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Research Article

Complete Moment Convergence for Sung's Type Weighted Sums of B-Valued Random Elements

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Let $p \geq 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed B-valued random elements and let $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ be an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ for some $q > p$. We give necessary and sufficient conditions for complete moment convergence of the form $\sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\{\|\sum_{i=1}^{n} a_{ni}X_i\| - \varepsilon n^{\alpha}\}_{+}^{v} < \infty,\ \forall \varepsilon > 0$, where $0 < v < p$. A strong law of large numbers for weighted sums of independent B-valued random elements is also obtained.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables (or random elements) and let $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ be an array of real numbers. The weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ include many useful linear statistical estimators, such as least squares estimators, nonparametric regression function estimators, and jackknife estimators. So it is interesting and meaningful to study the limiting behavior for them. In fact, many authors have studied some limiting properties. We refer to Bai and Cheng [1], Chen et al. [2], Cuzick [3], Sung [4, 5], Wang et al. [6], Wu [7], and Zhang [8].

Recently, Sung [5] obtained a complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables (we call them Sung's type weighted sums).

Theorem A. *Let $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables and let $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ be an array of real numbers with*

$$\sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} |a_{ni}|^q < \infty \tag{1}$$

for some $q > p$. Then, $EX = 0$ and $E|X|^p < \infty$ imply that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq m\leq n} \left|\sum_{i=1}^{m} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0. \tag{2}$$

Conversely, if (2) holds for any array $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ with (1) for some $q > p$, then $E|X|^p < \infty$.

The weights satisfying (1) are very general. For example, set $a_{ni} = 1$ for all $1 \leq i \leq n$ and $n \geq 1$. Then, (1) holds for any $q > 0$ and therefore the weighted sums include the partial sums. Set $a_{ni} = 1$ if $1 \leq i \leq n-1$ and $a_{nn} = n^{1/q}$ for some $q > 0$. Then, (1) holds; meanwhile, (1) does not hold for any $q' > q$, and obviously the weights are unbounded in this case. So Sung's type weights are very rich and interesting, but very few authors continue to study the kind of weighted sums except Zhang [8] who obtained Theorem A for END random variables.

Chow [9] first investigated the complete moment convergence as follows.

Theorem B. *Let $\{X, X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables with $EX = 0$ and*

$E\{|X|^p + |X|\log(1+|X|)\} < \infty$, *where* $p > 1/\alpha$, $1/2 < \alpha \le 1$. *Then,*

$$\sum_{n=1}^{\infty} n^{(p-1)\alpha-2} E\left\{\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_i\right| - \varepsilon n^{\alpha}\right\}_{+} < \infty, \quad \forall \varepsilon > 0, \tag{3}$$

where x_+ *means* $\max\{0, x\}$ *for any real number* x.

Chen and Wang [10] pointed out that (3) is equivalent to

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_i\right| > x n^{\alpha}\right) dx < \infty, \quad \forall \varepsilon > 0, \tag{4}$$

or

$$\int_{1}^{\infty} \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_i\right| > \varepsilon x n^{\alpha}\right) dx < \infty, \quad \forall \varepsilon > 0. \tag{5}$$

Li and Spătaru [11] called (4) the refined result of complete convergence. For some applications in the theory of branching processes, Spătaru [12] obtained (4) for the special case $p = 2$ and $\alpha = 1$.

Obviously, (4) or (5) implies that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0. \tag{6}$$

Formula (6) is called complete convergence which was introduced by Hsu and Robbins [13]. They first obtained (6) for the special case $p = 2$ and $\alpha = 1$. Therefore, the complete moment convergence and the refined result of complete convergence are more exact than the complete convergence.

The complete convergence, the complete moment convergence, and the refined result of complete convergence have attracted many authors. We refer to Bai and Su [14], Baum and Katz [15], Chen [16], Chen et al. [17, 18], Chen and Wang [10], Katz [19], Li and Spătaru [11, 20], Qiu et al. [21], Rosalsky et al. [22], Sung [23], Wang and Su [24], and Wu et al. [25].

The purpose of this paper is to extend Theorem A to complete moment convergence for independent and identically distributed random elements taking values in a Banach space B. We also consider the case $p\alpha = 1$. No geometric conditions are imposed on the Banach space. Our results also partially extend the results of Chen [16] and Li and Spătaru [20] from the partial sums to the weighted sums.

2. Preliminaries

Let B be a real separable Banach space with norm $\|\cdot\|$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. A random element X taking values in B is defined as a Borel measurable function from $(\Omega, \mathcal{F})$ into B with the Borel sigma-algebra. The expected value of a B-valued random element X is defined as the Bochner integral and denoted by EX.

A Banach space is said to be of Rademacher type p, $1 \le p \le 2$, if there exists a constant C such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \le C\sum_{i=1}^{n} E\|X_i\|^p \tag{7}$$

for all $n \ge 1$ and each sequence $\{X_n,\ n \ge 1\}$ of independent random elements taking values in B with mean zero and finite pth moments. It is well known that if B is of Rademacher type p, $p > 1$, then B is of Rademacher type r, $1 \le r \le p$. Set

$$L^p(a,b) = \left\{x(t): \text{measurable function } x(t) \text{ from } [a,b] \text{ to } R \text{ with } \int_a^b |x(t)|^p\, dt < \infty\right\} \tag{8}$$

for some $-\infty < a < b < \infty$ and $1 \le p \le 2$. It is well known that $L^p(a,b)$ is a Banach space with norm $\|x\| = (\int_a^b |x(t)|^p dt)^{1/p}$. Then, $L^p(a,b)$ is of Rademacher type p (see, e.g., Ledoux and Talagrand [26]).

The following assertion gives us a useful contraction principle and can be found in Lemma 6.5 of Ledoux and Talagrand [26].

Lemma 1. *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of symmetric* B*-valued random elements. Let* $\{\xi_n,\ n \ge 1\}$ *and* $\{\zeta_n,\ n \ge 1\}$ *be real random variables such that* $\xi_n = \varphi_n(X_n)$, *where* $\varphi_n : B \to R$ *is symmetric (even), similarly for* ζ_n. *If* $|\xi_n| \le |\zeta_n|$ *almost surely for every* n, *then for any* $n \ge 1$

$$E\left\|\sum_{i=1}^{n} \xi_i X_i\right\| \le E\left\|\sum_{i=1}^{n} \zeta_i X_i\right\|,$$
$$P\left(\left\|\sum_{i=1}^{n} \xi_i X_i\right\| > t\right) \le 2P\left(\left\|\sum_{i=1}^{n} \zeta_i X_i\right\| > t\right), \quad \forall t > 0. \tag{9}$$

Checking carefully the arguments of (2.15)–(2.17) and (2.21)–(2.23) in Sung [5], we have the following lemma.

Lemma 2. *Let* $p > 1/\alpha$ *and* $1/2 < \alpha \le 1$. *Let* Y *be a nonnegative random variable with* $EY^p < \infty$. *Assume that* $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ *is an array of real numbers with*

$$\sup_{n\ge 1} n^{-1} \sum_{i=1}^{n} |a_{ni}|^q \le 1 \tag{10}$$

for some $q > p$ *and* $a_{ni} = 0$ *or* $|a_{ni}| > 1$. *Then, there exist two positive constants* C_0 *and* C_1 *not depending on* Y *such that*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\left(|a_{ni}Y| > n^{\alpha}\right) \le C_0 EY^p,$$
$$\sum_{n=1}^{\infty} n^{p\alpha-s\alpha-2} \sum_{i=1}^{n} E|a_{ni}Y|^s I\left(|a_{ni}Y| \le n^{\alpha}\right) \le C_1 EY^p, \tag{11}$$

where $s > \max\{2(p\alpha-1)/(2\alpha-1), q\}$ *if* $p \ge 2$ *and* $s = 2$ *if* $p < 2$.

Lemma 3. *Let $1/2 < \alpha \le 1$. Let $\{X, X_n,\ n \ge 1\}$ be a sequence of independent and identically distributed B-valued random elements with $E\|X\|^{1/\alpha} < \infty$ and let $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of real numbers with (10) for some $q > 1/\alpha$. Assume that X is symmetric and $n^{-\alpha}\sum_{i=1}^{n} a_{ni}X_i \to 0$ in probability. Then,*

$$\lim_{n\to\infty} n^{-\alpha} E\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| = 0. \tag{12}$$

Proof. Note that

$$\begin{aligned}\sum_{i=1}^{n} a_{ni}X_i &= \sum_{i=1}^{n} a_{ni}X_i I\left(\|a_{ni}X_i\| > n^{\alpha}\right)\\ &\quad + \sum_{i=1}^{n} a_{ni}X_i I\left(\|a_{ni}X_i\| \le n^{\alpha}\right).\end{aligned} \tag{13}$$

From (10), $|a_{ni}| \le n^{1/q}$ for all $1 \le i \le n$ and $n \ge 1$. Then, by (10) and Hölder's inequality,

$$\begin{aligned}&En^{-\alpha}\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|a_{ni}X_i\| > n^{\alpha}\right)\right\|\\ &\le n^{-\alpha}\sum_{i=1}^{n} E\|a_{ni}X_i\| I\left(\|a_{ni}X_i\| > n^{\alpha}\right)\\ &= n^{-\alpha}\sum_{i=1}^{n} E\|a_{ni}X\|^{1/\alpha}\cdot\|a_{ni}X\|^{1-1/\alpha} I\left(\|a_{ni}X\| > n^{\alpha}\right)\\ &\le n^{-1}\left(\sum_{i=1}^{n}|a_{ni}|^{1/\alpha}\right) E\|X\|^{1/\alpha} I\left(\|X\| > n^{\alpha-1/q}\right)\\ &\le \left(\sup_{n\ge1} n^{-1}\sum_{i=1}^{n}|a_{ni}|^{q}\right)^{1/(q\alpha)} E\|X\|^{1/\alpha}\\ &\quad\cdot I\left(\|X\| > n^{\alpha-1/q}\right) \le E\|X\|^{1/\alpha} I\left(\|X\| > n^{\alpha-1/q}\right)\\ &\longrightarrow 0\end{aligned} \tag{14}$$

as $n \to \infty$ since $E\|X\|^{1/\alpha} < \infty$ and $\alpha > 1/q$. By Lemma 7.2 in Ledoux and Talagrand [26],

$$n^{-\alpha} E\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|a_{ni}X_i\| \le n^{\alpha}\right)\right\| \longrightarrow 0 \tag{15}$$

as $n \to \infty$. Hence, combining (13)–(15) gives (12). □

The following moment inequality is due to de Acosta [27].

Lemma 4. *For every $s \ge 1$, there exists a positive constant C_s such that, for any separable Banach space B and any finite sequence $\{X_i,\ 1 \le i \le n\}$ of independent B-valued random elements with $E\|X_i\|^s < \infty$ for every $1 \le i \le n$, the following inequalities hold:*

(i) *For $1 \le s < 2$,*

$$E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^{s} \le C_s\sum_{i=1}^{n} E\|X_i\|^{s}. \tag{16}$$

(ii) *For $s \ge 2$,*

$$\begin{aligned}&E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^{s}\\ &\le C_s\left\{\left(\sum_{i=1}^{n} E\|X_i\|^{2}\right)^{s/2} + \sum_{i=1}^{n} E\|X_i\|^{s}\right\}.\end{aligned} \tag{17}$$

In the following, C will be used to denote various positive constants whose exact value is immaterial.

3. Main Results

We now state the main results and give the proofs.

Theorem 5. *Let $p > 1/\alpha$, $1/2 < \alpha \le 1$, and $0 < v < p$. Let $\{X, X_n,\ n \ge 1\}$ be a sequence of independent and identically distributed B-valued random elements and let $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of real numbers with (1) for some $q > p$. Then, $E\|X\|^{p} < \infty$ and $n^{-\alpha}\sum_{i=1}^{n} a_{ni}X_i \to 0$ in probability imply that*

$$\sum_{n=1}^{\infty} n^{(p-v)\alpha-2} E\left\{\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| - \varepsilon n^{\alpha}\right\}_{+}^{v} < \infty, \quad \forall\varepsilon > 0, \tag{18}$$

and hence

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall\varepsilon > 0. \tag{19}$$

Conversely, if (18) or (19) holds for any array $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ with (1) for some $q > p$, then $E\|X\|^{p} < \infty$.

Proof.

Sufficiency. By $n^{-\alpha}\sum_{i=1}^{n} a_{ni}X_i \to 0$ in probability, for any fixed $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$, such that for all $n \ge n_0$

$$\begin{aligned}&\sup_{x\ge1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| \le \varepsilon n^{\alpha}x^{1/v}\right)\\ &\quad\ge 1 - P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) \ge \frac{1}{2}.\end{aligned} \tag{20}$$

Let $\{X_n',\ n \ge 1\}$ be an independent copy of $\{X_n,\ n \ge 1\}$. By formula (6.1) of Ledoux and Talagrand [26], for all $n \ge n_0$ and $x \ge 1$,

$$\begin{aligned}&P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > 2\varepsilon n^{\alpha}x^{1/v}\right)\\ &\quad\le 2P\left(\left\|\sum_{i=1}^{n} a_{ni}\left(X_i - X_i'\right)\right\| > \varepsilon n^{\alpha}x^{1/v}\right).\end{aligned} \tag{21}$$

By Proposition 1.1 in Chen and Wang [10], (18) is equivalent to

$$\int_{1}^{\infty}\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}x^{1/v}\right)dx < \infty, \quad \forall\varepsilon > 0. \tag{22}$$

Hence, to prove (22), by (21), it is enough to prove that

$$\int_1^\infty \sum_{n=1}^\infty n^{p\alpha-2} P\left(\left\|\sum_{i=1}^n a_{ni}\left(X_i - X_i'\right)\right\| > \varepsilon n^\alpha x^{1/\nu}\right) dx < \infty, \quad \forall \varepsilon > 0. \tag{23}$$

Therefore, we can assume that X is symmetric. Without loss of generality, we can assume that $\sum_{i=1}^n |a_{ni}|^q \le n$ for all $n \ge 1$. Set $a_{ni}' = a_{ni}$ and $a_{ni}'' = 0$ if $|a_{ni}| \le 1$ and $a_{ni}' = 0$ and $a_{ni}'' = a_{ni}$ if $|a_{ni}| > 1$. Then, $a_{ni} = a_{ni}' + a_{ni}''$. Hence, to prove (22), it is enough to prove that

$$\int_1^\infty \sum_{n=1}^\infty n^{p\alpha-2} P\left(\left\|\sum_{i=1}^n a_{ni}' X_i\right\| > \varepsilon n^\alpha x^{1/\nu}\right) dx < \infty, \quad \forall \varepsilon > 0, \tag{24}$$

$$\int_1^\infty \sum_{n=1}^\infty n^{p\alpha-2} P\left(\left\|\sum_{i=1}^n a_{ni}'' X_i\right\| > \varepsilon n^\alpha x^{1/\nu}\right) dx < \infty, \quad \forall \varepsilon > 0. \tag{25}$$

By Lemma 1, $P(\|\sum_{i=1}^n a_{ni}' X_i\| > \varepsilon n^\alpha) \le 2P(\|\sum_{i=1}^n a_{ni} X_i\| > \varepsilon n^\alpha)$ for any $\varepsilon > 0$. It follows from the assumption that $n^{-\alpha}\sum_{i=1}^n a_{ni}' X_i \to 0$ in probability. By the same argument as in Chen [16] or Li and Spătaru [20], it is not hard to prove (24). Here, we omit the details.

Set $Y_{ni}(x) = X_i I(\|a_{ni}'' X_i\| \le n^\alpha x^{1/\nu})$ for all $x > 0$, $1 \le i \le n$, and $n \ge 1$. Note that

$$\begin{aligned} P&\left(\left\|\sum_{i=1}^n a_{ni}'' X_i\right\| > \varepsilon n^\alpha x^{1/\nu}\right) \\ &\le P\left\{\max_{1\le i\le n}\left\|a_{ni}'' X_i\right\| > n^\alpha x^{1/\nu}\right\} \\ &\quad + P\left(\left\|\sum_{i=1}^n a_{ni}'' Y_{ni}(x)\right\| > \varepsilon n^\alpha x^{1/\nu}\right). \end{aligned} \tag{26}$$

Therefore, in order to prove (25), it is enough to prove that

$$\begin{aligned} I_1 &= \sum_{n=1}^\infty n^{p\alpha-2}\int_1^\infty P\left(\max_{1\le i\le n}\left\|a_{ni}'' X_i\right\| > n^\alpha x^{1/\nu}\right) dx < \infty, \\ I_2 &= \sum_{n=1}^\infty n^{p\alpha-2}\int_1^\infty P\left(\left\|\sum_{i=1}^n a_{ni}'' Y_{ni}(x)\right\| > \varepsilon n^\alpha x^{1/\nu}\right) dx < \infty. \end{aligned} \tag{27}$$

We first prove that $I_1 < \infty$. Taking $Y = \|X\|/x^{1/\nu}$ in Lemma 2, we have

$$\begin{aligned} I_1 &\le \sum_{n=1}^\infty n^{p\alpha-2}\int_1^\infty \sum_{i=1}^n P\left(\left\|a_{ni}'' X\right\| > n^\alpha x^{1/\nu}\right) dx \\ &= \int_1^\infty \left(\sum_{n=1}^\infty n^{p\alpha-2}\sum_{i=1}^n P\left(\left\|\frac{a_{ni}'' X}{x^{1/\nu}}\right\| > n^\alpha\right)\right) dx \\ &\le C\int_1^\infty E\left\|\frac{X}{x^{1/\nu}}\right\|^p dx = CE\|X\|^p \int_1^\infty x^{-p/\nu} dx \\ &< \infty. \end{aligned} \tag{28}$$

Now, we prove that $I_2 < \infty$. By Lemma 1, $P(\|\sum_{i=1}^n a_{ni}'' X_i\| > \varepsilon n^\alpha) \le 2P(\|\sum_{i=1}^n a_{ni} X_i\| > \varepsilon n^\alpha)$ for any $\varepsilon > 0$. It follows that $n^{-\alpha}\sum_{i=1}^n a_{ni}'' X_i \to 0$ in probability, and hence $n^{-\alpha}E\|\sum_{i=1}^n a_{ni}'' X_i\| \to 0$ by Lemma 3. By Lemma 1 again,

$$\sup_{x\ge1} n^{-\alpha}x^{-1/\nu}E\left\|\sum_{i=1}^n a_{ni}'' Y_{ni}(x)\right\| \le n^{-\alpha}E\left\|\sum_{i=1}^n a_{ni}'' X_i\right\|, \tag{29}$$

and so

$$\lim_{n\to\infty}\sup_{x\ge1} n^{-\alpha}x^{-1/\nu}E\left\|\sum_{i=1}^n a_{ni}'' Y_{ni}(x)\right\| = 0. \tag{30}$$

Hence, to prove $I_2 < \infty$, it is enough to prove that

$$I_2^* = \sum_{n=1}^\infty n^{p\alpha-2}\int_1^\infty P\left(\left\|\sum_{i=1}^m a_{ni}'' Y_{ni}(x)\right\| - E\left\|\sum_{i=1}^n a_{ni}'' Y_{ni}(x)\right\| > \frac{\varepsilon n^\alpha x^{1/\nu}}{2}\right) dx < \infty. \tag{31}$$

By the Markov inequality and Lemma 4, we have, for any $s \ge 2$,

$$\begin{aligned} I_2^* &\le C\sum_{n=1}^\infty n^{p\alpha-s\alpha-2}\int_1^\infty x^{-s/\nu}\left(\sum_{i=1}^n \left|a_{ni}''\right|^2 E\|Y_{ni}(x)\|^2\right)^{s/2} dx \\ &\quad + C\sum_{n=1}^\infty n^{p\alpha-s\alpha-2}\int_1^\infty x^{-s/\nu}\sum_{i=1}^n \left|a_{ni}''\right|^s E\|Y_{ni}(x)\|^s dx \\ &= CI_{21}^* + CI_{22}^*. \end{aligned} \tag{32}$$

If $p \ge 2$, then $E\|X\|^2 < \infty$. Choose s such that $s > \max\{2(p\alpha-1)/(2\alpha-1), q\}$. Then,

$$I_{21}^* \le \left(E\|X\|^2\right)^{s/2}\sum_{n=1}^\infty n^{p\alpha+s/2-s\alpha-2}\int_1^\infty x^{-s/\nu} dx < \infty, \tag{33}$$

and, by taking $Y = \|X\|/x^{1/\nu}$ in Lemma 2,

$$\begin{aligned} I_{22}^{*} &= \int_1^{\infty}\left(\sum_{n=1}^{\infty} n^{p\alpha-s\alpha-2}\sum_{i=1}^{n}\left|a_{ni}''\right|^{s} E\left\|\frac{X}{x^{1/\nu}}\right\|^{s} I\left(\left|\frac{a_{ni}''X}{x^{1/\nu}}\right| > n^{\alpha}\right)\right)dx \\ &\le C\int_1^{\infty} E\left\|\frac{X}{x^{1/\nu}}\right\|^{p} dx = CE\|X\|^{p}\int_1^{\infty} x^{-p/\nu}dx < \infty. \end{aligned} \tag{34}$$

If $p < 2$, then we choose $s = 2$. In this case, $I_{21}^{*} = I_{22}^{*}$. By Lemma 2 again, $I_{21}^{*} = I_{22}^{*} < \infty$.

Formula (22) implies that, for every $x > 1$ and $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right) < \infty, \quad \forall\varepsilon > 0. \tag{35}$$

Hence, (19) holds.

Necessity. Set $a_{ni} = 1$ for all $n \ge 1$ and $1 \le i \le n$. Then, (1) holds for all $q > 0$. In this case, (19) reduces to

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\left\|\sum_{i=1}^{n} X_i\right\| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall\varepsilon > 0, \tag{36}$$

which implies $E\|X\|^{p} < \infty$ by Yang and Wang [28]. So we complete the proof. □

Now, we consider the very interesting case $p\alpha = 1$ in Theorem 5.

Theorem 6. *Let $1/2 < \alpha \le 1$ and $0 < \nu < 1/\alpha$. Let $\{X, X_n,\ n \ge 1\}$ be a sequence of independent and identically distributed B-valued random elements and let $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of real numbers with (1) for some $q > 1/\alpha$. Then, $E\|X\|^{1/\alpha} < \infty$ and $n^{-\alpha}\sum_{i=1}^{n} a_{ni}X_i \to 0$ in probability imply that*

$$\sum_{n=1}^{\infty} n^{-1-\nu\alpha} E\left\{\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| - \varepsilon n^{\alpha}\right\}_{+}^{\nu} < \infty, \quad \forall\varepsilon > 0, \tag{37}$$

and hence

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall\varepsilon > 0. \tag{38}$$

Conversely, if (37) or (38) holds for any array $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ with (1) for some $q > 1/\alpha$, then $E\|X\|^{1/\alpha} < \infty$.

Proof. By the same argument as in the proof of Theorem 5, we can assume that X is symmetric. By the Hölder inequality, we can also assume that $\sup_{n\ge1} n^{-1}\sum_{i=1}^{n}|a_{ni}|^{q} \le 1$ for $1/\alpha < q < 2$. By Proposition 1.1 in Chen and Wang [10], (37) is equivalent to

$$\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right)dx < \infty, \quad \forall\varepsilon > 0. \tag{39}$$

Hence, it is enough to prove that

$$\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right)dx < \infty, \quad \forall\varepsilon > 0 \tag{40}$$

$$\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| \le n^{\alpha}\right)\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right)dx < \infty, \quad \forall\varepsilon > 0. \tag{41}$$

If $\alpha = 1$, then we take s such that $\nu < s < 1$. Then, we have by the Markov inequality, the C_r-inequality, the Hölder inequality, and a standard computation

$$\begin{aligned} &\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right)dx \\ &\quad\le C\sum_{n=1}^{\infty} n^{-1-s\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^{s} E\|X\|^{s} I\left(\|X\| > n^{\alpha}\right) \\ &\quad\cdot\int_1^{\infty} x^{-s/\nu}dx \le C\sum_{n=1}^{\infty} n^{-s\alpha} E\|X\|^{s} I\left(\|X\| > n^{\alpha}\right) \\ &\quad\le CE\|X\|^{1/\alpha} < \infty. \end{aligned} \tag{42}$$

If $1/2 < \alpha < 1$, then we take s such that $\max\{1, \nu\} < s < 1/\alpha$. Note that, by Lemmas 1 and 3, we have

$$\begin{aligned} n^{-\alpha} E\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\| &\le n^{-\alpha} E\left\|\sum_{i=1}^{n} a_{ni}X_i\right\| \\ &\longrightarrow 0 \end{aligned} \tag{43}$$

as $n \to \infty$. Then, by the Markov inequality, Lemma 4, the Hölder inequality, and a standard computation,

$$\begin{aligned} &\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\| > \varepsilon n^{\alpha}x^{1/\nu}\right)dx \le C \\ &\quad+\int_1^{\infty}\sum_{n=1}^{\infty} n^{-1} P\left(\left|\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\| - E\left\|\sum_{i=1}^{n} a_{ni}X_i I\left(\|X_i\| > n^{\alpha}\right)\right\|\right| > \frac{\varepsilon n^{\alpha}x^{1/\nu}}{2}\right)dx \le C \\ &\quad+C\sum_{n=1}^{\infty} n^{-1-s\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^{s} E\|X\|^{s} I\left(\|X\| > n^{\alpha}\right)\int_1^{\infty} x^{-s/\nu}dx \le C + C\sum_{n=1}^{\infty} n^{-s\alpha} E\|X\|^{s} I\left(\|X\| > n^{\alpha}\right) \le C + CE\|X\|^{1/\alpha} < \infty. \end{aligned} \tag{44}$$

Therefore, (40) holds. By Lemmas 1 and 3 again,

$$n^{-\alpha}E\left\|\sum_{i=1}^{n}a_{ni}X_iI\left(\|X_i\|\le n^{\alpha}\right)\right\|\le n^{-\alpha}E\left\|\sum_{i=1}^{n}a_{ni}X_i\right\| \longrightarrow 0 \tag{45}$$

as $n\to\infty$. Then, by the Markov inequality, Lemma 4, and a standard computation,

$$\begin{aligned}&\int_1^{\infty}\sum_{n=1}^{\infty}n^{-1}P\left(\left\|\sum_{i=1}^{n}a_{ni}X_iI\left(\|X_i\|\le n^{\alpha}\right)\right\|>\varepsilon n^{\alpha}x^{1/\nu}\right)dx\le C\\&+\int_1^{\infty}\sum_{n=1}^{\infty}n^{-1}P\left(\left|\left\|\sum_{i=1}^{n}a_{ni}X_iI\left(\|X_i\|\le n^{\alpha}\right)\right\|-E\left\|\sum_{i=1}^{n}a_{ni}X_iI\left(\|X_i\|\le n^{\alpha}\right)\right\|\right|>\frac{\varepsilon n^{\alpha}x^{1/\nu}}{2}\right)dx\le C\\&+C\sum_{n=1}^{\infty}n^{-1-q\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^{q}E\|X\|^{q}I\left(\|X\|\le n^{\alpha}\right)\int_1^{\infty}x^{-q/\nu}dx\le C+C\sum_{n=1}^{\infty}n^{-q\alpha}E\|X\|^{q}I\left(\|X\|\le n^{\alpha}\right)\le C+CE\|X\|^{1/\alpha}<\infty.\end{aligned}\tag{46}$$

Therefore, (41) holds. The rest of the proof is similar to that of Theorem 5. So we complete the proof. □

Remark 7. The condition $q>p$ cannot be weakened to $q>0$. For example, when $1/\alpha<q<p$, set $a_{ni}=0$ if $1\le i\le n-1$ and $a_{nn}=n^{1/q}$. Then, (19) reduces to

$$\sum_{n=1}^{\infty}n^{p\alpha-2}P\left(\left\|n^{1/q}X_n\right\|>\varepsilon n^{\alpha}\right)<\infty,\quad\forall\varepsilon>0,\tag{47}$$

which is equivalent to $E\|X\|^{q(p\alpha-1)/(q\alpha-1)}<\infty$. Note that $q(p\alpha-1)/(q\alpha-1)>p$, so the moment condition $E\|X\|^{q(p\alpha-1)/(q\alpha-1)}<\infty$ is stronger than the moment condition $E\|X\|^{p}<\infty$. When $0<q\le 1/\alpha$, set $a_{ni}=0$ if $1\le i\le n-1$ and $a_{nn}=n^{1/q}$, and set $P(X=-1)=P(X=1)=1/2$. Then, for $0<\varepsilon<1$,

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{p\alpha-2}P\left(\left\|\sum_{i=1}^{n}a_{ni}X_i\right\|>\varepsilon n^{\alpha}\right)\\&\quad=\sum_{n=1}^{\infty}n^{p\alpha-2}P\left(\|X_n\|>\varepsilon n^{\alpha-1/q}\right)=\sum_{n=1}^{\infty}n^{p\alpha-2}=\infty;\end{aligned}\tag{48}$$

that is, (19) does not hold. When $p\alpha=1$ and $q=p=1/\alpha$, Sung [29] studied the complete convergence under NA setup by taking $n^{\alpha}\log^{\beta}n$ instead of n^{α}, where $\beta>0$. But, as far as we know, there are no results when $p\alpha>1$ and $q=p$.

Remark 8. In the proof of Theorem 5, our method uses not only the truncation of random elements but also the truncation of weights. But in the proof of Theorem 6 we only truncate the random elements. Since the proof of Lemma 2 depends on the condition $p\alpha>1$, the method of the proof of Theorem 5 cannot be applied to that of Theorem 6. If we only truncate the random elements in the proof of Theorem 5, then it is hard to estimate I_{22}^{*} when $p\ge 2$ and q is not large enough. So we need two different methods to prove Theorems 5 and 6.

By Theorem 6, we have a strong law of large numbers.

Theorem 9. *Let $1/2<\alpha\le 1$. Let $\{X,X_n,\ n\ge 1\}$ be a sequence of independent and identically distributed B-valued random elements and let $\{a_n,\ n\ge 1\}$ be a sequence of real numbers with*

$$\sup_{n\ge 1}n^{-1}\sum_{i=1}^{n}\left|a_i\right|^{q}<\infty\tag{49}$$

for some $q>1/\alpha$. Then, $E\|X\|^{1/\alpha}<\infty$ and $n^{-\alpha}\sum_{i=1}^{n}a_iX_i\to 0$ in probability imply that

$$n^{-\alpha}\sum_{i=1}^{n}a_iX_i\longrightarrow 0\ a.s.\tag{50}$$

Conversely, if (50) holds for any sequence $\{a_n,\ n\ge 1\}$ with (49) for some $q>1/\alpha$, then $E\|X\|^{1/\alpha}<\infty$.

Proof.

Sufficiency. Since $n^{-\alpha}\sum_{i=1}^{n}a_iX_i\to 0$ in probability, by a standard symmetric argument, we can assume that X is symmetric. Set $a_{ni}=a_i$ for $1\le i\le n$ and $n\ge 1$. Then, (1) holds by (49). Therefore, by Theorem 6,

$$\sum_{n=1}^{\infty}n^{-1}P\left(\left\|\sum_{i=1}^{n}a_iX_i\right\|>\varepsilon n^{\alpha}\right)<\infty,\quad\forall\varepsilon>0.\tag{51}$$

By the Lévy inequality (see Proposition 2.3 in Ledoux and Talagrand [26]),

$$P\left(\max_{1\le m\le n}\left\|\sum_{i=1}^{m}a_iX_i\right\|>\varepsilon n^{\alpha}\right)\le 2P\left(\left\|\sum_{i=1}^{n}a_iX_i\right\|>\varepsilon n^{\alpha}\right),\tag{52}$$

which, together with (51), implies that

$$\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\le m\le n}\left\|\sum_{i=1}^{m}a_iX_i\right\|>\varepsilon n^{\alpha}\right)<\infty,\quad\forall\varepsilon>0.\tag{53}$$

Then, (50) holds by a standard argument (see, e.g., the proof of Theorem 2 in Chen et al. [18]).

Necessity. Set $a_n = 1$ for all $n \geq 1$. Then, (50) reduces to

$$n^{-\alpha}\sum_{i=1}^{n} X_i \longrightarrow 0 \text{ a.s.} \tag{54}$$

Hence, $E\|X\|^{1/\alpha} < \infty$. □

Remark 10. Set $a_n = 1$ when n is not a power of 2, and set $a_n = n^{1/q}$ when n is a power of 2, where $q > 1/\alpha$. Then, (49) holds. Hence, we give a nontrivial example of $\{a_n,\ n \geq 1\}$ satisfying (49).

By Theorems 5 and 6, we have the following corollaries.

Corollary 11. *Let $p \geq 1/\alpha$, $1/2 < \alpha \leq 1$, and $0 < \nu < p$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed B-valued random elements with $EX = 0$. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of real numbers with (1) for some $q > p$ and B is of Rademacher type $1/\alpha$. If $E\|X\|^p < \infty$, then (18) and (19) hold. Conversely, if (18) or (19) holds for any array $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ with (1) for some $q > p$, then $E\|X\|^p < \infty$.*

Proof. Since X is Radon, then, for any $\varepsilon > 0$, there exists a compact subset K_ε of B such that $E\|X\|^{1/\alpha} I(X \notin K_\varepsilon) < \varepsilon$. By a finite dimensional approximation argument (see, e.g., Lemma 2.1 in Chen and Wang [30]), there exists a sequence $\{Y_n,\ n \geq 1\}$ of independent, identically distributed, and bounded random elements with finite values such that, for all $n \geq 1$,

$$\left\|X_n I\left(X_n \in K_\varepsilon\right) - Y_n\right\| \leq \varepsilon. \tag{55}$$

Note that, for any $n \geq 1$,

$$\begin{aligned}
X_n &= \left(X_n I\left(X_n \notin K_\varepsilon\right) - EX_n I\left(X_n \notin K_\varepsilon\right)\right) \\
&\quad + \left[\left(X_n I\left(X_n \in K_\varepsilon\right) - Y_n\right) - E\left(X_n I\left(X_n \in K_\varepsilon\right)\right.\right. \\
&\quad \left.\left. - Y_n\right)\right] + \left(Y_n - EY_n\right), \\
&E\left\|n^{-\alpha}\sum_{i=1}^{n} a_{ni}\left(X_i I\left(X_i \notin K_\varepsilon\right) - EX_i I\left(X_i \notin K_\varepsilon\right)\right)\right\|^{1/\alpha} \\
&\quad \leq C\varepsilon, \\
&E\left\|n^{-\alpha}\sum_{i=1}^{n} a_{ni}\left[\left(X_i I\left(X_i \in K_\varepsilon\right) - Y_i\right)\right.\right. \\
&\quad \left.\left. - E\left(X_i I\left(X_i \in K_\varepsilon\right) - Y_i\right)\right]\right\|^{1/\alpha} \leq C\varepsilon, \\
&E\left\|n^{-\alpha}\sum_{i=1}^{n} a_{ni}\left(Y_i - EY_i\right)\right\|^{t} \leq Cn^{1-\alpha t} \longrightarrow 0,
\end{aligned} \tag{56}$$

where $1/\alpha < t < \min\{2, q\}$. Since $\varepsilon > 0$ was arbitrary,

$$n^{-\alpha}\sum_{i=1}^{n} a_{ni} X_i \longrightarrow 0 \text{ in probability.} \tag{57}$$

Then, the result follows from Theorems 5 and 6. □

Corollary 12. *Let $p \geq 1/\alpha$, $1/2 < \alpha \leq 1$, and $0 < \nu < p$. Let $\{U_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables with uniform distribution on $[0, 1]$. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of real numbers with (1) for some $q > p$. Then,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{(p-\nu)\alpha-2} E\left\{\left(\int_0^1 \left|\sum_{i=1}^{n} a_{ni}\left(I\left(U_i \leq t\right) - t\right)\right|^{1/\alpha} dt\right)^{\alpha}\right. \\
&\quad \left. - \varepsilon n^{\alpha}\right\}_{+}^{\nu} < \infty, \quad \forall \varepsilon > 0,
\end{aligned} \tag{58}$$

and hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\int_0^1 \left|\sum_{i=1}^{n} a_{ni}\left(I\left(U_i \leq t\right) - t\right)\right|^{1/\alpha} dt > \varepsilon n\right) \\
&\quad < \infty, \quad \forall \varepsilon > 0.
\end{aligned} \tag{59}$$

Proof. Since $\{I(U_n \leq t) - t,\ n \geq 1\}$ is a sequence of independent and identically distributed random elements taking values in $L^{1/\alpha}(0, 1)$ and $L^{1/\alpha}(0, 1)$ is of Rademacher type $1/\alpha$, we have the desired result by Corollary 11. □

Competing Interests

The authors declare that they have no competing interests.

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