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Research Article

New Results on Reachable Set Estimation and Controller Design for Linear Systems with Mixed Delays via Triple Integral Functionals

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This paper is concerned with the problem of reachable set estimation and controller design for a class of linear systems with mixed delays and bounded disturbance inputs. By constructing a newly augmented Lyapunov-Krasovskii functional, combining with triple integral technique and reciprocally convex approach, some new reachable set estimation conditions are derived in terms of linear matrix inequalities. Based on these conditions, a state-feedback controller has been designed to ensure the reachable set of the closed-loop system bounded by a given ellipsoid. Finally, two numerical examples are given to illustrate the effectiveness of the proposed analysis and designed method.

1. Introduction

For a dynamic system, the reachable set is defined as the set of all the states starting from the origin by inputs with peak values. The problem of reachable set bounding was first considered in the late 1960s in the context of state estimation. Owing to its extensive applications in peak-to-peak gain minimization, parameter estimation, and control systems with actuator saturation [1–4], a large number of previous works have been devoted to the study of reachable set estimation and its related areas [5–17]. It is well known that time delays are frequently encountered in various practical systems such as biological and engineering systems. Their existence may lead to poor performance, oscillation, or even instability. Therefore, the problem of stability analysis and control synthesis for systems with time delays have attracted remarkable attention of researchers; lots of related results have been achieved in the literature [18–30].

Recently, the reachable set of systems with time-varying delay have been investigated by many researchers. In [3], Fridam and Shaked derived delay-dependent conditions for an ellipsoid that contains the reachable set for a linear system with time-varying delay and bound peak input via Lyapunov-Razumikhin approach. In [6], Nam and Pathirana proposed an improved condition by using an enhanced Lyapunov-Krasovskii functional method and the delay decomposition technique. In [7], Zuo et al. proposed the maximal Lyapunov-Krasovskii functional approach to construct a pointwise maximum of a family of Lyapunov functionals to obtain reachable set conditions for polytopic systems with time-varying delay. In [11], the authors studied the problem of state bounding for discrete-time systems with time-varying delay and bounded disturbance inputs; delay-dependent conditions are obtained in terms of matrix inequalities by using delay decomposition technique and reciprocally convex approach. New explicit delay-independent conditions in terms of the Metzler matrix have been derived in [12] by using a novel way which does not involve the Lyapunov-Krasovskii functional method. In [13], Feng and Lam firstly studied the problem of reachable set estimation of singular systems. Moreover, the authors focused on reachable set bounding for linear systems with discrete and distributed delays in [15, 16]. On the other hand, the reachable set synthesis problem is an important issue because the state of a system should be restricted within a safety area to make the system operation safe. More recently, in [17], Feng and Lam investigated the problem of reachable set estimation and synthesis of time-delay system by introducing the nonuniform delay-partitioning method and the triple integral technique. To the best of the authors' knowledge, the reachable set synthesis problem for linear systems with both discrete and distributed delays has not been considered so far.

In this paper, we study the problem of reachable set estimation and controller design for a class of linear systems with mixed time delays and bounded disturbance inputs. Firstly, by choosing an improved Lyapunov-Krasovskii functional, based on triple integral technique and reciprocally convex approach, new reachable set estimation conditions are derived in terms of linear matrix inequality. Then, a sufficient condition for the existence of a state-feedback controller is designed to render the reachable set of the closed-loop system to be bounded. Finally, some numerical examples are given to illustrate the effectiveness and less conservatism of the proposed method.

Notations. Throughout this paper, the superscripts M^T stand for the transpose of the matrix M; $P > 0$($P \geqslant 0$, $P < 0$, and $P \leqslant 0$) means that the matrix P is symmetric positive definite (positive-semidefinite, negative definite, and negative-semidefinite); $\|\cdot\|$ refers to the Euclidean vector norm; $R^{m\times n}$ is the set of $m\times n$ real matrices; $*$ denotes the symmetric block in symmetric matrix; $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ denote, respectively, the maximal and minimal eigenvalue of matrix Q.

2. Problem Statement and Preliminaries

Consider the following linear systems with discrete and distributed delays:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h x(t-h(t)) + A_d\int_{t-d}^{t} x(s)\,ds \\ &\quad + Bu(t) + D\omega(t), \\ x(t) &= 0, \quad t \in [-\max\{h,d\},0],\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ is state vector of the system, $u(t)$ is the control input, A, A_h, A_d, B, and D are constant matrices with appropriate dimensions, and $h(t)$ and d are, respectively, time-varying discrete delay and time invariant distributed delay with

$$\begin{aligned}0 \le h(t) \le h, \\ \dot{h}(t) \le \mu.\end{aligned} \tag{2}$$

$\omega(t) \in R^l$ represents a disturbance which satisfies

$$\omega^T(t)\,\omega(t) \le \omega_m^2. \tag{3}$$

The reachable set of system (1) is denoted as follows:

$$\mathcal{R}_x = \{x(t) \mid x(t), \omega(t) \text{ satisfy } (1)-(3),\ t \ge 0\}. \tag{4}$$

An ellipsoid used to bound the reachable set of the system in (1) is given in the form of

$$\varepsilon(P,1) = \{x(t) \in R^n \mid x^T(t)\,Px(t) \le 1,\ P > 0\}. \tag{5}$$

One of the aims in this paper is to derive an ellipsoid to bound the reachable set $\mathcal{R}_x$ of system (1) with $u(t) = 0$ and the other is to design a state-feedback controller K; that is, $u(t) = Kx(t)$, such that the reachable set of closed-loop system

$$\begin{aligned}\dot{x}(t) &= (A+BK)\,x(t) + A_h x(t-h(t)) \\ &\quad + A_d\int_{t-d}^{t} x(s)\,ds + D\omega(t), \quad t \ge 0,\end{aligned} \tag{6}$$

is bounded by a given ellipsoid $\varepsilon(P,1)$.

In addition, we give some lemmas which will be used in deriving our results.

Lemma 1 (see [18]). *Let $f_1, f_2, \ldots, f_n : R^m \mapsto R$ have positive values in an open subset D and R^m; then, the reciprocally convex combination of f_i over D satisfies*

$$\begin{aligned}&\min_{\{\alpha_i \mid \alpha_i>0\ \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\ne j} g_{ij}(t) \\ &\text{subject to } \left\{ g_{i,j} R^m \longrightarrow R,\ g_{j,i}(t) \right. \\ &\quad \left. = g_{i,j}(t),\ \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}.\end{aligned} \tag{7}$$

Lemma 2 (see [17]). *For any constant matrix $M > 0$, scalars $b > a > 0$, and vector function $w : [a,b] \to R^n$, then*

$$\begin{aligned}&-(b-a)\int_a^b w^T(s)\,Mw(s)\,ds \le -\left(\int_a^b w(s)\,ds\right)^T \\ &\quad \cdot M\left(\int_a^b w(s)\,ds\right), \\ &-\frac{(b-a)^2}{2}\int_{-b}^{-a}\int_{t+\theta}^{t-a} w^T(s)\,Mw(s)\,ds\,d\theta \\ &\quad \le -\left(\int_{-b}^{-a}\int_{t+\theta}^{t-a} w(s)\,ds\,d\theta\right)^T \\ &\quad \cdot M\left(\int_{-b}^{-a}\int_{t+\theta}^{t-a} w(s)\,ds\,d\theta\right).\end{aligned} \tag{8}$$

Lemma 3 (see [14]). *Let $V(x(t))$ be a Lyapunov function for linear system (1) with $V(x(0)) = 0$ and $\omega^T(t)\omega(t) \le \omega_m^2$. If*

$$\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{\omega_m^2}\omega^T(t)\,\omega(t) \le 0, \quad \alpha > 0, \tag{9}$$

then one has $V(x(t)) \le 1$.

3. Main Results

In this section, the reachable set estimation and synthesis of linear systems with both discrete and distributed delays are addressed by employing some novel approaches. Firstly, the reachable set estimation problem can be resolved in terms of matrix inequalities as follows.

Theorem 4. *If there exist a scalar $\alpha > 0$, positive definite matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $T_1 > 0$, $T_2 > 0$, $R > 0$, and $Z_1 > 0$, and any matrices $Z_2 > 0$, S, H_1, and H_2 with appropriate dimensions, such that the following inequalities hold*

$$\begin{bmatrix} R+Z_1 & S \\ * & R+Z_2 \end{bmatrix} > 0, \quad (10)$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & H_1^T A_d + e^{-\alpha d} d T_2 & \Omega_{17} & H_1^T D \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & 0 & A_h^T H_2 & 0 \\ * & * & \Omega_{33} & 0 & \Omega_{35} & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & A_d^T H_2 & 0 \\ * & * & * & * & * & * & \Omega_{77} & H_2^T D \\ * & * & * & * & * & * & * & -\dfrac{\alpha}{\omega_d^2} I \end{bmatrix} < 0, \quad (11)$$

where

$$\begin{aligned} \Omega_{11} &= Q_1 + Q_2 + H_1^T A + A^T H_1 + d^2 T_1 - e^{-\alpha h} R \\ &\quad - 2e^{-\alpha h} Z_1 - e^{-\alpha d} d^2 T_2 + \alpha P, \\ \Omega_{12} &= H_1^T A_h + e^{-\alpha h} R - e^{-\alpha h} S, \\ \Omega_{13} &= e^{-\alpha h} S, \\ \Omega_{14} &= 2e^{-\alpha h} Z_1, \\ \Omega_{17} &= P - H_1^T + A^T H_2, \\ \Omega_{22} &= -(1-\mu) e^{-\alpha h} - 2e^{-\alpha h} R + e^{-\alpha h} \left(S^T + S\right) \\ &\quad - 2e^{-\alpha h} Z_1 - 2e^{-\alpha h} Z_2, \\ \Omega_{23} &= -e^{-\alpha h} S + e^{-\alpha h} R, \\ \Omega_{24} &= 2e^{-\alpha h} Z_2, \\ \Omega_{25} &= 2e^{-\alpha h} Z_1, \\ \Omega_{33} &= -e^{-\alpha h} Q_2 - e^{-\alpha h} R - 2e^{-\alpha h} Z_2, \\ \Omega_{35} &= 2e^{-\alpha h} Z_2, \\ \Omega_{44} &= -2e^{-\alpha h} Z_1 - 2e^{-\alpha h} Z_2, \\ \Omega_{55} &= -2e^{-\alpha h} Z_1 - 2e^{-\alpha h} Z_2, \\ \Omega_{66} &= -T_1 - e^{-\alpha d} T_2, \\ \Omega_{77} &= h^2 R + \frac{h^2}{2} Z_1 + \frac{h^2}{2} Z_2 + \frac{d^4}{4} T_2 - H_2 - H_2^T, \end{aligned} \quad (12)$$

then the ellipsoid $\varepsilon(P, 1)$ is an estimation of the reachable set of system (1) with $u(t) = 0$.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = \sum_{i=1}^{6} V_i(t), \quad (13)$$

where

$$\begin{aligned} V_1(t) &= x^T(t) P x(t), \\ V_2(t) &= \int_{t-h}^{t} e^{\alpha(s-t)} x^T(s) Q_1 x(s)\, ds \\ &\quad + \int_{t-h(t)}^{t} e^{\alpha(s-t)} x^T(s) Q_2 x(s)\, ds, \\ V_3(t) &= d \int_{-d}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} x^T(s) T_1 x(s)\, ds\, d\theta \\ &\quad + \frac{d^2}{2} \int_{-d}^{0} \int_{\eta}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) T_2 \dot{x}(s)\, ds\, d\theta\, d\eta, \\ V_4(t) &= h \int_{-h}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) R \dot{x}(s)\, ds\, d\theta, \\ V_5(t) &= \int_{-h}^{0} \int_{\eta}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) Z_1 \dot{x}(s)\, ds\, d\theta\, d\eta, \\ V_6(t) &= \int_{-h}^{0} \int_{-h}^{\eta} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) Z_2 \dot{x}(s)\, ds\, d\theta\, d\eta. \end{aligned} \quad (14)$$

Now, calculating the derivative of $V_i(t)$, it yields that

$$\begin{aligned} \dot{V}_1(t) &= -\alpha V_1(t) + 2\dot{x}^T(t) P x(t) + \alpha x^T(t) P x(t), \\ \dot{V}_2(t) &= -\alpha V_2(t) + x^T(t) Q_1 x(t) + x^T(t) Q_2 x(t) \end{aligned}$$

$$
\begin{aligned}
&-e^{-\alpha h}x^{T}(t-h)Q_{1}x(t-h)\\
&-\left(1-\dot{h}(t)\right)e^{-\alpha h(t)}x^{T}(t-h(t))Q_{2}x^{T}(t-h(t))\\
\le&-\alpha V_{2}(t)+x^{T}(t)Q_{1}x(t)+x^{T}(t)Q_{2}x(t)\\
&-e^{-\alpha h}x^{T}(t-h)Q_{2}x(t-h)\\
&-(1-\mu)e^{-\alpha h}x^{T}(t-h(t))Q_{2}x^{T}(t-h(t)),
\end{aligned}
$$

$$
\begin{aligned}
&\dot{V}_{3}(t)\\
=&-\alpha V_{3}(t)+d^{2}x^{T}(t)T_{1}x(t)+\frac{d^{4}}{4}\dot{x}^{T}(t)T_{2}\dot{x}(t)\\
&-d\int_{t-d}^{t}e^{\alpha(s-t)}x^{T}(s)T_{1}x(s)\,ds\\
&-\frac{d^{2}}{2}\int_{-d}^{0}\int_{t+\theta}^{t}e^{\alpha(s-t)}\dot{x}^{T}(s)T_{2}\dot{x}(s)\,ds\,d\theta\\
\le&-\alpha V_{3}(t)+d^{2}x^{T}(t)T_{1}x(t)+\frac{d^{4}}{4}\dot{x}^{T}(t)T_{2}\dot{x}(t)\\
&-e^{-\alpha d}d\int_{t-d}^{t}x^{T}(s)T_{1}x(s)\,ds\\
&-e^{-\alpha d}\frac{d^{2}}{2}\int_{-d}^{0}\int_{t+\theta}^{t}\dot{x}^{T}(s)T_{2}\dot{x}(s)\,ds\,d\theta,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{V}_{4}(t)\\
=&-\alpha V_{4}(t)+h^{2}\dot{x}^{T}(t)R\dot{x}(t)\\
&-h\int_{t-h}^{t}e^{\alpha(s-t)}\dot{x}^{T}(s)R\dot{x}(s)\,ds\\
\le&-\alpha V_{4}(t)+h^{2}\dot{x}^{T}(t)R\dot{x}(t)\\
&-he^{-\alpha h}\int_{t-h(t)}^{t}\dot{x}^{T}(s)R\dot{x}(s)\,ds\\
&-he^{-\alpha h}\int_{t-h}^{t-h(t)}\dot{x}^{T}(s)R\dot{x}(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{V}_{5}(t)\\
=&-\alpha V_{5}(t)+\frac{1}{2}h^{2}\dot{x}^{T}(t)Z_{1}\dot{x}(t)\\
&-\int_{-h}^{0}\int_{t+\theta}^{t}e^{\alpha(s-t)}\dot{x}^{T}(s)Z_{1}\dot{x}(s)\,ds\,d\theta\\
\le&-\alpha V_{5}(t)+\frac{1}{2}h^{2}\dot{x}^{T}(t)Z_{1}\dot{x}(t)\\
&-e^{-\alpha h}\int_{-h(t)}^{0}\int_{t+\theta}^{t}\dot{x}^{T}(s)Z_{1}\dot{x}(s)\,ds\,d\theta\\
&-e^{-\alpha h}\int_{-h}^{-h(t)}\int_{t+\theta}^{t-h(t)}\dot{x}^{T}(s)Z_{1}\dot{x}(s)\,ds\,d\theta\\
&-e^{-\alpha h}(h-h(t))\int_{t-h(t)}^{t}\dot{x}(s)^{T}Z_{1}\dot{x}(s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{V}_{6}(t)\\
=&-\alpha V_{6}(t)+\frac{1}{2}h^{2}\dot{x}^{T}(t)Z_{2}\dot{x}(t)\\
&-\int_{-h}^{0}\int_{t-h}^{t+\theta}e^{\alpha(s-t)}\dot{x}^{T}(s)Z_{2}\dot{x}(s)\,ds\,d\theta\\
\le&-\alpha V_{6}(t)+\frac{1}{2}h^{2}\dot{x}^{T}(t)Z_{2}\dot{x}(t)\\
&-e^{-\alpha h}\int_{-h(t)}^{0}\int_{t-h(t)}^{t+\theta}\dot{x}^{T}(s)Z_{2}\dot{x}(s)\,ds\,d\theta\\
&-e^{-\alpha h}\int_{-h}^{-h(t)}\int_{t-h}^{t+\theta}\dot{x}^{T}(s)Z_{2}\dot{x}(s)\,ds\,d\theta\\
&-e^{-\alpha h}(h(t))\int_{t-h}^{t-h(t)}\dot{x}^{T}(s)Z_{2}\dot{x}(s)\,ds.
\end{aligned}
\tag{15}
$$

By using Lemma 2, one can obtain

$$
\begin{aligned}
\dot{V}_{3}(t)\le&-\alpha V_{3}(t)+d^{2}x^{T}(t)T_{1}x(t)+\frac{d^{4}}{4}\dot{x}^{T}(t)T_{2}\dot{x}(t)\\
&-e^{-\alpha d}\int_{t-d}^{t}x^{T}(s)\,ds\,T_{1}\int_{t-d}^{t}x(s)\,ds\\
&-e^{-\alpha d}\left[dx(t)-\int_{t-d}^{t}x(s)\,ds\right]^{T}\\
&\cdot T_{2}\left[dx(t)-\int_{t-d}^{t}x(s)\,ds\right],
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\dot{V}_{4}(t)\le&-\alpha V_{4}(t)+h^{2}\dot{x}^{T}(t)Rx(t)-e^{-\alpha h}\frac{h}{h(t)}\\
&\cdot\int_{t-h(t)}^{t}\dot{x}^{T}(s)\,ds\,R\int_{t-h(t)}^{t}\dot{x}(s)\,ds-e^{-\alpha h}\frac{h}{h-h(t)}\\
&\cdot\int_{t-h}^{t-h(t)}\dot{x}^{T}(s)\,ds\,R\int_{t-h}^{t-h(t)}\dot{x}^{T}(s)\,ds=-\alpha V_{4}(t)\\
&+h^{2}\dot{x}^{T}(t)Rx(t)-e^{-\alpha h}\frac{1}{\beta_{1}}\zeta_{1}^{T}R\zeta_{1}-e^{-\alpha h}\frac{1}{\beta_{2}}\zeta_{2}^{T}R\zeta_{2},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\dot{V}_{5}(t)\le&-\alpha V_{5}(t)+\frac{1}{2}h^{2}\dot{x}^{T}(t)Z_{1}\dot{x}(t)-e^{-\alpha h}\\
&\cdot\frac{h-h(t)}{h(t)}\left[\int_{t-h(t)}^{t}\dot{x}(s)\,ds\right]^{T}Z_{1}\int_{t-h(t)}^{t}\dot{x}(s)\,ds\\
&-2e^{-\alpha h_{2}}\frac{1}{h(t)^{2}}\left[\int_{-h(t)}^{0}\int_{t+\theta}^{t}\dot{x}(s)\,ds\,d\theta\right]^{T}\\
&\cdot Z_{1}\int_{-h(t)}^{0}\int_{t+\theta}^{t}\dot{x}(s)\,ds\,d\theta-2e^{-\alpha h_{2}}
\end{aligned}
$$

$$\begin{aligned}
&\cdot \frac{1}{(h-h(t))^2}\left[\int_{-h}^{0}\int_{t+\theta}^{t-h(t)}\dot{x}(s)\,ds\,d\theta\right]^T \\
&\cdot Z_1\int_{-h}^{0}\int_{t+\theta}^{t-h(t)}\dot{x}(s)\,ds\,d\theta = -\alpha V_5(t)+\frac{1}{2}h^2\dot{x}(t)^T \\
&\cdot Z_1\dot{x}(t) - e^{-\alpha h}\frac{\beta_2}{\beta_1}\zeta_1^T Z_1\zeta_2 \\
&-2e^{-\alpha h}\left[x(t)-\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right]^T \\
&\cdot Z_1\left[x(t)-\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds\right] \\
&-2e^{-\alpha h}\left[x(t-h(t))-\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds\right]^T \\
&\cdot Z_1\left[x(t-h(t))-\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds\right],
\end{aligned} \tag{18}$$

$$\begin{aligned}
\dot{V}_6(t) &\le -\alpha V_6(t)+\frac{1}{2}h^2\dot{x}(t)^T Z_2\dot{x}(t) - e^{-\alpha h} \\
&\cdot \frac{h(t)}{h-h(t)}\left[\int_{t-h}^{t-h(t)}\dot{x}(s)\,ds\right]^T Z_2\int_{t-h(t)}^{t}\dot{x}(s)\,ds \\
&-2e^{-\alpha h}\frac{1}{h(t)^2}\left[\int_{-h(t)}^{0}\int_{t-h(t)}^{t+\theta}\dot{x}(s)\,ds\,d\theta\right]^T \\
&\cdot Z_2\int_{-h(t)}^{0}\int_{t-h(t)}^{t+\theta}\dot{x}(s)\,ds\,d\theta - 2e^{-\alpha h} \\
&\cdot \frac{1}{(h-h(t))^2}\left[\int_{-h}^{-h(t)}\int_{t-h}^{t+\theta}\dot{x}(s)\,ds\,d\theta\right]^T \\
&\cdot Z_2\int_{-h}^{-h(t)}\int_{t-h}^{t+\theta}\dot{x}(s)\,ds\,d\theta = -\alpha V_6(t)+\frac{1}{2}h^2\dot{x}(t)^T \\
&\cdot Z_2\dot{x}(t) - e^{-\alpha h}\frac{\beta_1}{\beta_2}\zeta_2^T Z_2\zeta_2 \\
&-2e^{-\alpha h}\left[\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds - x(t-h(t))\right]^T \\
&\cdot Z_2\left[\frac{1}{h(t)}\int_{t-h(t)}^{t}x(s)\,ds - x(t-h(t))\right] \\
&-2e^{-\alpha h}\left[\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds - x(t-h)\right]^T \\
&\cdot Z_2\left[\frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}x(s)\,ds - x(t-h)\right],
\end{aligned} \tag{19}$$

where $\zeta_1 = [x(t) - x(t-h(t))]$, $\zeta_2 = [x(t-h(t)) - x(t-h)]$, $\beta_1 = h(t)/h$, $\beta_2 = (h-h(t))\ /h$, and $\beta_1 + \beta_2 = 1$.

From Lemma 1, there exists matrix S satisfying $\begin{bmatrix} R_2+Z_1 & S \\ * & R_2+Z_2 \end{bmatrix} > 0$, and then it holds that

$$\begin{aligned}
&-e^{-\alpha h}\frac{1}{\beta_1}\zeta_1^T R\zeta_1 - e^{-\alpha h}\frac{1}{\beta_2}\zeta_2^T R\zeta_2 - e^{-\alpha h}\frac{\beta_2}{\beta_1}\zeta_1^T Z_1\zeta_1 \\
&\quad - e^{-\alpha h}\frac{\beta_1}{\beta_2}\zeta_2^T Z_2\zeta_2 = -e^{-\alpha h}\frac{1}{\beta_1}\zeta_1^T R\zeta_1 - e^{-\alpha h}\frac{1}{\beta_2}\zeta_2^T R\zeta_2 \\
&\quad - e^{-\alpha h}\left(\frac{1}{\beta_1}-1\right)\zeta_1^T Z_1\zeta_1 - e^{-\alpha h}\left(\frac{1}{\beta_2}-1\right)\zeta_2^T Z_2\zeta_2 \\
&\quad \le -e^{-\alpha h}\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix}^T\begin{bmatrix} R & S \\ * & R\end{bmatrix}\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix}.
\end{aligned} \tag{20}$$

When $h(t) = 0$ or $h(t) = h$, inequality (20) still holds.

Using two free-weighing matrices H_1 and H_2, we have

$$\begin{aligned}
&2\left(x^T(t)H_1^T + \dot{x}^T(t)H_2^T\right)\Big[-\dot{x}(t) + Ax(t) \\
&\quad + A_h(x(t-h(t))) + A_d\int_{t-d}^{t}x(s)\,ds + D\omega(t)\Big] \\
&\quad = 0.
\end{aligned} \tag{21}$$

Considering (15)–(21), we obtain

$$\dot{V}(t) + \alpha V(t) - \frac{\alpha}{\omega_d^2}\omega^T(t)\,\omega(t) \le \xi^T(t)\,\Omega\xi(t) < 0, \tag{22}$$

where

$$\begin{aligned}
\xi(t) = \mathrm{col}\Big\{&x(t), x(t-h(t)), x(t-h), \frac{1}{h(t)} \\
&\cdot\int_{t-h(t)}^{t}x(s)\,ds, \frac{1}{h-h(t)} \\
&\cdot\int_{t-h}^{t-h(t)}x(s)\,ds, \int_{t-d}^{t}x(s)\,ds, \dot{x}(t), \omega(t)\Big\}.
\end{aligned} \tag{23}$$

By Lemma 3, we have $V(t) \le 1$, which implies that $x^T(t)Px(t) \le 1$. The proof is completed. □

Remark 5. Different from the method in [15, 16], we choose a new Lyapunov-Krasovskii functional containing three triple integral terms in the proof of Theorem 4; the metrics and effectiveness of the triple integral terms have been proved in terms of reducing the conservatism, which will be demonstrated by the following numerical examples.

Remark 6. To deal with the β_2/β_1- and β_1/β_2-dependent terms in (18) and (19), in [24], the authors have introduced an approximation as $-\beta_2/\beta_1 \le \beta_2$ and $-\beta_1/\beta_2 \le \beta_1$, which contains considerable conservatism. However, by using the relations $\beta_2/\beta_1 = -1+1/\beta_1$ and $\beta_1/\beta_2 = -1+1/\beta_2$, it can also be regarded as one of the reciprocally convex combinations, which can be treated by the simple variation of Lemma 1. The proposed method is effective in reducing the conservatism, which will be verified by numerical examples.

Remark 7. It should be noted that the inequality $\dot{V}(t)+\alpha V(t)-(\alpha/\omega_d^2)\omega^T(t)\omega(t) \leq \xi^T(t)\Omega\xi(t) \leq 0$ can be obtained from the conditions in Theorem 4; when $\omega(t) = 0$, we can get $\dot{V}(t) < 0$, for nonzero $x(t)$ due to the positive definiteness of $V(t)$ which guarantees the stability of system (1).

Remark 8. In order to obtain the smallest possible ellipsoid with the shortest major axis, the matrix P is considered as a decision variable and satisfied $0 < \delta I \leq P$ which is equivalent to

$$\begin{bmatrix} -\overline{\delta} I & I \\ I & -P \end{bmatrix} \leq 0, \tag{24}$$

where $\overline{\delta} = 1/\delta$; then the smallest possible ellipsoid can be obtained when $\overline{\delta}$ is minimum.

Next, based on Theorem 4, the synthesis result by using state-feedback control is given in the following theorem.

Theorem 9. *Given scalars β_1 and β_2, if there exist a scalar $\alpha > 0$, positive definite matrices $\overline{P} > 0$, $\overline{Q}_1 > 0$, $\overline{Q}_2 > 0$, $\overline{R} > 0$, $\overline{V} > 0$, and $\overline{Z}_1 > 0$, and any matrices $\overline{S}$, M, and G with appropriate dimensions, such that the following inequalities hold*

$$\begin{bmatrix} \overline{R}+\overline{Z}_1 & \overline{S} \\ * & \overline{R}+\overline{Z}_2 \end{bmatrix} > 0, \tag{25}$$

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & 0 & \beta_1 A_d M + e^{-\alpha d} d T_2 & \Phi_{17} & \beta_1 DM \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & 0 & \beta_2 M^T A_h^T & 0 \\ * & * & \Phi_{33} & 0 & \Phi_{35} & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & -\beta_2 M^T A_d^T & 0 \\ * & * & * & * & * & * & \Phi_{77} & \beta_2 DM \\ * & * & * & * & * & * & * & -\dfrac{\alpha}{\omega_d^2} I \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned}
\Phi_{11} &= \overline{Q}_1 + \overline{Q}_2 + \beta_1 M^T A^T + \beta_1 AM + \beta_1 G^T B^T \\
&\quad + \beta_1 BG + d^2 \overline{T}_1 - e^{-\alpha d} d^2 \overline{T}_2 - e^{-\alpha h} \overline{R} \\
&\quad - 2e^{-\alpha h} \overline{Z}_1 + \alpha P, \\
\Phi_{12} &= \beta_1 A_h M + e^{-\alpha h} \overline{R} - e^{-\alpha h} \overline{S}, \\
\Phi_{13} &= e^{-\alpha h} \overline{S}, \\
\Phi_{14} &= 2e^{-\alpha h} \overline{Z}_1, \\
\Phi_{17} &= \overline{P} - \beta_1 M + \beta_2 M^T A^T + \beta_2 G^T B^T, \\
\Phi_{22} &= -(1-\mu) e^{-\alpha h} \overline{Q}_2 - 2e^{-\alpha h} \overline{R} + e^{-\alpha h} \left(\overline{S}^T + \overline{S} \right) \\
&\quad - 2e^{-\alpha h} \overline{Z}_1 - e^{-\alpha h} \overline{Z}_2, \\
\Phi_{23} &= -e^{-\alpha h} \overline{S} + e^{-\alpha h} \overline{R}, \\
\Phi_{24} &= 2e^{-\alpha h} \overline{Z}_2, \\
\Xi_{25} &= 2e^{-\alpha h} \overline{Z}_1, \\
\Phi_{33} &= -e^{-\alpha h} \overline{Q}_1 - e^{-\alpha h} \overline{R} - 2e^{-\alpha h} \overline{Z}_2, \\
\Phi_{35} &= 2e^{-\alpha h} \overline{Z}_2, \\
\Phi_{44} &= -2e^{-\alpha h} \overline{Z}_1 - 2e^{-\alpha h} \overline{Z}_2, \\
\Phi_{55} &= -2e^{-\alpha h} \overline{Z}_1 - 2e^{-\alpha h} \overline{Z}_2, \\
\Phi_{66} &= -\overline{T}_1 - e^{-\alpha d} \overline{T}_2, \\
\Phi_{77} &= h^2 \overline{R} + \frac{h^2}{2} \overline{Z}_1 + \frac{h^2}{2} \overline{Z}_2 + \frac{d^4}{4} \overline{T}_2 - \beta_2 M^T - \beta_2 M,
\end{aligned} \tag{27}$$

then the reachable set of the closed-loop system (6) is bounded by the prescribed ellipsoid in $\varepsilon(M^{-T}\overline{P}M^{-1}, 1)$ and the desired state-feedback controller can be obtained as $K = GM^{-1}$.

Proof. Denote $X_1 = \mathrm{diag}\{M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, M^{-1}, I\}$ and $X_2 = \mathrm{diag}\{M^{-1}, M^{-1}\}$; pre- and postmultiplying Φ_1 by X_1^T and X_1, we have $X_1^T \Phi X_1 < 0$.

Then by employing congruence transformation for (25) with X_2, we will get the following inequalities:

$$\begin{bmatrix} M^{-T}\left(\overline{R}+\overline{Z}_1\right)M^{-1} & M^{-T}\overline{S}M^{-1} \\ * & M^{-T}\left(\overline{R}+\overline{Z}_2\right)M^{-1} \end{bmatrix} > 0. \quad (28)$$

We obtain that there exist matrices $P = M^{-T}\overline{P}M^{-1}$, $Q_1 = M^{-T}\overline{Q}_1M^{-1}$, $Q_2 = M^{-T}\overline{Q}_2M^{-1}$, $R = M^{-T}\overline{R}M^{-1}$, $Z_1 = M^{-T}\overline{Z}_1M^{-1}$, $Z_2 = M^{-T}\overline{Z}_2M^{-1}$, $S = M^{-T}\overline{S}M^{-1}$, $H_1 = \beta_1 M^{-1}$, and $H_2 = \beta_2 M^{-1}$ satisfying the conditions in Theorem 4. Therefore the reachable set of closed-loop system in (6) is bounded by the ellipsoid $\varepsilon(M^{-T}\overline{P}M^{-1}, 1)$. The proof is completed. □

Remark 10. In order to guarantee negative definite, μ is required to be less than 1 in [4]. However, the value of μ is not necessarily less than 1 in Theorems 4 and 9 since the terms Ω_{22}, Φ_{22} can be negative definite when $\mu > 1$. Obviously, the results in this paper are more general than the ones in [4].

Remark 11. To get the reachable set of closed-loop system to be bounded by a given ellipsoid $x^T(t)P_r x(t) \le 1$, we need the inequality $0 < P_r \le P = M^{-T}\overline{P}M^{-1}$ which is equivalent to

$$\begin{bmatrix} -\overline{P} & M^T \\ M & -P_r \end{bmatrix} \le 0. \quad (29)$$

Remark 12. It should be pointed out that the matrix inequalities (11) and (26) contain a nonconvex scalar α; when α is fixed, these inequations will become LMIs, and MATLAB's toolbox is employed to solve the matrix inequalities in Theorems 4 and 9.

4. Numerical Examples

In this section, two numerical examples are proposed to show the effectiveness of the results obtained in this paper.

Example 1. Consider system (1) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0.2 \\ 0 & -0.2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = 0, \quad \omega_m = 1. \quad (30)$$

In order to compare with previous results, the resulting $\overline{\delta}'s$ for different values of d with $\mu = 0.5$ and $h = 0.2$ are listed in Table 1. It can be seen that much tighter bounds are obtained than the ones in [15, 16] by the proposed method in this paper. This is mainly because we consider the additional useful term and introduce the reciprocally convex combination approach to deal with single integral term and double integral terms when estimating the upper bound of the derivative of Lyapunov-Krasovskii functional. Tables 2 and 3 are some comparisons for different values of h and μ, respectively, which also implies that our method is less conservative than the ones in [15, 16].

Table 1: Computed $\overline{\delta}'s$ for different d with $\mu = 0.5$ and $h = 0.2$.

d	0.2	0.4	0.6	0.7	0.8
[15, Theorem 1]	10.0502	11.5762	13.7431	14.9492	16.3729
[16, Theorem 1]	5.8645	6.4724	7.3010	7.7887	8.3381
Theorem 4	2.2621	2.9137	3.0292	3.2147	3.2855

Table 2: Computed $\overline{\delta}'s$ for different h with $\mu = 0.5$ and $d = 0.1$.

h	0.1	0.3	0.5	0.7	0.9
[15, Theorem 1]	9.3269	9.5769	9.8531	10.1593	10.5000
[16, Theorem 1]	5.5456	5.7058	5.9109	6.1423	6.4505
Theorem 4	3.0658	3.9427	4.1006	4.4615	4.9985

Table 3: Computed $\overline{\delta}'s$ for different μ with $d = 0.1$ and $h = 0.2$.

μ	0.1	0.5	0.8	0.9	2
[15, Theorem 1]	8.2338	9.4488	13.4135	21.2778	—
[16, Theorem 1]	5.5992	5.6198	5.6198	5.6198	5.6198
Theorem 4	4.0305	4.0821	4.1802	4.3839	5.1657

Example 2. Consider system (1) with the following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, \quad A_h = \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \omega_m = 1. \quad (31)$$

For this example, when $u(t) = 0$, the computed $\overline{\delta}$ for different values of h with $\mu = 0$ is listed in Table 4. By using Theorem 4, it is clear to see that the bounds obtained

Table 4: Computed $\overline{\delta}'s$ for different h with $\mu = 0$.

h	0.1	0.3	0.5	0.7	0.9
[4, Theorem 1]	0.83	1.28	1.94	2.90	4.46
[6, Theorem 1]	0.74	0.92	1.36	2.30	3.51
[16, Theorem 1]	0.66	0.75	0.94	1.61	3.14
Theorem 4	0.40	0.71	0.92	1.24	2.73

in this paper are better than the ones of literatures [4, 6, 16]. When $h = 0.5$, $u = 0.1$, $\beta_1 = 1$, and $\beta_2 = 1$, by solving the conditions in Theorem 9, the controller can be obtained, $K = [6.1472 - 2.3741]$ and $P = M^{-T}\overline{P}M^{-1} = \begin{bmatrix} 4.3726 & 0.1148 \\ 0.1148 & 0.9278 \end{bmatrix}$, and the obtained ellipsoid $\varepsilon(P, 1)$ can bound the reachable set of the closed-loop system.

5. Conclusions

In this paper, the problem of reachable set estimation and controller design for linear systems with discrete and distributed delays has been studied. Based on the reciprocally convex approach and triple integral technique, improved delay-dependent conditions for the considered system have been presented in terms of linear matrix inequalities. It should be pointed out that triple integral terms are firstly introduced to research the problem of reachable set estimation and synthesis for addressed system. In addition, a state-feedback controller is designed to guarantee the reachable set of closed-loop system to be bounded by a given ellipsoid. Finally, two numerical examples are given to show the usefulness and effectiveness of the proposed criteria.

Further, it should be worth mentioning that the proposed method in this paper can be extensively applicable in many other areas, such as nonlinear systems, singular systems, Markov jump systems, and switch systems, which deserve further investigation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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