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Research Article

On Antiperiodic Nonlocal Three-Point Boundary Value Problems for Nonlinear Fractional Differential Equations

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We introduce boundary value conditions involving antiperiodic and nonlocal three-point boundary conditions. We solve a nonlinear fractional differential equation supplemented with those conditions. We obtain some existence results for the given problem by applying some standard tools of fixed point theory. These results are well illustrated with the aid of examples.

1. Introduction

In recent years, several kinds of boundary value problems of nonlinear fractional differential equations, supplemented with a variety of boundary conditions (including Dirichlet, Neumann, mixed, periodic, antiperiodic, multipoint, integral type, and nonlocal), have been investigated by several researchers. This investigation includes a wide collection of results ranging from theoretical to analytic and numerical methods. For details and examples, see [1–9] and the references therein. This surge has been mainly due to the extensive applications of fractional operators in basic and technical sciences and engineering. One can easily witness from literature (special issues and books) on the topic that the tools of fractional calculus have helped in improving the mathematical modeling of several phenomena of practical nature; for instance, see [10–17].

In this paper, we study a new class of problems of fractional differential equations supplemented with antiperiodic and three-point nonlocal boundary conditions. Precisely, we consider the following fractional problem:

$$\begin{gathered} {}^{c}D^{q}x(t)+f(t,x(t))=0, \quad t\in[0,1], \ 1<q\le 2, \\ x(0)=-x(1), \\ \alpha x'(0)+\beta x'(1)=-\gamma x(\theta), \quad 0<\theta<1, \end{gathered} \tag{1}$$

where ${}^{c}D^{q}$ is the usual Caputo fractional derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, α,β are positive real constants, and γ is a nonnegative constant. Further, it is assumed that $\gamma(1-2\theta)-2(\alpha+\beta)\neq 0$.

We emphasize that the second boundary condition in (1) can be interpreted as the sum of scalar multiples of the values of the derivative of the unknown function at $x=0$ and $x=1$ is proportional to the value of the unknown function at an arbitrary value $\theta\in(0,1)$. In case of $\alpha=\beta=1$, $\gamma=0$, problem (1) reduces to the one with antiperiodic boundary conditions. Thus the proposed problem generalizes antiperiodic problems to semiantiperiodic three-point nonlocal problems.

2. Auxiliary Lemma and Notations

In order to define the solutions for the given problem, we consider the following lemma.

Lemma 1. *Let $g\in C[0,1]$. Then the following linear semi-antiperiodic three-point fractional boundary value problem*

$$\begin{gathered} {}^{c}D^{q}x(t)+g(t)=0, \quad t\in[0,1], \ 1<q\le 2, \\ x(0)=-x(1), \\ \alpha x'(0)+\beta x'(1)=-\gamma x(\theta), \quad 0<\theta<1 \end{gathered} \tag{2}$$

has a unique solution given by

$$\begin{aligned} x(t) = &-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)\,ds \\ &+ \frac{1}{[\gamma(1-2\theta)-2(\alpha+\beta)]}\Bigg[(\gamma(t-\theta)-\alpha-\beta) \\ &\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s)\,ds + (1-2t) \\ &\cdot \Bigg(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} g(s)\,ds \\ &+ \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds\Bigg)\Bigg]. \end{aligned} \tag{3}$$

Proof. It is well known that the solution of equation ${}^cD^q x(t) + g(t) = 0,\ 1 < q \le 2$ can be written as

$$x(t) = -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)\,ds + b_1 + b_2 t, \tag{4}$$

where $b_1, b_2 \in \mathbb{R}$ are unknown arbitrary constants. Since $x(0) = -x(1)$, (4) gives

$$2b_1 + b_2 = \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s)\,ds. \tag{5}$$

Using the boundary conditions $\alpha x'(0) + \beta x'(1) = -\gamma x(\theta)$ in (4), we get

$$\begin{aligned} \gamma b_1 + (\alpha+\beta+\gamma\theta) b_2 = &\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} g(s)\,ds \\ &+ \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds. \end{aligned} \tag{6}$$

Solving system of (5) and (6) for b_1 and b_2, it is found that

$$\begin{aligned} b_1 = &\frac{1}{[\gamma(1-2\theta)-2(\alpha+\beta)]}\Bigg[-(\theta\gamma+\alpha+\beta) \\ &\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s)\,ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \\ &+ \gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} g(s)\,ds\Bigg], \\ b_2 = &\frac{1}{[\gamma(1-2\theta)-2(\alpha+\beta)]}\Bigg[\gamma \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s)\,ds \\ &- 2\beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \\ &- 2\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} g(s)\,ds\Bigg]. \end{aligned} \tag{7}$$

Substituting these values of b_1 and b_2 in (4) completes solution (3).

Let $\mathscr{D} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual supremum norm defined by $\|x\| = \sup\{|x(t)|,\ t \in [0,1]\}$.

In view of Lemma 1, we consider a fixed point problem equivalent to the nonlinear antisymmetric problem (1) as follows:

$$\mathscr{H} x = x, \tag{8}$$

where the operator $\mathscr{H} : \mathscr{D} \to \mathscr{D}$ is defined as

$$\begin{aligned} (\mathscr{H} x)(t) = &-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\,ds \\ &+ \frac{1}{[\gamma(1-2\theta)-2(\alpha+\beta)]}\Bigg[(\gamma(t-\theta)-\alpha-\beta) \\ &\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))\,ds + (1-2t) \\ &\cdot \Bigg(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} f(s, x(s))\,ds \\ &+ \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))\,ds\Bigg)\Bigg]. \end{aligned} \tag{9}$$

Next we set

$$\sigma = \frac{1}{\Gamma(q+1)}\left(1 + \frac{|\gamma(1-\theta)-\alpha-\beta| + \beta q + \gamma\theta^q}{|\gamma(1-2\theta)-2(\alpha+\beta)|}\right). \tag{10}$$

□

3. Main Results

In this section, we present our main results. The first result relies on classical Banach's contraction mapping principle.

Theorem 2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition; that is, it exists $\ell > 0$ such that*

$$(H_1)\ |f(t,x) - f(t,y)| \le \ell |x - y|,\ \forall t \in [0,1], x, y \in \mathbb{R}.$$

Then problem (1) has a unique solution if $\ell\sigma < 1$, where σ is given by (10).

Proof. In the first step, it will be shown that $\mathscr{H}B_r \subset B_r$, where $B_r = \{x \in \mathscr{D} : \|x\| \le r\}$ with $r > \sigma\mu(1-\sigma\ell)^{-1}$, and $\sup_{t\in[0,1]}|f(t,0)| = \mu$. For $x \in B_r, t \in [0,1]$, using $|f(s,x(s))| = |f(s,x(s)) - f(s,0) + f(s,0)| \le \ell r + \mu$, we get

$$\begin{aligned}
\|(\mathscr{H}x)\| \le \sup_{t\in[0,1]} \Bigg\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,0) + f(s,0) \right| ds \\
& + \frac{1}{|\gamma(1-2\theta) - 2(\alpha+\beta)|} \Bigg[|\gamma(t-\theta) - \alpha - \beta| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,0) + f(s,0) \right| ds \\
& + |1-2t| \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,0) + f(s,0) \right| ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left| f(s,x(s)) - f(s,0) + f(s,0) \right| ds \right) \Bigg] \Bigg\} \\
\le (\ell r + \mu) \sup_{t\in[0,1]} \Bigg\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{|\gamma(1-2\theta) - 2(\alpha+\beta)|} \Bigg[|\gamma(t-\theta) - \alpha - \beta| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds \\
& + |1-2t| \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} ds \right) \Bigg] \Bigg\} \le (\ell r + \mu)\sigma \le r,
\end{aligned} \tag{11}$$

which implies that $\mathscr{H}B_r \subset B_r$, where we have used (10). Now, for $x, y \in \mathscr{D}$, and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\|\mathscr{H}x - \mathscr{H}y\| \le \sup_{t\in[0,1]} \Bigg\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
& + \frac{1}{|\gamma(1-2\theta) - 2(\alpha+\beta)|} \Bigg[|\gamma(t-\theta) - \alpha - \beta| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
& + |1-2t| \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) - f(s,y(s)) \right| ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \right) \Bigg] \Bigg\} \le \ell\sigma \|x \\
& - y\|.
\end{aligned} \tag{12}$$

Since $\ell\sigma < 1$ by the given assumption, the operator $\mathscr{H}$ is a contraction. Thus, by Banach's contraction mapping principle, there exists a unique solution for problem (1). This completes the proof. □

The next existence result is based on the following Schaefer's fixed point theorem [18], Th. 4.3.2.

Theorem 3. *Let X be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$V = \{u \in X \mid u = \nu T u,\ 0 < \nu < 1\} \tag{13}$$

is bounded. Then T has a fixed point in X.

Theorem 4. *Assume that there exists a positive constant M such that $|f(t,x)| \le M$ for $t \in [0,1]$, $x \in \mathbb{R}$. Then problem (1) has at least one solution.*

Proof. We first show that the operator $\mathscr{H}$ is completely continuous. Obviously continuity of the operator $\mathscr{H}$ follows from continuity of f. Let $\mathscr{B} \subset \mathscr{D}$ be a bounded set. By the assumption $|f(t,x)| \le M$, for $x \in \mathscr{B}$, we have

$$\begin{aligned}
|(\mathscr{H}x)(t)| \le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s,x(s)) \right| ds \\
& + \frac{1}{|\gamma(1-2\theta) - 2(\alpha+\beta)|} \Bigg[|\gamma(t-\theta) - \alpha - \beta|
\end{aligned}$$

$$\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds + |1-2t| \cdot \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left| f(s, x(s)) \right| ds \right) \Bigg] \le M\sigma = M_1, \tag{14}$$

which implies that $\|(\mathscr{H}x)\| \le M_1$. Further, we find that

$$\begin{aligned} \left|(\mathscr{H}x)'(t)\right| &\le \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \left| f(s, x(s)) \right| ds \\ &+ \frac{1}{\left|\gamma(1-2\theta) - 2(\alpha+\beta)\right|} \left[\gamma \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds \right. \\ &+ 2\left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds \right. \\ &\left. \left. + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left| f(s, x(s)) \right| ds \right) \right] \le \frac{M}{\Gamma(q+1)} \left(q \right. \\ &\left. + \frac{\left|\gamma + 2(\beta q + \gamma\theta^q)\right|}{\left|\gamma(1-2\theta) - 2(\alpha+\beta)\right|} \right) = M_2. \end{aligned} \tag{15}$$

Hence, for $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned} &\left|(\mathscr{H}x)(t_2) - (\mathscr{H}x)(t_1)\right| \\ &\quad \le \int_{t_1}^{t_2} \left|(\mathscr{H}x)'(s)\right| ds \le M_2(t_2 - t_1). \end{aligned} \tag{16}$$

This implies that $\mathscr{H}$ is equicontinuous on $[0, 1]$. Thus, by the Arzelà-Ascoli theorem, the operator $\mathscr{H} : \mathscr{D} \to \mathscr{D}$ is completely continuous.

Next, we consider the set

$$\mathscr{A} = \{x \in \mathscr{D} \mid x = \nu\mathscr{H}x,\ 0 < \nu < 1\}, \tag{17}$$

and show that the set $\mathscr{A}$ is bounded. Let $x \in \mathscr{A}$, and then $x = \nu\mathscr{H}x, 0 < \nu < 1$. For any $t \in [0, 1]$, we have

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\ &+ \frac{1}{\left[\gamma(1-2\theta) - 2(\alpha+\beta)\right]} \left[(\gamma(t-\theta) - \alpha - \beta) \right. \\ &\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds + (1-2t) \\ &\cdot \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \right. \\ &\left. \left. + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))\, ds \right) \right] \end{aligned} \tag{18}$$

and $|x(t)| = \nu|(\mathscr{H}x)(t)| \le |(\mathscr{H}x)(t)| \le M\sigma = M_1$ for any $t \in [0, 1]$. So the set $\mathscr{A}$ is bounded. Thus, the conclusion of Theorem 3 applies and the operator $\mathscr{H}$ has at least one fixed point. This, in turn, implies that problem (1) has at least one solution on $[0, 1]$. □

Now we show the existence of solutions for problem (1) by means of Leray-Schauder degree theory.

Theorem 5. *Suppose that there exist constants $0 \le \varrho < 1/\sigma$ and $N > 0$ such that $|f(t, x)| \le \varrho|x| + N$ for all $t \in [0, 1], x \in \mathscr{D}$. Then problem (1) has at least one solution.*

Proof. Define a ball $B_R \subset \mathscr{D}$ with radius $R > 0$ by

$$B_R = \left\{ x \in \mathscr{D} : \max_{t \in [0,1]} |x(t)| < R \right\}, \tag{19}$$

where R will be fixed later. Then, it is enough to show that the operator $\mathscr{H} : \overline{B}_R \to \mathscr{D}$ (given by (10)) is such that

$$x \ne \lambda\mathscr{H}x, \quad \forall x \in \partial B_R,\ \forall \lambda \in [0, 1]. \tag{20}$$

Now we set

$$\mathscr{K}(\lambda, x) = \lambda\mathscr{H}x, \quad \lambda \in [0, 1]. \tag{21}$$

Then, by Arzelà-Ascoli theorem, $\omega_\lambda(x) = x - \mathscr{K}(\lambda, x) = x - \lambda\mathscr{H}x$ is completely continuous. If condition (20) holds true, then the following Leray-Schauder degrees are well defined and, by the homotopy invariance of topological degree, we have that

$$\begin{aligned} \deg(\omega_\lambda, B_R, 0) &= \deg(I - \lambda\mathscr{H}, B_R, 0) \\ &= \deg(\omega_1, B_R, 0) = \deg(\omega_0, B_R, 0) \\ &= \deg(I, B_R, 0) = 1 \ne 0, \quad 0 \in B_R, \end{aligned} \tag{22}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $\omega_1(t) = x - \lambda\mathscr{H}x = 0$ for at least one $x \in B_R$. In order to justify condition (20), it is assumed that $x = \lambda\mathscr{H}x$ for some $\lambda \in [0, 1]$ and for all $t \in [0, 1]$ so that

$$\begin{aligned} |x(t)| = |\lambda\mathscr{H}x(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds \\ &+ \frac{1}{\left|\gamma(1-2\theta) - 2(\alpha+\beta)\right|} \left[\left|\gamma(t-\theta) - \alpha - \beta\right| \right. \\ &\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds + |1-2t| \\ &\cdot \left(\gamma \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) \right| ds \right. \\ &\left. \left. + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left| f(s, x(s)) \right| ds \right) \right] \le (\varrho\|x\| + N) \\ &\cdot \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds \right. \end{aligned}$$

$$+\frac{1}{|\gamma(1-2\theta)-2(\alpha+\beta)|}\left[|\gamma(t-\theta)-\alpha-\beta|\right.$$
$$\cdot\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}ds+|1-2t|$$
$$\left.\cdot\left(\gamma\int_0^\theta\frac{(\theta-s)^{q-1}}{\Gamma(q)}ds+\beta\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}ds\right)\right]\Bigg\}, \tag{23}$$

which, on taking norm over the interval $[0,1]$, yields

$$\|x\|\le(\varrho\|x\|+N)\sigma, \tag{24}$$

where σ is given by (10). In consequence we have

$$\|x\|\le\frac{\sigma N}{1-\varrho\sigma}. \tag{25}$$

Letting $R=\sigma N/(1-\varrho\sigma)+1$, (20) holds. This completes the proof. □

The next result is based on Krasnoselskii's fixed point theorem [18], Th. 4.4.1.

Theorem 6. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (H_1) and*

(H_2) $|f(t,x)|\le\mu(t)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$, *and* $\mu\in C([0,1],\mathbb{R}^+)$.

Then problem (1) has at least one solution on $[0,1]$ *if*

$$\frac{|\gamma(1-\theta)-\alpha-\beta|+\beta q+\gamma\theta^q}{\Gamma(q+1)|\gamma(1-2\theta)-2(\alpha+\beta)|}<1. \tag{26}$$

Proof. For $r\ge\|\mu\|\sigma$, let us define a closed set (ball) $B_r=\{x\in\mathcal{D}:\|x\|\le r\}$ and introduce the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on B_r as

$$(\mathcal{H}_1x)(t)=-\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds,$$
$$(\mathcal{H}_2x)(t)=+\frac{1}{[\gamma(1-2\theta)-2(\alpha+\beta)]}\left[(\gamma(t-\theta)\right.$$
$$-\alpha-\beta)\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds+(1-2t) \tag{27}$$
$$\cdot\left(\gamma\int_0^\theta\frac{(\theta-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\right.$$
$$\left.\left.+\beta\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))\,ds\right)\right].$$

For $x,y\in B_r$, it is easy to show that $\|(\mathcal{H}_1x)+(\mathcal{H}_2y)\|\le\|\mu\|\sigma\le r$, where σ is given by (10). This implies that $\mathcal{H}_1x+\mathcal{H}_2y\in B_r$.

In view of condition (26), the operator $\mathcal{H}_2$ is a contraction. Continuity of the operator $\mathcal{H}_1$ follows from that of f. Also, $\|\mathcal{H}_1x\|\le\|\mu\|/\Gamma(q+1)$ implies that $\mathcal{H}_1$ is uniformly bounded on B_r. Furthermore, with $\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)|=f_b<\infty$ and $t_1,t_2\in(0,1]$, we have

$$\|(\mathcal{H}_1x)(t_2)-(\mathcal{H}_1x)(t_1)\|$$
$$\le\frac{f_b}{\Gamma(q+1)}\left(|t_2^q-t_1^q|+2(|t_2-t_1|)^q\right)\longrightarrow 0, \tag{28}$$

independent of x as $t_1\to t_2$. This shows that $\mathcal{H}_1$ is relatively compact on B_r. Hence, we infer by the Arzelà-Ascoli theorem that $\mathcal{H}_1$ is compact on B_r. Thus all the conditions of Krasnoselskii's fixed point theorem hold true. Hence, problem (1) has at least one solution on $[0,1]$. This completes the proof. □

Finally, we make use of Leray-Schauder nonlinear alternative to show the existence of solutions for problem (1).

Lemma 7 (nonlinear alternative for single valued maps [19]). *Let E_1 be a closed, convex subset of a Banach space E and V be an open subset of E_1 with $0\in V$. Suppose that $\mathcal{U}:\overline{V}\to E_1$ is continuous, compact (i.e., $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1) map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{V}$ or*

(ii) *there is $x\in\partial V$ (the boundary of V in E_1) and $\kappa\in(0,1)$ with $x=\kappa\mathcal{U}(x)$.*

Theorem 8. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(H_3) *there exist a function $p\in\mathcal{C}([0,1],\mathbb{R}^+)$ and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\le p(t)\psi(\|x\|)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$;*

(H_4) *there exists a constant $M>0$ such that*

$$\frac{M}{\psi(M)\|p\|\sigma}>1, \tag{29}$$

where σ is given by (10).

Then problem (1) has at least one solution on $[0,1]$.

Proof. As a first step, we show that the operator $\mathcal{H}:\mathcal{D}\to\mathcal{D}$ defined by (10) maps bounded sets into bounded sets in $\mathcal{D}$. For a positive number r, let $B_r=\{x\in\mathcal{D}:\|x\|\le r\}$ be a bounded set in $\mathcal{D}$. Then, for $x\in B_r$ together with (H_3) and (H_4), we obtain (as before)

$$|(\mathcal{H}x)(t)|\le\|p\|\sigma\psi(M)<M. \tag{30}$$

Next, it will be shown that $\mathscr{H}$ maps bounded sets into equicontinuous sets of $\mathscr{D}$. Let $t_1, t_2 \in [0,1]$ and $x \in B_r$. Then

$$\begin{aligned}
&\left|(\mathscr{H}x)(t_2) - (\mathscr{H}x)(t_1)\right| \\
&\quad = \left| -\int_0^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \right. \\
&\quad + \int_0^{t_1} \frac{(t_1 - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\
&\quad + \frac{1}{[\gamma(1-2\theta) - 2(\alpha+\beta)]} \left[\gamma(t_2 - t_1) \right. \\
&\quad \cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds - (t_2 - t_1) \\
&\quad \cdot \left(\gamma \int_0^{\theta} \frac{(\theta - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \right. \\
&\quad \left. \left. \left. + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))\, ds \right) \right] \right| \\
&\quad \le \frac{\psi(r)\|p\|}{\Gamma(q+1)} \left(\left|t_2^q - t_1^q\right| + 2\left(|t_2 - t_1|\right)^q + \delta\left|t_2 - t_1\right| \right),
\end{aligned} \tag{31}$$

where

$$\delta = \frac{\gamma(1+\theta^q) + \beta q}{\left|\gamma(1-2\theta) - 2(\alpha+\beta)\right|}. \tag{32}$$

Clearly, the right-hand side of the above inequality tends to zero independent of $x \in B_r$ as $t_2 \to t_1$. Thus, by the Arzelà-Ascoli theorem, the operator $\mathscr{H}$ is completely continuous.

Let x be a solution for the given problem. Then, for $\lambda \in (0,1)$, as before, we obtain

$$\|x\| = \sup_{t\in[0,1]} |\lambda(\mathscr{H}x)(t)| \le \psi(\|x\|)\|p\|\sigma. \tag{33}$$

In view of (H_4), there exists M such that $\|x\| \neq M$. Let us choose $\mathscr{P} = \{x \in \mathscr{D} : \|x\| < M + 1\}$.

Notice that the operator $\mathscr{H} : \overline{\mathscr{P}} \to \mathscr{D}$ is continuous and completely continuous. From the choice of $\mathscr{P}$, there is no $x \in \partial\mathscr{P}$ such that $x = \lambda\mathscr{H}(x)$ for some $\lambda \in (0,1)$. Consequently, the conclusion of Lemma 7 applies and hence the operator $\mathscr{H}$ has a fixed point $x \in \overline{\mathscr{P}}$ which is a solution of problem (1). This completes the proof. □

3.1. Examples. Consider a three-point boundary value problem of nonlinear fractional differential equations given by

$$^cD^{3/2}x(t) + f(t, x(t)) = 0, \quad t \in [0,1], \tag{34}$$

$$\begin{aligned}
x(0) &= -x(1), \\
x'(0) + x'(1) &= -x\left(\frac{1}{2}\right).
\end{aligned} \tag{35}$$

Here $q = 3/2$, $\alpha = \beta = \gamma = 1$, and $\theta = 1/2$. With the given values, it is found that

$$\begin{aligned}
\sigma &= \frac{1}{\Gamma(q+1)} \left(1 + \frac{\left|\gamma(1-\theta) - \alpha - \beta\right| + \beta q + \gamma\theta^q}{\left|\gamma(1-2\theta) - 2(\alpha+\beta)\right|} \right) \\
&= \frac{14\sqrt{2}+1}{6\sqrt{2\pi}} \simeq 1.3829.
\end{aligned} \tag{36}$$

(a) In (34), let us choose a continuous function

$$f(t, x(t)) = \frac{e^{-t}}{\sqrt{49+t^2}} \left(\frac{\pi}{2} + x + \tan^{-1} x \right), \quad t \in [0,1]. \tag{37}$$

Clearly $|f(t,x) - f(t,y)| \le (2/7)|x-y|$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$, and $\ell/\sigma < 1$, where $\ell = 2/7$ and σ is given by (36). Thus problem (34)-(35) with $f(t,x)$ given by (37) has a unique solution on $[0,1]$ by Theorem 2.

(b) Taking

$$f(t,x) = \frac{(t+1)^2}{16} \left(\left(e^{-2t} + 1\right) \sin|x| + 3\cos^2 t \right) \tag{38}$$

in (34), one can notice that $|f(t,x)| \le (1/2)|x| + 3/4$. Thus, with $\varrho = 1/2$, $N = 3/4$, the assumptions of Theorem 5 are satisfied and consequently its conclusion applies to problem (34)-(35) with $f(t,x)$ given by (38).

(c) Choosing

$$f(t,x) = \frac{\cos\left(t^2+1\right)}{\sqrt{4+t}} \left(\frac{|x|^3}{1+|x|^3} + |x| + \frac{1}{2} \right) \tag{39}$$

in (34), we find that $|f(t,x)| \le p(t)\psi(\|x\|)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$, where

$$\begin{aligned}
p(t) &= \frac{\cos\left(t^2+1\right)}{\sqrt{4+t}}, \\
\psi(\|x\|) &= \|x\| + \frac{3}{2}.
\end{aligned} \tag{40}$$

Using condition (H_4), that is, $M/\psi(M)\|p\|\sigma > 1$, we get $M > M_1$, $M_1 \simeq 1.1205$. Clearly all the conditions of Theorem 8 are satisfied. Hence there exists a solution of problem (34)-(35) with $f(t,x)$ given by (39).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] B. Ahmad and J. J. Nieto, "Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory," *Topological Methods in Nonlinear Analysis*, vol. 35, no. 2, pp. 295–304, 2010.

[2] B. Ahmad and J. J. Nieto, "Sequential fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3046–3052, 2012.

[3] R. Sakthivel, N. I. Mahmudov, and J. J. Nieto, "Controllability for a class of fractional-order neutral evolution control systems," *Applied Mathematics and Computation*, vol. 218, no. 20, pp. 10334–10340, 2012.

[4] R. P. Agarwal, D. O'Regan, and S. Stanek, "Positive solutions for mixed problems of singular fractional differential equations," *Mathematische Nachrichten*, vol. 285, no. 1, pp. 27–41, 2012.

[5] C. Zhai and M. Hao, "Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems," *Boundary Value Problems*, vol. 2013, article 85, 13 pages, 2013.

[6] J. R. Wang, Y. Zhou, and M. Feckan, "On the nonlocal Cauchy problem for semilinear fractional order evolution equations," *Central European Journal of Mathematics*, vol. 12, no. 6, pp. 911–922, 2014.

[7] J. R. Graef, L. Kong, and M. Wang, "Existence and uniqueness of solutions for a fractional boundary value problem on a graph," *Fractional Calculus and Applied Analysis*, vol. 17, no. 2, pp. 499–510, 2014.

[8] B. Ahmad and J. J. Nieto, "A class of differential equations of fractional order with multi-point boundary conditions," *Georgian Mathematical Journal*, vol. 21, no. 3, pp. 243–248, 2014.

[9] J. Henderson and R. Luca, "Positive solutions for a system of nonlocal fractional boundary value problems," *Fractional Calculus and Applied Analysis*, vol. 16, no. 4, pp. 985–1008, 2013.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[11] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[12] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, London, UK, 2010.

[13] J. Klafter, S. C. Lim, and R. Metzler, Eds., *Fractional Dynamics in Physics*, World Scientific, Singapore, 2011.

[14] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston, Mass, USA, 2012.

[15] J. Wang, Y. Zhou, and W. Wei, "Fractional sewage treatment models with impulses at variable times," *Applicable Analysis*, vol. 92, no. 9, pp. 1959–1979, 2013.

[16] X. Liu, Z. Liu, and X. Fu, "Relaxation in nonconvex optimal control problems described by fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 446–458, 2014.

[17] F. Punzo and G. Terrone, "On the Cauchy problem for a general fractional porous medium equation with variable density," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 98, pp. 27–47, 2014.

[18] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1980.

[19] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, NY, USA, 2003.