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Research Article

Analytical Solutions for Systems of Singular Partial Differential-Algebraic Equations

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This paper proposes power series method (PSM) in order to find solutions for singular partial differential-algebraic equations (SPDAEs). We will solve three examples to show that PSM method can be used to search for analytical solutions of SPDAEs. What is more, we will see that, in some cases, Padé posttreatment, besides enlarging the domain of convergence, may be employed in order to get the exact solution from the truncated series solutions of PSM.

1. Introduction

The importance of research on partial differential-algebraic equations (PDAEs) is that they are used in the mathematical modeling of many phenomena, both practical and theoretical. These systems arise, for example, in nanoelectronics, electrical networks, and mechanical systems, among many others. Despite the importance of this topic, it may be considered relatively new and little known.

Although the case of constant-coefficient linear PDAEs has been investigated by means of numerical methods, for instance, in [1, 2], perhaps the more relevant aspect of PDAEs, both linear and nonlinear, is the concept of index. The differentiation index is defined as the minimum number of times that all or part of the PDAEs must be differentiated with respect to time, in order to obtain the time derivative of the solution, as a continuous function of the solution and its space derivatives [3]. A fact that justifies the search for other methods of solution to these equations is that the solutions of higher index PDAEs (index greater than one) become very complicated, even for numerical methods, and many application problems lead to PDAEs with different indices. A further difficulty to be considered that arises and affects also other kinds of systems of differential equations, as well as differential equations, is the presence of singularities, which are related to points at which some terms of the differential equations become infinite or undefined.

In recent years, several methods focused on approximating nonlinear and linear problems, as an alternative to classical methods, have been reported, such as those based on variational approaches [4–7], tanh method [8], exp-function [9, 10], Adomian's decomposition method [11–16], parameter expansion [17], homotopy perturbation method [7, 16, 18–46], homotopy analysis method [47], homotopy asymptotic method [48], series method [49, 50], and perturbation method [51–54], among many others. Also, a few exact solutions to nonlinear differential equations have been reported occasionally [55].

This study shows that power series method (PSM) [56, 57] is able to address the above difficulties to obtain power series solutions for singular partial differential-algebraic equations (SPDAEs), that is, PDAEs with singular points. These systems turn out to be difficult even for numerical methods. More generally, we will see that the combination of PSM and Padé posttreatment could be effective to improve the PSM's truncated series solutions in convergence rate; what is more, sometimes it ends up giving the exact solution of the system, such as what will happen in our third case study.

This paper is organized as follows. In Section 2, we introduce the basic idea of power series method. Section 3 provides a brief explanation of application of PSM to solve SPDAEs. Section 4 presents three case studies: one singular nonlinear index-one system, one singular linear index-two system, and one singular nonlinear index-two system. Besides, a discussion on the results is presented in Section 5. Finally, a brief conclusion is given in Section 6.

2. Basic Concept of Power Series Method

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(t) = 0, \quad t \in \Omega, \tag{1}$$

with the following boundary condition:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad t \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(t)$ is a known analytical function, and Γ is the domain boundary for Ω.

PSM [49, 50] assumes that the solution of a differential equation can be written in the following form:

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \tag{3}$$

where $u_0, u_1, \ldots$ are unknown functions to be determined by series method.

The method of solution for differential equations can be summarized as follows.

(1) Equation (3) is substituted into (1), and then we regroup the resulting polynomial equation in terms of powers of t.

(2) We equate each coefficient of the above-mentioned polynomial to zero.

(3) As a consequence, a linear algebraic system for the unknowns of (3) is obtained.

(4) To conclude, the solution of the above system allows obtaining the coefficients $u_0, u_1, \ldots$.

3. Application of PSM to Solve PDAE Systems

Since many applications problems in science and engineering are often modeled by semiexplicit PDAEs, we consider therefore the following class of PDAEs:

$$u_{1t} = \phi(u, u_x, u_{xx}), \tag{4}$$

$$0 = \psi(u, u_x, u_{xx}), \quad (t, x) \in (0, T) \times (a, b), \tag{5}$$

where $u_k : [0, T] \times [a, b] \rightarrow R^{m_k}$, $k = 1, 2$, and $b > a$; in other words $u = (u_1, u_2)$.

For clarification, the method is described for the general system (4)-(5) where the number of unknowns is given by $m_1 + m_2$. In this notation, u_1 (differential unknown) has m_1 components and u_2 (algebraic unknown) has m_2 components. In fact m_1 and m_2 can take any values greater than or equal to one, so that the number of unknowns in (4)-(5) is greater than or equal to 2.

System (4)-(5) is subject to the initial condition

$$u_1(0, x) = g(x), \quad a \le x \le b, \tag{6}$$

and some suitable boundary conditions

$$B(u(t,a), u(t,b), u_x(t,a), u_x(t,b)) = 0, \quad 0 \le t \le T, \tag{7}$$

where $g(x)$ is a given function.

We assume that the solution to initial boundary-value problem (4)–(7) exists and is unique and sufficiently smooth.

To simplify the exposition of the PSM, we integrate first (4) with respect to t and use the initial condition (6) to obtain

$$u_1(t, x) - g(x) - \int_0^t \phi(u, u_x, u_{xx})\, dt = 0. \tag{8}$$

It is important to note that the time integration of (4) is not relevant to the solution procedure presented here, so one can apply the PSM directly to (4).

A fact that justifies the use of PSM is that, in general terms, getting solutions for PDAEs becomes very complicated, even for numerical methods. Moreover, there are not systematic analytical or numerical methods to solve these problems.

In view of PSM, we assume the solution components $u_k(t, x)$, $k = 1, 2$, have the form

$$u_k(t, x) = u_{k,0}(x) + u_{k,1}(x)\, t + u_{k,2}(x)\, t^2 + \cdots, \tag{9}$$

where $u_{k,n}(x)$, $k = 1, 2$, $n = 0, 1, 2, \ldots$, are unknown functions to be determined later on by the PSM.

Then substitute (9) into system (4)-(5) and equate the coefficients of powers of t in the resulting equation to zero, to obtain an algebraic linear system for the coefficients, whose solution is employed in (9), with the end of obtaining a solution for (4)–(7) in series form. These series may have limited regions of convergence, even if we take a large number of terms. Therefore, in some cases, it will be convenient to apply the Padé resummation method to PSM truncated series to enlarge the convergence region as depicted in the next section. A relevant fact is that the steps outlined in this section will be also sufficient to obtain satisfactory solutions for the most difficult case of SPDAEs.

4. Case Studies

The objective of this section is employing PSM, in order to solve three SPDAE systems.

Our results will show the efficiency of the presented method.

4.1. Nonlinear Index-One SPDAE (following Section 3, $m_1 = 1$ and $m_2 = 1$). Consider the following:

$$u_{1t} - u_{1xx} + u_1 u_{1x} + \frac{u_2}{x} = x^3 - 6xt + 3t^2x^5 + \frac{t^4}{x^2}, \quad (10)$$

$$u_1 + u_2 = tx^3 + \frac{t^4}{x}, \quad t > 0, \quad (11)$$

subject to the initial conditions

$$u_1(0, x) = 0. \quad (12)$$

In order to apply PSM, we integrate (10) with respect to t and use the initial condition (12) to obtain

$$u_1(t, x) = \int_0^t \left[u_{1xx} - u_1 u_{1x} - \frac{u_2}{x} + x^3 - 6xt + 3t^2x^5 + \frac{t^4}{x^2} \right] dt. \quad (13)$$

PSM assumes that $u(t, x)$ and $v(t, x)$ can be written as

$$u_1(t, x) = u_{10}(x) + u_{11}(x)t + u_{12}(x)t^2 + \cdots, \quad (14)$$

$$u_2(t, x) = u_{20}(x) + u_{21}(x)t + u_{22}(x)t^2 + \cdots, \quad (15)$$

where $u_{10}(x), u_{11}(x), u_{12}(x), u_{20}(x), u_{21}(x), u_{22}(x), \ldots$ are unknown functions.

This case study is simplified, substituting (14) and (15) into (11), to get

$$\sum_{n=0}^{\infty} u_{2n}t^n = \frac{t^4}{x} + tx^3 - \sum_{n=0}^{\infty} u_{1n}t^n. \quad (16)$$

On the other hand, substituting (14) through (16) into (13) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} u_{1n}t^n = \int_0^t \Bigg[& \sum_{n=0}^{\infty} u''_{1n}t^n - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} u_{1n}u'_{1m}t^{n+m} \\ & - \frac{1}{x}\left(\frac{t^4}{x} + tx^3 - \sum_{n=0}^{\infty} u_{1n}t^n \right) \\ & + x^3 - 6xt + 3t^2x^5 + \frac{t^4}{x^2} \Bigg] dt. \end{aligned} \quad (17)$$

From here on, the dash notation in u' denotes the ordinary derivative with respect to x.

Integrating the above result, it is obtained that

$$\begin{aligned} \sum_{n=0}^{\infty} u_{1n}t^n = & \sum_{n=0}^{\infty} u''_{1n}\frac{t^{n+1}}{n+1} - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} u_{1n}u'_{1m}\frac{t^{n+m+1}}{n+m+1} \\ & - \frac{x^2t^2}{2} + x^3t - 3xt^2 + x^5t^3 + \frac{1}{x}\sum_{n=0}^{\infty}\frac{u_{1n}t^{n+1}}{n+1}. \end{aligned} \quad (18)$$

Standardizing the summation index and grouping, we get the recursive formula

$$\begin{aligned} & u_{10}t^0 - x^3t + \left(3x + \frac{x^2}{2}\right)t^2 - x^5t^3 \\ & + \sum_{k=1}^{\infty}\left[u_{1k} - \frac{u''_{1k-1}}{k} + \sum_{m=0}^{\infty}\frac{u_{1k-m-1}u'_{1m}}{k} - \frac{u_{1k-1}}{kx} \right] t^k = 0. \end{aligned} \quad (19)$$

Equating the coefficients of powers of t to zero in (19), we obtain

$$\begin{aligned} & k = 0, \\ & u_0 = 0, \\ & k = 1, \\ & u_{11} = u''_{10} - u'_{10}u_{10} + \frac{1}{x}u_{10} + x^3; \end{aligned} \quad (20)$$

after employing (20), it is obtained that

$$\begin{aligned} & u_{11} = x^3, \\ & k = 2, \\ & u_{12} = \frac{u''_{11}}{2} - \frac{u'_{10}u_{11}}{2} - \frac{u'_{11}u_{10}}{2} + \frac{u_{11}}{2x} - \frac{x^2}{2} - 3x; \end{aligned} \quad (21)$$

substituting (20) and (21) in the above equation, it is obtained that

$$\begin{aligned} & u_{12} = 0, \\ & k = 3, \\ & u_{13} = \frac{u''_{12}}{3} - \frac{u'_{10}u_{12}}{3} - \frac{u'_{11}u_{11}}{3} - \frac{u'_{12}u_{10}}{3} + \frac{u_{12}}{3x} + x^5; \end{aligned} \quad (22)$$

after substituting (20), (21), and (22) in the last equation, we get

$$\begin{aligned} & u_{13} = 0, \\ & k = 4, \\ & u_{14} = \frac{u''_{13}}{4} - \frac{u'_{10}u_{13}}{4} - \frac{u'_{11}u_{12}}{4} - \frac{u'_{12}u_{11}}{4} - \frac{u'_{13}u_{10}}{4} + \frac{u_{13}}{4x}; \end{aligned} \quad (23)$$

after employing (20), (21), (22), and (23), we get

$$\begin{aligned} & u_{14} = 0, \\ & k = 5, \\ & u_{15} = \frac{u''_{14}}{5} - \frac{u'_{10}u_{14}}{5} - \frac{u'_{11}u_{13}}{5} - \frac{u'_{12}u_{12}}{5} \\ & \qquad - \frac{u'_{13}u_{11}}{5} - \frac{u'_{14}u_{10}}{5} + \frac{u_{14}}{5x}; \end{aligned} \quad (24)$$

the substitution of (20), (21), (22), (23), and (24) leads to

$$u_{15} = 0; \tag{25}$$

in the same way we obtain

$$u_{16} = u_{17} = u_{18} = \cdots = 0. \tag{26}$$

Substituting (20) through (26) into (14) leads us to

$$u_1(t, x) = x^3 t. \tag{27}$$

Finally, substituting (27) into (11) leads to

$$u_2(t, x) = \frac{t^4}{x}. \tag{28}$$

Thus, (27) and (28) are the exact solution for SPDAE system (10)–(12).

4.2. Linear Index-Two SPDAE with Variable Coefficients ($m_1 = 2$, $m_2 = 1$). Consider the following:

$$u_{1t} = x^2 u_{1xx} - 3u_1 + u_3 + \frac{x^2}{1+t}, \tag{29}$$

$$u_{2t} = x^2 u_{2xx} - 3u_2 + u_3 + \frac{x^2}{1+t}, \tag{30}$$

$$0 = u_1 + u_2 - 2x^2 \ln(1+t), \tag{31}$$

subject to the initial conditions

$$\begin{aligned} &u_1(0, x) = 0, \qquad u_2(0, x) = 0, \\ &-1 < t \le 1, \qquad -\infty < x < \infty. \end{aligned} \tag{32}$$

The integration of (29) and (30) with respect to t and using the initial conditions (32) lead to

$$u_1(t, x) = \int_0^t \left[x^2 u_{1xx} - 3u_1 + u_3\right] dt + x^2 \ln(1+t), \tag{33}$$

$$u_2(t, x) = \int_0^t \left[x^2 u_{2xx} - 3u_2 + u_3\right] dt + x^2 \ln(1+t), \tag{34}$$

assuming that $u_1(t, x)$, $u_2(t, x)$, and $u_3(t, x)$ can be written as

$$u_1(t, x) = u_{10}(x) + u_{11}(x) t + u_{12}(x) t^2 + \cdots, \tag{35}$$

$$u_2(t, x) = u_{20}(x) + u_{21}(x) t + u_{22}(x) t^2 + \cdots, \tag{36}$$

$$u_3(t, x) = u_{30}(x) + u_{31}(x) t + u_{32}(x) t^2 + \cdots, \tag{37}$$

where $u_{10}(x), u_{11}(x), \ldots, u_{20}(x), u_{21}(x), \ldots, u_{30}(x), u_{31}(x), \ldots$ are unknown functions to be determined later on by the PSM method.

After substituting (35) and (37) into (33), we get

$$\begin{aligned} u_{10} t^0 + \sum_{k=1}^{\infty} \frac{1}{k} \Big[k u_{1k} - x^2 u''_{1k-1} + 3u_{1k-1} \\ - u_{3k-1} - x^2 (-1)^{k-1}\Big] t^k = 0, \end{aligned} \tag{38}$$

where we have standardized the summation index and employed the following Taylor series expansion:

$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n, \quad -1 < t \le 1. \tag{39}$$

In the same way, the substitution of (36) and (37) into (34) leads to

$$\begin{aligned} u_{20} t^0 + \sum_{k=1}^{\infty} \frac{1}{k} \Big[k u_{2k} - x^2 u''_{2k-1} + 3u_{2k-1} \\ - u_{3k-1} - x^2 (-1)^{k-1}\Big] t^k = 0. \end{aligned} \tag{40}$$

On the other hand, after substituting (35), (36), and (39) into (31), we have

$$\sum_{k=1}^{\infty} \left[u_{1k} + u_{2k} - \frac{2x^2}{k} (-1)^{k-1}\right] t^k = 0, \tag{41}$$

where we have employed the following results, deduced from (38) and (40):

$$u_{10} = u_{20} = 0. \tag{42}$$

Equations (38), (40), and (41) give rise to the following formulas:

$$u_{1n} = \frac{x^2 u''_{1n-1} - 3u_{1n-1} + u_{3n-1} + (-1)^{n-1} x^2}{n}, \quad n \ge 1, \tag{43}$$

$$u_{2n} = \frac{x^2 u''_{2n-1} - 3u_{2n-1} + u_{3n-1} + (-1)^{n-1} x^2}{n}, \quad n \ge 1, \tag{44}$$

$$u_{1n} + u_{2n} = \frac{2x^2 (-1)^{n-1}}{n}, \quad n \ge 1. \tag{45}$$

Combining the result of adding (43) and (44), with (45), we obtain

$$u_{3n-1} = -\frac{1}{2}\left(u''_{1n-1} + u''_{2n-1}\right) x^2 + \frac{3}{2}\left(u_{1n-1} + u_{2n-1}\right), \quad n \ge 1. \tag{46}$$

The substitution of (46) into (43) and (44), respectively, leads us to

$$\begin{aligned} u_{1n} &= \frac{1}{2n}\Big(x^2 u''_{1n-1} - 3u_{1n-1} + 3u_{2n-1} \\ &\qquad - x^2 u''_{2n-1} + 2(-1)^{n-1} x^2\Big), \quad n \ge 1, \\ u_{2n} &= \frac{1}{2n}\Big(x^2 u''_{2n-1} - 3u_{2n-1} + 3u_{1n-1} \\ &\qquad - x^2 u''_{1n-1} + 2(-1)^{n-1} x^2\Big), \quad n \ge 1. \end{aligned} \tag{47}$$

From recursion formulas (46) and (47), we get the functions

$$u_{10}(x)=0,\qquad u_{11}(x)=x^2,\qquad u_{12}(x)=\frac{-x^2}{2},$$
$$u_{13}=\frac{x^2}{3},\qquad u_{14}=\frac{-x^2}{4}\cdots,\tag{48}$$

$$u_{20}(x)=0,\qquad u_{21}(x)=x^2,\qquad u_{22}(x)=\frac{-x^2}{2},$$
$$u_{23}=\frac{x^2}{3},\qquad u_{24}=\frac{-x^2}{4}\cdots,\tag{49}$$

$$u_{30}(x)=0,\qquad u_{31}(x)=x^2,\qquad u_{32}(x)=\frac{-x^2}{2},$$
$$u_{33}=\frac{x^2}{3},\qquad u_{34}=\frac{-x^2}{4}\cdots.\tag{50}$$

After substituting (48) through (50) into series (35), (36), and (37), respectively, we get

$$u_1(t,x)=x^2\left[t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right],\tag{51}$$

$$u_2(t,x)=x^2\left[t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right],\tag{52}$$

$$u_3(t,x)=x^2\left[t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right].\tag{53}$$

After identifying the nth terms of the series (51), (52), and (53) as $((-1)^{n-1}/n)t^n$, we conclude that

$$\begin{aligned}u_1(t,x)&=x^2\ln(1+t),\\ u_2(t,x)&=x^2\ln(1+t),\\ u_3(t,x)&=x^2\ln(1+t)\end{aligned}\tag{54}$$

which is the exact solution of (29)–(32) (see (39)).

4.3. Nonlinear Index-Two SPDAE with Variable Coefficients ($m_1=2$, $m_2=1$). Finally, consider the following:

$$u_{1t}=f(x)u_{1xx}+u_1u_{1x}-\frac{1-t}{1+t}u_3,\tag{55}$$

$$u_{2t}=g(x)u_{2xx}-u_2u_{2x}+\frac{1+t}{1-t}u_3,\tag{56}$$

$$0=u_1(1+t)-u_2(1-t),\quad -\infty<x<\infty,\ -1<t<1,\tag{57}$$

subject to the initial conditions

$$u_1(0,x)=x,\qquad u_2(0,x)=x,\qquad u_3(0,x)=2x,\tag{58}$$

where $f(x)$ and $g(x)$ are analytical functions on $-\infty<x<\infty$.

The integration of (55) and (56) with respect to t and using the initial conditions (58) lead to

$$u_1(t,x)=x+\int_0^t\left[f(x)u_{1xx}+u_1u_{1x}-\frac{1-t}{1+t}u_3\right]dt,\tag{59}$$

$$u_2(t,x)=x+\int_0^t\left[g(x)u_{2xx}-u_2u_{2x}+\frac{1+t}{1-t}u_3\right]dt.\tag{60}$$

PSM assumes once again that $u_1(t,x)$, $u_2(t,x)$, and $u_3(t,x)$ can be written as

$$u_1(t,x)=u_{10}(x)+u_{11}(x)t+u_{12}(x)t^2+\cdots,\tag{61}$$

$$u_2(t,x)=u_{20}(x)+u_{21}(x)t+u_{22}(x)t^2+\cdots,\tag{62}$$

$$u_3(t,x)=u_{30}(x)+u_{31}(x)t+u_{32}(x)t^2+\cdots,\tag{63}$$

where $u_{10}(x),u_{11}(x),\ldots,u_{20}(x),u_{21}(x),\ldots,u_{30}(x),u_{31}(x),\ldots$ are unknown functions to be determined later on by the PSM method.

Substituting (61) and (63) into (59) and also (62) and (63) into (60), respectively, we get

$$\begin{aligned}\sum_{n=0}^{\infty}u_{1n}t^n&=x+\int_0^t f(x)\sum_{n=0}^{\infty}u_{1n}''t^n dt+\int_0^t\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{1n}u_{1m}'t^{n+m}dt\\ &\quad-\int_0^t(1-t)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n u_{3m}t^{n+m}dt,\end{aligned}\tag{64}$$

$$\begin{aligned}\sum_{n=0}^{\infty}u_{2n}t^n&=x+\int_0^t g(x)\sum_{n=0}^{\infty}u_{2n}''t^n dt+\int_0^t\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{2n}u_{2m}'t^{n+m}dt\\ &\quad-\int_0^t(1+t)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{3m}t^{n+m}dt,\end{aligned}\tag{65}$$

where we have employed the Taylor series expansions

$$\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n,\qquad \frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n.\tag{66}$$

After integrating and standardizing the summation index, we get the following recursion formulas, from (64) and (65), respectively:

$$\begin{aligned}&-u_{10}+x-u_{30}t-\frac{1}{2}(u_{31}-2u_{30})t^2\\ &-\frac{1}{3}(u_{32}-2u_{31}+2u_{30})t^3\\ &-\frac{1}{4}(u_{33}-2u_{32}+2u_{31}-2u_{30})t^4\\ &+\sum_{k=1}^{\infty}\left[\frac{f(x)u_{1k-1}''}{k}+\sum_{m=0}^{\infty}\frac{u_{1m}'u_{1k-m-1}}{k}-u_{1k}\right]t^k=0,\end{aligned}$$

$$\begin{aligned}&-u_{20}+x+u_{30}t+\frac{1}{2}(u_{31}+2u_{30})t^2\\ &+\frac{1}{3}(u_{32}+2u_{31}+2u_{30})t^3\end{aligned}$$

$$+\frac{1}{4}\left(u_{33}+2u_{32}+2u_{31}+2u_{30}\right)t^{4}+\sum_{k=1}^{\infty}\left[\frac{g(x)u_{2k-1}''}{k}-\sum_{m=0}^{\infty}\frac{u_{2m}'u_{2k-m-1}}{k}-u_{2k}\right]t^{k}=0. \tag{67}$$

From (57) we obtain

$$\sum_{m=0}^{\infty}u_{2m}t^{m}=(1+t)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}t^{n+j}u_{1j}, \tag{68}$$

after using again the first series of (66).

After standardizing the summation index, we get a third recurrence formula from (68):

$$u_{2k}=\sum_{n=0}^{\infty}\left[u_{1k-n}+u_{1k-n-1}\right],\quad \text{where } k=0,1,2,3,\ldots. \tag{69}$$

From recursion formulas (67) and (69), we get the following coupled equations:

$$u_{10}=u_{10}(0,x), \tag{70}$$

$$u_{11}=f(x)u_{10}''+u_{10}'u_{10}-u_{30}, \tag{71}$$

$$u_{12}=f(x)\frac{u_{11}''}{2}+\frac{u_{10}'u_{11}+u_{11}'u_{10}}{2}+\frac{u_{31}}{2}, \tag{72}$$

$$u_{13}=f(x)\frac{u_{12}''}{3}+\frac{u_{10}'u_{12}+u_{11}'u_{11}+u_{12}'u_{10}}{3}-\frac{u_{32}+2u_{30}-2u_{31}}{3}, \tag{73}$$

$$u_{14}=\frac{u_{13}''}{4}+\frac{u_{10}'u_{13}+u_{11}'u_{12}+u_{12}'u_{11}+u_{13}'u_{10}}{4}-\frac{u_{33}-2u_{32}+2u_{31}-2u_{30}}{4}, \tag{74}$$

$$\vdots$$

$$u_{20}=u_{20}(0,x), \tag{75}$$

$$u_{21}=g(x)u_{20}''-u_{20}'u_{20}+u_{30}, \tag{76}$$

$$u_{22}=g(x)\frac{u_{21}''}{2}-\frac{u_{20}'u_{21}+u_{21}'u_{20}}{2}+\frac{u_{31}+2u_{30}}{2}, \tag{77}$$

$$u_{23}=g(x)\frac{u_{22}''}{3}-\frac{u_{20}'u_{22}+u_{21}'u_{21}+u_{22}'u_{20}}{3}+\frac{u_{32}+2u_{30}+2u_{31}}{3}, \tag{78}$$

$$u_{24}=g(x)\frac{u_{23}''}{4}-\frac{u_{20}'u_{23}+u_{21}'u_{22}+u_{22}'u_{21}+u_{23}'u_{20}}{4}+\frac{u_{33}+2u_{32}+2u_{31}+2u_{30}}{4}, \tag{79}$$

$$\vdots$$

$$u_{20}=u_{10}, \tag{80}$$

$$u_{21}=2u_{10}+u_{11}, \tag{81}$$

$$u_{22}=u_{12}+2u_{11}+2u_{10}, \tag{82}$$

$$u_{23}=u_{13}+2u_{12}+2u_{11}+2u_{10}, \tag{83}$$

$$u_{24}=u_{14}+2u_{13}+2u_{12}+2u_{11}+2u_{10}, \tag{84}$$

$$\vdots$$

From (70) through (84), we get the functions

$$u_{10}=x,\quad u_{11}=-x,\quad u_{12}=x,\quad u_{13}=-x,\quad u_{14}=x\cdots, \tag{85}$$

$$u_{20}=x,\quad u_{21}=x,\quad u_{22}=x,\quad u_{23}=x,\quad u_{24}=x\cdots, \tag{86}$$

$$u_{30}=2x,\quad u_{31}=0,\quad u_{32}=2x,\quad u_{33}=0,\quad u_{34}=2x. \tag{87}$$

Substituting (85) through (87) into series (61), (62), and (63), respectively, we get

$$u_1(t,x)=x\left(1-t+t^2-t^3+t^4+\cdots\right), \tag{88}$$

$$u_2(t,x)=x\left(1+t+t^2+t^3+t^4+\cdots\right), \tag{89}$$

$$u_3(t,x)=2x\left(1+t^2+t^4+t^6+\cdots\right). \tag{90}$$

After identifying the nth terms of the above series as $(-1)^n t^n$, t^n, and t^{2n}, respectively, we conclude that series (88) through (90) admit the following closed forms:

$$u_1(t,x)=\frac{x}{1+t},\quad u_2(t,x)=\frac{x}{1-t},\quad u_3(t,x)=\frac{2x}{1-t^2}, \tag{91}$$

which is the exact solution of (55)–(58), where we have employed (66) and

$$\frac{1}{1-t^2}=\sum_{n=0}^{\infty}t^{2n}. \tag{92}$$

This case admits an alternative way to obtain the closed solution (91) by using Padé posttreatment [58, 59]. In general terms, Padé technology is employed, in order to obtain solutions for differential equations, handier and computationally more efficient. Also, it is employed to improve the convergence of truncated series. As a matter of fact, the application of Padé [2/2] to series (88)–(90) leads to the exact solution (91).

5. Discussion

In this study we presented the power series method (PSM) as a useful tool in the search for analytical solutions for singular partial differential-algebraic equations (SPDAEs). To this end, two SPDAE problems of index-two and another of index-one were solved by this technique, leading (for these cases) to the exact solutions. For each of the cases studied, PSM essentially transformed the SPDAE into an easily solvable algebraic system for the coefficient functions of the proposed power series solution.

Since not all the SPDAEs have exact solutions, it is possible that, in some cases, the series solution obtained from PSM may have limited regions of convergence, even taking a large number of terms; our case study three suggests the use of a Padé posttreatment, as a possibility to improve the domain of convergence for the PSM's truncated series. In fact, the mentioned example showed that, sometimes, Padé approximant leads to the exact solution. It should be mentioned that Laplace-Padé resummation is another known method, employed in the literature [53] to enlarge the domain of convergence of solutions or is inclusive to find exact solutions. This technique, which combines Laplace transform and Padé posttreatment, may be used in the future research of SPDAEs.

One of the important features of our method is that the high complexity of SPDAE problems was effectively handled by this method. This is clear if one notes that our examples were chosen to include higher-order-index PDAEs (differentiation index greater than one), linear and nonlinear cases, even with variable coefficients. In addition the last example proposed the case of a system of equations containing two functions entirely arbitrary. The above makes this system completely inaccessible to numerical methods; also we add singularities, which gave rise to the name of SPDAEs.

Finally, the fact that there are not any standard analytical or numerical methods to solve higher-index SPDAEs converts the PSM method into an attractive tool to solve such problems.

6. Conclusion

By solving the three examples, we presented PSM as a handy and useful tool, with high potential to find analytical solutions to SPDAEs. Since, on one hand, we proposed the way to improve the solutions obtained by this method if necessary and, on the other hand, it is based on a straightforward procedure, our proposal will be useful for practical applications and suitable for engineers and scientists. Finally, further research should be conducted to solve other SPDAEs systems, above all of index greater than one, combining PSM and Laplace-Padé resummation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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