamma^+(T,\xi)\| = 0$ in the transverse section S_0, it means that the homoclinic orbit is kept (see Figure 1(a)) but it may not be of codimension-1.

Moreover, the system (5) still has other solutions $\gamma_i(t,\xi)$; i is a natural number. Set the time of the orbit $\gamma_i(t,\xi)$ from S_0 to S_1 and from S_1 to S_0 to be τ_i and T_i, respectively; there are

$$\begin{gathered}\dot{\gamma}_i = f(\gamma_i,\xi),\\ \gamma_i(T+T_i+\tau_i,\xi), \quad \gamma_i(T,\xi) \in S_0,\\ \left\|\gamma_i(T+T_i+\tau_i,\xi) - \gamma_i(T,\xi)\right\| \ll 1.\end{gathered} \tag{8}$$

Actually $\gamma_i(t,\xi)$ is a regular orbit and will be periodic if the gap $\|\gamma_i(T+T_i+\tau_i,\xi)-\gamma_i(T,\xi)\| = 0$; namely, the orbit starting in S_0 will return to S_0 after the time $T_i+\tau_i$; see Figure 1(b).

From above, we see that the gap in the transverse section S_0 of some orbits is crucial to study bifurcations of the system. So in the next section we try to quantitate the gap size.

2. Main Method

To well carry out our discussion, we give some hypotheses for the system (5) here.

(A_1) The spectrum $\sigma(D_x f(p,\xi)) = \{\lambda_1(\xi), \lambda_2(\xi), -\rho_1(\xi), -\rho_2(\xi)\}$, and $\lambda_2(\xi) > \lambda_1(\xi) > 0 > -\rho_1(\xi) > -\rho_2(\xi)$.

(A_2) As $t \to +\infty$, the homoclinic orbit $\gamma(t) \to p$ along the strong stable manifold $W^{ss}(p)$.

(A_3) Vectors in the strong unstable (resp., stable) manifold W^{uu} (resp., W^{ss}) return to the saddle p in the direction along W^u (resp., W^s).

We know that the discontinuity of the functions $\gamma^{\pm}(t,\xi)$ or $\gamma_i(t,\xi)$ is confined in a special position in S_0. Since $\dim(T_{\gamma(t)}W^s \cap T_{\gamma(t)}W^u) = 1$, the space

$$\left(T_{\gamma(t)}W^s \cap T_{\gamma(t)}W^u\right)^{\perp} = \operatorname{span}\{\varphi_1\} \tag{9}$$

is of one dimension, where φ_1 can be taken as the solution of the linear variational system

$$\dot{y} = D_x f(\gamma(t),0)\, y, \tag{10}$$

and $\varphi_1(T) = (0,\omega_{12},1,0)$, $\varphi_1(-T) = (\omega_{11},0,\omega_{13},\omega_{14})$ based on the assumptions of (A_2) and (A_3); refer to [13] for the details.

Beside this, by the theories of matrix, the other three solutions denoted by φ_2, φ_3, and φ_4 of the system (10) can also be taken in the following ways:

$$\begin{gathered}\varphi_2 = \frac{-\dot{\gamma}(t)}{|\dot{\gamma}(T)|} \in T_{\gamma(t)}W^u \cap T_{\gamma(t)}W^s,\\ \varphi_3 \in T_{\gamma(t)}W^u, \quad \varphi_4 \in T_{\gamma(t)}W^s,\end{gathered} \tag{11}$$

satisfying

$$\begin{gathered}\varphi_2(-T) = (\omega_{21},0,0,0), \qquad \varphi_2(T) = (0,0,0,1),\\ \varphi_3(-T) = (0,0,1,0), \qquad \varphi_3(T) = (\omega_{31},\omega_{32},0,\omega_{34}),\\ \varphi_4(-T) = (\omega_{41},\omega_{42},\omega_{43},\omega_{44}), \qquad \varphi_4(T) = (0,1,0,0).\end{gathered} \tag{12}$$

Now take a transformation

$$x(t) = \gamma(t) + \Phi(t)\,\Xi = \gamma(t) + \varphi_1(t)\,\chi_1 + \varphi_3(t)\,\chi_3 + \varphi_4(t)\,\chi_4, \tag{13}$$

where $\Phi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t))$ and $\Xi = (\chi_1, 0, \chi_3, \chi_4)^*$. Then system (5) becomes the following ODE in the new variable Ξ; namely,

$$\dot{\Xi} = \Phi^{-1} D_\xi f(\gamma(t), 0)\,\xi + \Phi^{-1} D^2_{x\xi} f(\gamma(t), 0)\,\Phi\Xi\xi + O\left(|\Phi|\,|\Xi|^2\right) + O\left(|\Phi|^{-1}|\xi|^2\right). \tag{14}$$

By (14),

$$\int_{-T}^{T} \dot{\Xi}\,\mathrm{d}t = \int_{-T}^{T} \Phi^{-1} D_\xi f(\gamma(t), 0)\,\xi\,\mathrm{d}t + \text{h.o.t.} \tag{15}$$

gives

$$\Xi(T) = \Xi(-T) + M\xi + \text{h.o.t.}, \tag{16}$$

where $M = (M_1, 0, M_3, M_4)^* = \int_{-T}^{T} \Phi^{-1} D_\xi f(\gamma(t), 0)\mathrm{d}t$.

Notice that in (13), Ξ represents in some meaning the deviation in the normal direction of the manifolds $T_{\gamma(t)}W^s \cap T_{\gamma(t)}W^u$, so $\Xi(-T) \in S_1$ and $\Xi(T) \in S_0$; in other words, (16) maps a point in S_1 to a point in S_0.

On the other hand, from assumption (A_1), system (5) admits a local linearization

$$D_x f(p, \xi) = \lambda_1(\xi)\,x_1\frac{\partial}{\partial x_1} - \rho_1(\xi)\,x_2\frac{\partial}{\partial x_2} + \lambda_2(\xi)\,x_3\frac{\partial}{\partial x_3} - \rho_2(\xi)\,x_4\frac{\partial}{\partial x_4}, \tag{17}$$

where $x = (x_1, x_2, x_3, x_4)$.

Suppose $\Xi(T) = \Xi_0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ and $\Xi(-T) = \Xi_1 = (x_1^1, x_2^1, x_3^1, x_4^1)$. Rescale the time $s = e^{-\lambda_1(\xi)\tau}$ and write $\alpha = \rho_1(\xi)/\lambda_1(\xi)$, $\beta = \rho_2(\xi)/\lambda_1(\xi)$, and $\nu = \lambda_2(\xi)/\lambda_1(\xi)$. Without loss of generality, we assume $\alpha \geq 1$ for sufficiently small ξ. Otherwise, we can set $s = e^{-\rho_1(\xi)\tau}$ and $\alpha = \lambda_1(\xi)/\rho_1(\xi)$. Clearly, $\beta > \alpha \geq 1$.

Then by the linear approximation solutions of system (17), we have thereby

$$\begin{aligned} x &= e^{\lambda_1(\xi)\tau}x_1^0\frac{\partial x}{\partial x_1} + e^{-\rho_1(\xi)\tau}x_2^0\frac{\partial x}{\partial x_2} + e^{\lambda_2(\xi)\tau}x_3^0\frac{\partial x}{\partial x_3} + e^{-\rho_2(\xi)\tau}x_4^0\frac{\partial x}{\partial x_4} \\ &= s^{-1}x_1^0\frac{\partial x}{\partial x_1} + s^{\alpha}x_2^0\frac{\partial x}{\partial x_2} + s^{-\nu}x_3^0\frac{\partial x}{\partial x_3} + s^{\beta}x_4^0\frac{\partial x}{\partial x_4}. \end{aligned} \tag{18}$$

Formula (18) indeed maps a point in S_0 to a point in S_1 in some subset of U if we substitute x by Ξ_1.

Now take $\Xi_0 \in S_0$ as the initial point. System (5) must have an orbit $\gamma(t, \xi)$ starting at Ξ_0, passing through S_1 with an intersection Ξ_1, and finally returning to S_0 at some point Ξ_2; see Figure 1. From (13), (16), and (18), we can derive $\Xi_2 - \Xi_0 = (\chi_1^2, 0, \chi_3^2, \chi_4^2) - (\chi_1^0, 0, \chi_3^0, \chi_4^0) \triangleq (\varepsilon_1(s,\xi), 0, \varepsilon_3(s,\xi), \varepsilon_4(s,\xi))$, where

$$\begin{aligned} \omega_{14}\varepsilon_1(s,\xi) &= \delta s^{\beta} - \omega_{14}x_3^1 s^{\nu} - \omega_{44}\omega_{42}^{-1}s^{\alpha}x_2^0 + \omega_{14}M_1\xi + \text{h.o.t.}, \\ \varepsilon_3(s,\xi) &= x_3^1 - \omega_{31}^{-1}\delta s + \left(\omega_{13}\omega_{44}\omega_{14}^{-1} - \omega_{43}\right)\omega_{42}^{-1}s^{\alpha}x_2^0 + M_3\xi + \text{h.o.t.}, \\ \varepsilon_4(s,\xi) &= \omega_{42}^{-1}s^{\alpha}x_2^0 - x_2^0 + \omega_{12}s^{\nu}x_3^1 + \omega_{32}\omega_{31}^{-1}\delta s + M_4\xi + \text{h.o.t.} \end{aligned} \tag{19}$$

Denote $a = \omega_{44}w_{42}^{-1}$, $b = \omega_{32}w_{31}^{-1}$, and $\omega = \omega_{14}$. Then the gap between the points Ξ_2 and Ξ_0 can be represented by (refer to [13–15])

$$\begin{aligned} &\gamma(2T + \tau, \xi) - \gamma(T, \xi) \\ &\quad = \varepsilon_\gamma(s,\xi) \\ &\quad = -aM_4\xi s^{\alpha} + \delta s^{\beta} + \omega M_3\xi s^{\nu} - ab\delta s^{1+\alpha} + \omega M_1\xi + O\left(s^{1+\nu}\right). \end{aligned} \tag{20}$$

Obviously, $\varepsilon_\gamma(s,\xi) = 0$ means that system (5) has closed orbits.

3. Saddle-Node Bifurcations

From this section, we analyze bifurcation construction of system (5). Firstly, set $r = s^{\alpha}$. Then $\varepsilon_\gamma(r,\xi) = 0$ equals

$$aM_4\xi r - \delta r^{\beta/\alpha} - \omega M_3\xi r^{\nu/\alpha} + ab\delta r^{1+(1/\alpha)} - \omega M_1\xi + O\left(r^{(1+\nu)/\alpha}\right) = 0 \tag{21}$$

or

$$r = \frac{\delta}{aM_4\xi}r^{\beta/\alpha} + \frac{\omega M_3\xi}{aM_4\xi}r^{\nu/\alpha} - \frac{ab\delta}{aM_4\xi}r^{1+(1/\alpha)} + \frac{\omega M_1\xi}{aM_4\xi} + \frac{1}{aM_4\xi}O\left(r^{(1+\nu)/\alpha}\right). \tag{22}$$

Define $S(\xi) = \omega M_1\xi/aM_4\xi$. When $\|S(\xi)\| = O(\|M_4\xi\|^{\alpha/(\beta-\alpha)})$, $aM_4\xi r$ is the leading term in (21), so (21) has a small solution $r = S(\xi) + \text{h.o.t.} > 0$; but as $\|S(\xi)\| \gg \|M_4\xi\|^{\alpha/(\beta-\alpha)}$, $r = (-\delta^{-1}\omega M_1\xi)^{\alpha/\beta} + \text{h.o.t.} > 0$. No matter which case, a periodic orbit of system (5) exists.

Theorem 1. *Under (A_1)–(A_3) and for $1 + \alpha > \beta$, system (5) has a 1-periodic orbit.*

To look for saddle-node bifurcations of 1-periodic orbits, it is enough to differentiate (22) with respect to r. Consider

$$1 = \frac{\beta\delta}{\alpha aM_4\xi}r^{(\beta/\alpha)-1} + \frac{\nu\omega M_3\xi}{\alpha aM_4\xi}r^{(\nu/\alpha)-1} + \frac{1}{aM_4\xi}O\left(r^{1/\alpha}\right). \tag{23}$$

Solving (23) for r, there is

$$r = \left(\frac{\alpha a M_4 \xi}{\beta\delta}\right)^{\alpha/(\beta-\alpha)} + O\left(\|M_3\xi\|^{\alpha/(\beta-\alpha)}\|M_4\xi\|^{\alpha(\nu-\alpha)/(\beta-\alpha)^2}\right). \tag{24}$$

Then substituting (24) into (22), an asymptotic expression for a saddle-node bifurcation is given by

$$S(\xi) = \frac{\beta-\alpha}{\beta}\left(\frac{\alpha a M_4\xi}{\beta\delta}\right)^{\alpha/(\beta-\alpha)} + O\left(\|M_3\xi\|\,\|M_4\xi\|^{(\nu-\beta+\alpha)/(\beta-\alpha)}\right). \tag{25}$$

Furthermore, if we continue to differentiate (23), there is

$$0 = \frac{(\beta-\alpha)\beta\delta}{\alpha^2 a M_4\xi} r^{(\beta/\alpha)-2} + \frac{(\nu-\alpha)\nu\omega M_3\xi}{\alpha^2 a M_4\xi} r^{(\nu/\alpha)-2} + \frac{1}{aM_4\xi} O\left(r^{(1-\alpha)/\alpha}\right). \tag{26}$$

Equation (26) is solvable for $\beta > \nu$ with

$$r = \left(-\frac{(\nu-\alpha)\nu\omega M_3\xi}{(\beta-\alpha)\delta\beta}\right)^{\alpha/(\beta-\nu)} + O\left(\|M_3\xi\|^{\alpha(1-\alpha)/(\beta-\nu)^2}\right). \tag{27}$$

This is a triple solution of (22). It means that a saddle-node bifurcation of a triple 1-periodic orbit exists. The asymptotic expression can be derived from (22) and (23):

$$S(\xi) = \left(-\frac{(\nu-\alpha)\nu\omega M_3\xi}{(\beta-\alpha)\delta\beta}\right)^{\alpha/(\beta-\nu)} + O\left(\|M_3\xi\|^{\alpha/(\beta-\nu)}\right), \tag{28}$$

where

$$\omega M_3\xi = -\frac{(\beta-\alpha)\delta\beta}{(\nu-\alpha)\nu}\left(\frac{\alpha(\nu-\alpha)aM_4\xi}{\delta\beta(\nu-\beta)}\right)^{(\beta-\nu)/(\beta-\alpha)} + o\left(\|M_4\xi\|^{(\beta-\nu)/(\beta-\alpha)}\right). \tag{29}$$

Theorem 2. *Under (A_1)–(A_3) and for $1+\alpha > \beta$, system (5) has a saddle-node bifurcation SN of a double 1-periodic orbit given by (25) in the parameter space; moreover, for $\beta > \nu$, system (5) has a saddle-node bifurcation SN^2 of a triple 1-periodic orbit given by (28).*

Remark 3. For the case $\beta < \nu$, (26) has no sufficiently small positive solution, so there does not exist n-multiple 1-periodic orbit bifurcation for $n \geq 3$.

Now we define a surface in the parameter space of ξ:

$$H_1(\xi) = \{\xi : M_1\xi + o(1) = 0\}. \tag{30}$$

On the surface H_1, (21) equals

$$r\left(aM_4\xi - \delta r^{(\beta/\alpha)-1} - \omega M_3\xi r^{(\nu/\alpha)-1} + ab\delta r^{1/\alpha} + O\left(r^{((1+\nu)/\alpha)-1}\right)\right) = 0. \tag{31}$$

Clearly, it has a zero solution $r_1 = 0$. If we differentiate the part in the parentheses in (31) for r, we get

$$-\frac{\beta-\alpha}{\alpha}\delta r^{(\beta/\alpha)-2} - \frac{\nu-\alpha}{\alpha}\omega M_3\xi r^{(\nu/\alpha)-2} + O\left(r^{(1/\alpha)-1}\right) = 0. \tag{32}$$

It has a solution for $\beta > \nu$:

$$r_2 = \left(-\frac{\nu-\alpha}{(\beta-\alpha)\delta}\omega M_3\xi\right)^{\alpha/(\beta-\nu)} + O\left(\|M_3\xi\|^{\alpha(1+\alpha-\nu)/(\beta-\nu)^2}\right). \tag{33}$$

Then we obtain another saddle-node bifurcation similarly:

$$R(\xi) = \frac{\beta-\nu}{\beta-\alpha}\left(-\frac{\nu-\alpha}{(\beta-\alpha)\delta}\omega M_3\xi\right)^{(\nu-\alpha)/(\beta-\nu)} + O\left(\|M_3\xi\|^{(1+\nu-\beta)/(\beta-\nu)}\right), \tag{34}$$

where $R(\xi) = aM_4\xi/\omega M_3\xi$.

Notice that (32) has no solution for $\beta < \nu$. But from (31), a small positive solution in the form $r_2' = (\delta^{-1}aM_4\xi)^{\alpha/(\beta-\alpha)} + O(\|M_3\xi\|^{\alpha/(\beta-\alpha)}\|M_4\xi\|^{\alpha(\nu-\alpha)/(\beta-\alpha)^2})$ exists.

So we can conclude the following.

Theorem 4. *Under (A_1)–(A_3) and for $1+\alpha > \beta > \nu$, system (5) has a homoclinic-saddle-node bifurcation HSN of a 1-homoclinic orbit and a double 1-periodic orbit confined on $H_1 \cap R(\xi)$ while, for $\beta < \nu$, system (5) has only a 1-homoclinic orbit and a 1-periodic orbit in the parameter space and the 1-homoclinic orbit is of codimension-1.*

Remark 5. In Theorem 4, the 1-homoclinic orbit may be nongeneral, that is, may be a flip orbit, because the orbit can connect the saddle along the weak unstable and strong stable directions if $M_4\xi = 0$.

4. Homoclinic-Doubling and Periodic-Doubling Bifurcations

Now we focus on 2-homoclinic orbits and 2-periodic orbits. Correspondingly, the gap functions are

$$\begin{aligned} &r_1 - \frac{\delta}{aM_4\xi}r_1^{\beta/\alpha} - \frac{\omega M_3\xi}{aM_4\xi}r_2^{\nu/\alpha} + \frac{ab\delta}{aM_4\xi}r_1^{1+(1/\alpha)} - S(\xi) + \frac{1}{aM_4\xi}O\left(r_1^{1/\alpha}r_2^{\nu/\alpha}\right) = 0,\\ &r_2 - \frac{\delta}{aM_4\xi}r_2^{\beta/\alpha} - \frac{\omega M_3\xi}{aM_4\xi}r_1^{\nu/\alpha} + \frac{ab\delta}{aM_4\xi}r_2^{1+(1/\alpha)} - S(\xi) + \frac{1}{aM_4\xi}O\left(r_2^{1/\alpha}r_1^{\nu/\alpha}\right) = 0. \end{aligned} \tag{35}$$

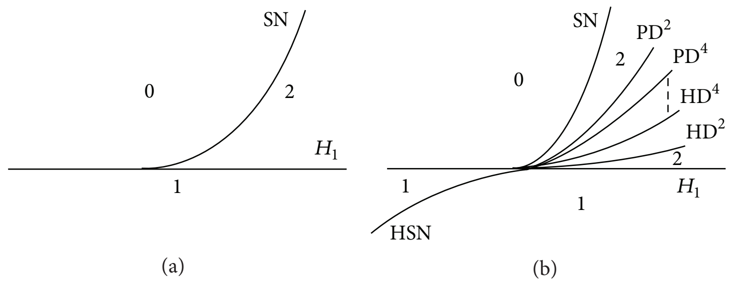

FIGURE 2: Bifurcation surfaces for $1+\alpha>\beta$, $\nu>\beta$ in (a) and for $1+\alpha>\beta>\nu$ in (b). 0 means no periodic orbits and n means n periodic orbits. Chaos occurs in the region bounded by HD^{2^n} and PD^{2^n}.

To find a 2-homoclinic orbit, the above two equations must have a kind of solutions with $r_1=0$ and $r_2>0$. That is,

$$\begin{aligned} &r_2^{\nu/\alpha}+\frac{\omega M_1\xi}{\omega M_3\xi}+\frac{1}{\omega M_3\xi}O\left(r_2^{(\nu+\beta)/\alpha}\right)=0, \\ &r_2-\frac{\delta}{aM_4\xi}r_2^{\beta/\alpha}+\frac{ab\delta}{aM_4\xi}r_2^{1+(1/\alpha)}-S(\xi) \\ &\quad+\frac{1}{aM_4\xi}O\left(r_2^{(1+\beta)/\alpha}\right)=0. \end{aligned} \tag{36}$$

Then we get

$$\begin{aligned} S(\xi)=&\left(-\frac{M_1\xi}{M_3\xi}\right)^{\alpha/\nu}-\frac{\delta}{aM_4\xi}\left(-\frac{M_1\xi}{M_3\xi}\right)^{\beta/\nu} \\ &+\frac{1}{aM_4\xi}O\left(\left\|\frac{M_1\xi}{M_3\xi}\right\|^{(1+\alpha)/\nu}\right), \end{aligned} \tag{37}$$

in the region defined by $\|M_1\xi\|\ll\|M_3\xi\|^{\beta/(\beta-\nu)}$.

To find a 2-periodic orbit, the gap functions will have two positive solutions r_1 and r_2. We suppose that $r_2=(1+\epsilon)r_1$ after the reparametrization $\xi=(\xi_1,\epsilon)$. Then there are

$$\begin{aligned} &r_1-\frac{\delta}{aM_4\xi}r_1^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}(1+\epsilon)^{\nu/\alpha}r_1^{\nu/\alpha}-S(\xi) \\ &\quad+\frac{1}{aM_4\xi}O\left(r_1^{(1+\alpha)/\alpha}\right)=0, \\ &(1+\epsilon)r_1-\frac{\delta}{aM_4\xi}(1+\epsilon)^{\beta/\alpha}r_1^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}r_1^{\nu/\alpha} \\ &\quad-S(\xi)+\frac{1}{aM_4\xi}O\left((1+\epsilon)^{(1+\alpha)/\alpha}r_1^{(1+\alpha)/\alpha}\right)=0. \end{aligned} \tag{38}$$

Subtracting the two equations, there is

$$\begin{aligned} r_1=&\left(\frac{\delta^{-1}\epsilon}{(1+\epsilon)^{\beta/\alpha}-1}\right)^{\alpha/(\beta-\alpha)}\left(aM_4\xi\right)^{\alpha/(\beta-\alpha)} \\ &+O\left(\|M_3\xi\|^{\alpha/(\beta-\alpha)}\|\epsilon\|\,r_1^{(\nu-\alpha)/(\beta-\alpha)}\right). \end{aligned} \tag{39}$$

Finally, we get the 2-periodic orbit bifurcation:

$$\begin{aligned} S(\xi)=&\left(1-\frac{\epsilon}{(1+\epsilon)^{\beta/\alpha}-1}\right)\left(\frac{\delta^{-1}\epsilon}{(1+\epsilon)^{\beta/\alpha}-1}\right)^{\alpha/(\beta-\alpha)} \\ &\times\left(aM_4\xi\right)^{\alpha/(\beta-\alpha)}+O\left(\|M_3\xi\|\,\|M_4\xi\|^{(\alpha+\nu-\beta)/(\beta-\alpha)}\right). \end{aligned} \tag{40}$$

Remark 6. Obviously, if $r_1=r_2$, the 2-periodic orbit is close to the double 1-periodic orbit perturbed from the saddle-node bifurcation. This is true by taking limit $\epsilon\to0$ in (40), and one may get the similar approximate expression given in (25).

If we continue the computation, we can finally get an asymptotic expression of the homoclinic-doubling bifurcation of 2^n-homoclinic orbit and the periodic-doubling bifurcation of 2^n-periodic orbit with the same leading terms as in (37) and (40), respectively. For example, for a 4-homoclinic orbit or a 4-periodic orbit, the gap functions are

$$\begin{aligned} &r_1-\frac{\delta}{aM_4\xi}r_1^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}r_2^{\nu/\alpha}+\frac{ab\delta}{aM_4\xi}r_1^{1+(1/\alpha)} \\ &\quad-S(\xi)+\frac{1}{aM_4\xi}o(1)=0, \\ &r_2-\frac{\delta}{aM_4\xi}r_2^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}r_3^{\nu/\alpha}+\frac{ab\delta}{aM_4\xi}r_2^{1+(1/\alpha)} \\ &\quad-S(\xi)+\frac{1}{aM_4\xi}o(1)=0, \\ &r_3-\frac{\delta}{aM_4\xi}r_3^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}r_4^{\nu/\alpha}+\frac{ab\delta}{aM_4\xi}r_3^{1+(1/\alpha)} \\ &\quad-S(\xi)+\frac{1}{aM_4\xi}o(1)=0, \\ &r_4-\frac{\delta}{aM_4\xi}r_4^{\beta/\alpha}-\frac{\omega M_3\xi}{aM_4\xi}r_1^{\nu/\alpha}+\frac{ab\delta}{aM_4\xi}r_4^{1+(1/\alpha)} \\ &\quad-S(\xi)+\frac{1}{aM_4\xi}o(1)=0. \end{aligned} \tag{41}$$

We need only to consider solutions $r_1=0$ and $r_i>0$, $i=2,3,4$, for the 4-homoclinic orbit or all the positive solutions for 4-periodic orbit. For concision, we omit the details here.

Now we can claim our last theorem.

Theorem 7. *Under* (A_1)–(A_3) *and for* $1 + \alpha > \beta > \nu$, *system* (5) *has a homoclinic-doubling bifurcation* HD^{2^n} *of* 2^n-*homoclinic orbit and a periodic-doubling bifurcation* PD^{2^n} *of* 2^n-*periodic orbit defined by* (37) *and* (40), *respectively, in the parameter region* $\|M_1\xi\| \ll \|M_3\xi\|^{\beta/(\beta-\nu)}$.

From the above analysis, one may see that all of these bifurcation surfaces have the same order $S(\xi) = O(\|M_4\xi\|^{\alpha/(\beta-\alpha)})$ except HSN and are tangent to H_1. To be clear, we illustrate these bifurcation surfaces in the parameter plane in Figure 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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