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Research Article

Multiple Solutions for Nonlinear Navier Boundary Systems Involving $(p_1(x), \ldots, p_n(x))$-Biharmonic Problem

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We improve some results on the existence and multiplicity of solutions for the $(p_1(x), \ldots, p_n(x))$-biharmonic system. Our main results are new. Our approach is based on general variational principle and the theory of the variable exponent Sobolev spaces.

1. Introduction

In this paper, we consider the existence of solutions for the following system:

$$\Delta\left(\left|\Delta u_i\right|^{p_i(x)-2}\Delta u_i\right) = \lambda F_{u_i}\left(x, u_1, u_2, \ldots, u_n\right) \quad \text{in } \Omega, \qquad u_i = \Delta u_i = 0 \quad \text{on } \partial\Omega, \tag{1}$$

for $1 \le i \le n$, where $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded domain with smooth boundary $\partial\Omega$. λ is a positive parameter and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \ldots, t_n) \to F(x, t_1, t_2, \ldots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in \Omega$, F_{t_i} denotes the partial derivative of F with respect to t_i, and F_{t_i} is continuous in $\Omega \times \mathbb{R}^n$, for $i = 1, 2, \ldots, n$. $p_i(x) \in C(\overline{\Omega})$ $(i = 1, 2, \ldots, n)$ with $N/2 < p_i^- := \inf_{x\in\overline{\Omega}} p_i(x) \le p_i^+ := \sup_{x\in\overline{\Omega}} p_i(x) < +\infty$.

In recent years, many authors considered the existence and multiplicity of solutions for some fourth order problems [1–10]. In [4], based on critical point theory, the existence of infinitely many solutions has been established for a class of nonlinear elliptic equations involving the p-biharmonic operator and under Navier boundary value conditions. The $p(x)$-Laplacian operator is more complicated nonlinearities than p-Laplacian; it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero. In [11], based on variational methods, the authors established the existence of an unbounded sequence of weak solutions for a class of differential equations with $p(x)$-Laplacian. In [12], when the nonlinearity f has the subcritical growth and via variational methods [13], the authors obtained the existence of at least one, two, or three weak solutions for a class of differential equations with $p(x)$-Laplacian whenever the parameter λ belongs to a precise positive interval. Recently, the $p(x)$-biharmonic problems have attracted more and more attention; we refer the reader to [11, 14–21]. In [16], El Amrouss and Ourraoui studied the $p(x)$-biharmonic equation with Navier and Neumann boundary condition; the technical approach is based on Ricceri's variational principle and local mountain pass theorem, without Palais-Smale condition. In [20], the authors established the existence of at least three solutions for elliptic systems involving the $(p(x), q(x))$-biharmonic operator. In [15], Allaoui et al. considered the existence of infinitely many solutions for the $(p(x), q(x))$-biharmonic problem by a general Ricceri's variational principle. However, there are rare results on $(p_1(x), \ldots, p_n(x))$-biharmonic problem.

Inspired by the aforementioned papers, our objective is to prove the existence and multiplicity solutions for problem (1); we study problem (1) by using the results as follows.

Theorem A (see [13, 22]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0. \tag{2}$$

Assume that there exist $r > 0$ and $\overline{u} \in X$, with $r < \Phi(\overline{u})$, such that

(i) $\sup_{\Phi(u)\leq r}\Psi(u)/r < \Psi(\overline{u})/\Phi(\overline{u})$;

(ii) *for each $\lambda \in \Lambda_r := (\Phi(\overline{u})/\Psi(\overline{u}), r/\sup_{\Phi(u)\leq r}\Psi(u))$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each compact interval $[\alpha, \beta] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [\alpha, \beta]$, the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0 \tag{3}$$

has at least three solutions in X whose norms are less than ρ.

Theorem B (see [23]). *Let X be a reflexive real Banach space; $\Phi, \Psi : X \to \mathbb{R}$ are two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let one put*

$$\begin{aligned}\varphi(r) &:= \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))}\Psi(v) - \Psi(u)}{r - \Phi(u)},\\ \gamma &:= \liminf_{r\to+\infty}\varphi(r),\\ \delta &:= \liminf_{r\to(\inf_X\Phi)^+}\varphi(r).\end{aligned} \tag{4}$$

Then, one has the following:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b1) *I_λ possesses a global minimum, or*

(b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty}\Phi(u_n) = +\infty$.*

(c) *If $\delta < +\infty$, then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(c1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c2) *there is a sequence of pairwise distinct critical points local minima of I_λ which weakly converges to a global minimum of Φ.*

This paper is organized as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces, some important properties of the $p(x)$-biharmonic operator. In Section 3, we establish the main results.

2. Preliminaries

In order to deal with the $p(x)$-biharmonic problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$ and introduce some notations used in the following.

Denote

$$\begin{aligned}C_+(\overline{\Omega}) &= \{h \in C(\overline{\Omega});\ h(x) > 1,\ \forall x \in \overline{\Omega}\},\\ L^{p(x)}(\Omega) &= \Big\{u :\ u \text{ is a measurable real-valued function},\ \int_\Omega |u|^{p(x)}\,dx < \infty\Big\}.\end{aligned} \tag{5}$$

We introduce a norm on $L^{p(x)}(\Omega)$:

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{6}$$

Then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space; we call it a generalized Lebesgue space.

Proposition 1 (see [24]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$, where $1/p(x) + 1/p^0(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, one has the following Hölder-type inequality:*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^0)^-}\right)|u|_{p(x)}|v|_{p^0(x)}. \tag{7}$$

The variable exponent Sobolev space $W^{m,p(x)}(\Omega)$ is defined by

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq m\}, \tag{8}$$

where α is the multi-index and $|\alpha|$ is the order, m is a positive integer, and it can be equipped with the norm

$$\|u\|_{m,p(x)} = \sum_{|\alpha|\leq m} |D^\alpha u|_{p(x)}. \tag{9}$$

From [24], we know that spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces.

We denote by $W_0^{m,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$.

Let $X := \prod_{i=1}^n (W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega))$ endow with the norm

$$\|(u_1, u_2, \ldots, u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i(x)}, \tag{10}$$

where

$$\|u_i\|_{p_i(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{\Delta u_i}{\lambda}\right|^{p_i(x)} + \left|\frac{\nabla u_i}{\lambda}\right|^{p_i(x)} + \left|\frac{u_i}{\lambda}\right|^{p_i(x)}\right)dx\right\}. \tag{11}$$

Remark 2. According to [25], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta\cdot|_{p(x)}$ in the space $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $|\Delta\cdot|_{p(x)}$, and $\|\cdot\|_{p(x)}$ are equivalent.

Proposition 3 (see [24]). *Put* $\rho(u)=\int_\Omega|\Delta u|^{p(x)}dx$, $\forall u\in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$; *then*

(1) $\|u\|_{p(x)}<1(=1;>1)\Leftrightarrow\rho(u)<1(=1;>1)$;

(2) $\|u\|_{p(x)}\ge1\Rightarrow\|u\|_{p(x)}^{p^-}\le\rho(u)\le\|u\|_{p(x)}^{p^+}$;

(3) $\|u\|_{p(x)}\le1\Rightarrow\|u\|_{p(x)}^{p^+}\le\rho(u)\le\|u\|_{p(x)}^{p^-}$;

(4) $\lim_{k\to+\infty}\|u_k\|_{p(x)}=0\Leftrightarrow\lim_{k\to+\infty}\rho(u_k)=0$.

Proposition 4 (see [20, 26]). *The embedding* $W_0^{1,p_i(x)}\cap W^{2,p_i(x)}\hookrightarrow C(\overline{\Omega})$ *is compact whenever* $p_i^->N/2$, $i=1,2,\dots,n$. *So there is a constant* $C>0$ *such that*

$$C\coloneqq\max\left\{\sup_{u_i\in W^{2,p_i(x)}(\Omega)\cap W_0^{1,p_i(x)}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|}{\|u_i(x)\|_{p_i(x)}}\right\}<+\infty. \tag{12}$$

3. Main Results

Definition 5. One says that $u=(u_1,u_2,\dots,u_n)\in X$ is a weak solution to the system (1) if $u=(u_1,u_2,\dots,u_n)\in X$ and

$$\int_\Omega\sum_{i=1}^n\left(|\Delta u_i(x)|^{p_i(x)-2}\Delta u_i(x)\Delta v_i(x)\right)dx-\lambda\int_\Omega\sum_{i=1}^nF_{u_i}(x,u_1,u_2,\dots,u_n)v_i(x)\,dx=0, \tag{13}$$

for every $v=(v_1,v_2,\dots,v_n)\in X$.

Let $\underline{p}=\min\{p_i^-;\ i=1,2,\dots,n\}$, $\widehat{p}=\max\{p_i^+;\ i=1,2,\dots,n\}$. For $\sigma>0$, one denotes the set

$$Q(\sigma)=\left\{(t_1,t_2,\dots,t_n)\in\mathbb{R}^n,\ \sum_{i=1}^n|t_i|\le\sigma\right\}. \tag{14}$$

Define the function $I_\lambda:X\to\mathbb{R}$ by

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u) \tag{15}$$

for all $u=(u_1,u_2,\dots,u_n)\in X$, where

$$\Phi(u)=\sum_{i=1}^n\int_\Omega\frac{1}{p_i(x)}|\Delta u_i|^{p_i(x)}dx,\qquad \Psi(u)=\int_\Omega F(x,u_1,\dots,u_n)\,dx. \tag{16}$$

Then the operator $\Phi':X\to X^*$, where X^* is the dual space of X, is defined by

$$\Phi'(u)(v)=\sum_{i=1}^n\int_\Omega|\Delta u_i|^{p_i(x)-2}\Delta u_i\Delta v_i\,dx, \tag{17}$$

for $v=(v_1,v_2,\dots,v_n)\in X$.

Proposition 6. *Φ' is continuous, coercive, and strictly monotone. (Φ') admits a continuous inverse on X^*.*

Proof. Since

$$\Phi'(u)(u)=\sum_{i=1}^n\int_\Omega|\Delta u_i|^{p_i(x)}dx\ge\sum_{i=1}^n\min\left\{\|u_i\|_{p_i(x)}^{p_i^+},\|u_i\|_{p_i(x)}^{p_i^-}\right\}, \tag{18}$$

and $p_i^->1$, then Φ' is coercive.

Using the elementary inequalities

$$\left\langle|x|^{p-2}x-|y|^{p-2}y,\ x-y\right\rangle\ge\begin{cases}\dfrac{1}{2^p}|x-y|^p, & p\ge2,\\ C_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & p<2.\end{cases} \tag{19}$$

We deduce that

$$\left\langle\Phi'(u)-\Phi'(v),\ u-v\right\rangle>0, \tag{20}$$

which means that Φ' is strictly monotone. The inverse operator $(\Phi')^{-1}$ of Φ' exists and the continuity of $(\Phi')^{-1}$ can be proved essentially by the same way as the latter part of the proof of [16, Proposition 2.5]; we omit the details. □

From Proposition 6, we see that $\Phi\in C^1(X,\mathbb{R})$. Since X is compactly embedded in $C(\overline{\Omega})\times\cdots\times C(\overline{\Omega})$, we can see that $\Phi:X\to\mathbb{R}$ are sequentially weakly lower semicontinuous.

The functional $\Psi:X\to\mathbb{R}$ is Gateaux differentiable functional and

$$\Psi'(u)(v)=\int_\Omega\sum_{i=1}^nF_{u_i}(x,u_1,u_2,\dots,u_n)v_i(x)\,dx, \tag{21}$$

for $v=(v_1,v_2,\dots,v_n)\in X$. Ψ is sequentially weakly upper semicontinuous. Furthermore, $\Psi':X\to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1,u_2,\dots,u_n)\in X$, let $(u_{1k},u_{2k},\dots,u_{nk})\to(u_1,u_2,\dots,u_n)$ weakly in X as $k\to+\infty$. Then we have $(u_{1k},u_{2k},\dots,u_{nk})$ converges uniformly to $(u_1,u_2,\dots,u_n)$ on Ω as $k\to+\infty$ [27]. Since $F(x,\cdot,\dots,\cdot)$ is C^1 in $\mathbb{R}^n$ for every $x\in\Omega$, so for $1\le i\le n$, $F_{u_i}(x,u_{1k},\dots,u_{nk})\to F_{u_i}(x,u_1,\dots,u_n)$ strongly as $k\to+\infty$, from which follows $\Psi'(x,u_{1k},\dots,u_{nk})\to\Psi'(x,u_1,\dots,u_n)$ strongly as $k\to+\infty$. Thus we have that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [27, Proposition 26.2].

Theorem 7. *Assume the following:*

(A1) $F(x,0,0,\dots,0)=0$ *for* $x\in\Omega$.

(A2) *There exist* $\alpha(x)\in L^1(\Omega)$ *and n positive constants* β_i *with* $\beta_i<p_i^-$ *for* $1\le i\le n$, *such that*

$$0\le F(x,t_1,\dots,t_n)\le\alpha(x)\left(1+\sum_{i=1}^n|t_i|^{\beta_i}\right) \tag{22}$$

for a.e. $x \in \Omega$, $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$, *where* $\mathbb{R}_+^n = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, $t_i \geq 0$, *for* $i = 1, 2, \ldots, n$.

(A3) *There exist* $x_0 \in \Omega$, $0 < R_1 < R_2$, $M_i > 0$, *and* $\delta \in \mathbb{R}$ *with* $\delta \geq (R_2^2 - R_1^2)/2N$ *and* $(2\delta N/(R_2^2 - R_1^2))^{p_i^-}(\pi^{N/2}/\Gamma(1 + N/2))(R_2^N - R_1^N) > 1$ *such that*

$$\int_\Omega \sup_{|t_1|\leq b_1,\ldots,|t_n|\leq b_n} F(x, t_1, \ldots, t_n)\, dx < \frac{\min\left\{(1/p_i^+)(b_i/C)^{p_i^+} : 1 \leq i \leq n\right\}}{\sum_{i=1}^n (1/p_i^-)(2\delta N/(R_2^2 - R_1^2))^{p_i^+}(\pi^{N/2}/\Gamma(1 + N/2))(R_2^N - R_1^N)} \cdot \int_{B(x_0,R_1)} F(x, \delta, \ldots, \delta)\, dx, \tag{23}$$

where $b_i = \min\{C, M_i\}$ *for* $1 \leq i \leq n$.

Then, setting

$$\Lambda := \left(\frac{\sum_{i=1}^n (1/p_i^-)\left(2\delta N/\left(R_2^2 - R_1^2\right)\right)^{p_i^+}\left(\pi^{N/2}/\Gamma(1 + N/2)\right)\left(R_2^N - R_1^N\right)}{\int_{B(x_0,R_1)} F(x, \delta, \ldots, \delta)\, dx}, \frac{\min\left\{(1/p_i^+)(b_i/C)^{p_i^+} : 1 \leq i \leq n\right\}}{\int_\Omega \sup_{|t_1|\leq b_1,\ldots,|t_n|\leq b_n} F(x, t_1, \ldots, t_n)\, dx}\right), \tag{24}$$

for each compact interval $[\alpha, \beta] \subseteq \Lambda$, *there exists a positive real number* ρ *with the following property: for every* $\lambda \in [\alpha, \beta]$, *problem (1) admits at least three weak solutions whose norms are less than* ρ.

Proof. To apply Theorem A to our problem, the functionals Φ, Ψ satisfy the conditions of Theorem A. Now, we show that the hypotheses of Theorem A are fulfilled.

Now we set $u_0 = (0, \ldots, 0)$; from (A1), we have $\Phi(u_0) = \Psi(u_0) = 0$. Let $x_0 \in \Omega$, $0 < R_1 < R_2$, and take

$$w(x) = \begin{cases} 0, & x \in \Omega \setminus B(x_0, R_2), \\ \delta, & x \in B(x_0, R_1), \\ \dfrac{\delta}{R_2^2 - R_1^2}\left(R_2^2 - \sum_{i=1}^N \left(x_i - x_i^0\right)^2\right), & x \in B(x_0, R_2) \setminus B(x_0, R_1), \end{cases}$$
$$\sum_{i=1}^N \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \Omega \setminus B(x_0, R_2) \cup B(x_0, R_1), \\ -\dfrac{2\delta N}{R_2^2 - R_1^2}, & x \in B(x_0, R_2) \setminus B(x_0, R_1). \end{cases} \tag{25}$$

Let $\overline{u} = (w(x), \ldots, w(x))$, and $r = \min\{(1/p_i^+)(b_i/C)^{p_i^+} : 1 \leq i \leq n\}$. Clearly, $\overline{u} \in X$, and we have

$$\begin{aligned} \Phi(\overline{u}) &= \sum_{i=1}^n \int_\Omega \frac{1}{p_i(x)} |\Delta w(x)|^{p_i(x)}\, dx \\ &\geq \sum_{i=1}^n \frac{1}{p_i^+} \int_\Omega |\Delta w|^{p_i(x)}\, dx \\ &\geq \sum_{i=1}^n \frac{1}{p_i^+}\left(\frac{2\delta N}{R_2^2 - R_1^2}\right)^{p_i^-} \frac{\pi^{N/2}}{\Gamma(1 + N/2)}\left(R_2^N - R_1^N\right) \\ &\geq r. \end{aligned} \tag{26}$$

On the other way, when $\Phi(u) \leq r$, we have

$$\sum_{i=1}^n \frac{1}{p_i^+} \int_\Omega |\Delta u_i|^{p_i(x)}\, dx \leq r. \tag{27}$$

So, by Proposition 3, we have

$$\frac{1}{p_i^+} \min\left\{\|u_i\|_{p_i(x)}^{p_i^+}, \|u_i\|_{p_i(x)}^{p_i^-}\right\} \leq r. \tag{28}$$

We deduce that

$$\|u_i\|_{p_i(x)} < \max\left\{\left(rp_i^+\right)^{1/p_i^+}, \left(rp_i^+\right)^{1/p_i^-}\right\}. \tag{29}$$

For $r = \min\{(1/p_i^+)(b_i/C)^{p_i^+} : 1 \leq i \leq n\}$, we have $\|u_i\|_{p_i(x)} \leq b_i/C$ for $1 \leq i \leq n$.

From (12) we have $\max|u_i(x)| \leq C\|u_i\|_{p_i(x)}$; we obtain for all $x \in \Omega$,

$$|u_i(x)| \leq b_i, \quad 1 \leq i \leq n. \tag{30}$$

It follows that, for every $u = (u_1, u_2, \ldots, u_n) \in X$,

$$\begin{aligned} \sup_{\Phi(u)\leq r} \Psi(u) &= \sup_{\Phi(u)\leq r} \int_\Omega F(x, u_1, \ldots, u_n)\, dx \\ &\leq \int_\Omega \sup_{|t_1|\leq b_1,\ldots,|t_n|\leq b_n} F(x, t_1, \ldots, t_n)\, dx. \end{aligned} \tag{31}$$

Since

$$\begin{aligned} \Phi(\overline{u}) &= \sum_{i=1}^{n} \int_\Omega \frac{1}{p_i(x)} |\Delta w(x)|^{p_i(x)} dx \\ &\leq \sum_{i=1}^{n} \frac{1}{p_i^-} \left(\frac{2\delta N}{R_2^2 - R_1^2}\right)^{p_i^+} \frac{\pi^{N/2}}{\Gamma(1+N/2)} \left(R_2^N - R_1^N\right), \\ \Psi(\overline{u}) &> \int_{B(x_0,R_1)} F(x, \delta, \ldots, \delta)\, dx, \end{aligned} \tag{32}$$

therefore, from (A3), we have

$$\begin{aligned} &\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \\ &\quad\leq \frac{\int_\Omega \sup_{|t_1|\leq b_1,\ldots,|t_n|\leq b_n} F(x, t_1, \ldots, t_n)\, dx}{\min\left\{(1/p_i^+)(b_i/C)^{p_i^+} : 1 \leq i \leq n\right\}} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}, \end{aligned} \tag{33}$$

and the assumption (i) of Theorem A is satisfied.

From Proposition 3, we know that if $\|u_i\|_{p_i(x)} < 1$, then

$$\frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^+} \leq \int_\Omega \frac{1}{p_i(x)} |\Delta u_i|^{p_i(x)} dx \leq \frac{1}{p_i^-} \|u_i\|_{p_i(x)}^{p_i^-}, \tag{34}$$

let $k_i > 0$, such that $k_i \geq (1/p_i^+)\|u_i\|_{p_i(x)}^{p_i^-} - (1/p_i^+)\|u_i\|_{p_i(x)}^{p_i^+}$, and then

$$\int_\Omega \frac{1}{p_i(x)} |\Delta u_i|^{p_i(x)} dx \geq \frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^-} - k_i. \tag{35}$$

If $\|u_i\|_{p_i(x)} \geq 1$, then

$$\frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^-} \leq \int_\Omega \frac{1}{p_i(x)} |\Delta u_i|^{p_i(x)} dx \leq \frac{1}{p_i^-} \|u_i\|_{p_i(x)}^{p_i^+}. \tag{36}$$

From (A2), (12), (35), and (36), we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= \sum_{i=1}^{n} \int_\Omega \frac{1}{p_i(x)} |\Delta u_i|^{p_i(x)} dx \\ &\quad - \lambda \int_\Omega F(x, u_1, \ldots, u_n)\, dx \\ &\geq \sum_{i=1}^{n} \left(\frac{1}{p_i^+} \|u_i\|_{p_i(x)}^{p_i^-} - k_i\right) \\ &\quad - \lambda \int_\Omega \alpha(x) \left(1 + \sum_{i=1}^{n} |u_i|^{\beta_i}\right) dx, \end{aligned} \tag{37}$$

noting that $p_i^- > \beta_i$; therefore for $\lambda \geq 0$, we see that

$$\lim_{\|u\|\to+\infty} \Phi(u) + \lambda\Psi(u) = \infty, \tag{38}$$

in particular, for every $\lambda \in \Lambda$. Then the assumption (ii) of Theorem A holds.

Then all the assumptions of Theorem A are fulfilled. By Theorem A, we know that there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that, for any $\lambda \in \Lambda$, problem (1) has at least three weak solutions whose norms are less than ρ. □

Remark 8. Graef et al. [5] studied the problem and established the existence of at least three solutions in the particular case when $p_i(x) = p_i (> 1)$.

Theorem 9. *Assume the following:*

(A4) $F(x, t_1, t_2, \ldots, t_n) \geq 0$, *for each* $(x, t_1, t_2, \ldots, t_n) \in \Omega \times \mathbb{R}_+^n$.

(A5) *There exist* $x_1 \in \Omega$, $0 < R_3 < R_4$ *such that, if one puts*

$$\begin{aligned} \alpha &:= \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,t_2,\ldots,t_n)\in Q(\xi)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\xi^{\underline{p}}}, \\ \beta &:= \limsup_{(t_1,t_2,\ldots,t_n)\to(+\infty,\ldots,+\infty)} \frac{\int_{B(x_1,R_3)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\sum_{i=1}^{n} \left(t_i^{p_i^+}/p_i^-\right)}, \end{aligned} \tag{39}$$

one has

$$\alpha < L\beta, \tag{40}$$

where $L := \min\{L_{p_i^+}, i = 1, 2, \ldots, n\}$,

$$L_{p_i^+} = \frac{\Gamma(1+N/2)}{C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}} \pi^{N/2}} \cdot \frac{1}{R_4^N - R_3^N} \left(\frac{R_4^2 - R_3^2}{2N}\right)^{p_i^+}. \tag{41}$$

Then, for every

$$\lambda \in \Lambda := \frac{1}{C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}}} \left(\frac{1}{L\beta}, \frac{1}{\alpha}\right) \tag{42}$$

problem (1) admits an unbounded sequence of weak solutions.

Proof. To apply Theorem B to our problem, the functionals Φ, Ψ satisfy the conditions of Theorem B. Now, let us verify that $\gamma < +\infty$. Let $\{\xi_k\}$ be a real sequence such that $\xi_k \to +\infty$ as $k \to +\infty$ and

$$\begin{aligned} &\lim_{k\to+\infty} \frac{\int_\Omega \sup_{(t_1,t_2,\ldots,t_n)\in Q(\xi_k)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\xi_k^{\underline{p}}} \\ &\quad= \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,t_2,\ldots,t_n)\in Q(\xi)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\xi^{\underline{p}}} \\ &\quad= \alpha < \infty. \end{aligned} \tag{43}$$

Put $r_k = \xi_k^{\underline{p}} / C^{\underline{p}} (\sum_{i=1}^n (p_i^+)^{1/p_i^-})^{\underline{p}}$ for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\Phi^{-1}((-\infty, r_k)) \\
&\quad = \{u = (u_1, u_2, \dots, u_n) \in X;\ \Phi(u) < r_k\} \\
&\quad \subseteq \left\{u \in X;\ \sum_{i=1}^n \int_\Omega \frac{1}{p_i(x)} |\Delta u_i|^{p_i(x)} dx < r_k \right\} \\
&\quad \subseteq \left\{u \in X;\ \sum_{i=1}^n \frac{1}{p_i^+} \int_\Omega |\Delta u_i|^{p_i(x)} dx < r_k \right\}.
\end{aligned} \tag{44}$$

So, by Proposition 3, we have

$$\frac{1}{p_i^+} \min \left\{ \|u_i\|_{p_i(x)}^{p_i^+}, \|u_i\|_{p_i(x)}^{p_i^-} \right\} < r_k. \tag{45}$$

Hence for k large enough ($r_k > 1$),

$$\|u_i\|_{p_i(x)} < (p_i^+ r_k)^{1/p_i^-}. \tag{46}$$

From (12) we have $\max|u_i(x)| \le C\|u_i\|_{p_i(x)}$; we obtain for all $x \in \Omega$,

$$|u_i(x)| \le C (p_i^+ r_k)^{1/p_i^-}. \tag{47}$$

Thus $\sum_{i=1}^n |u_i(x)| \le \sum_{i=1}^n C(p_i^+ r_k)^{1/p_i^-} \le \xi_k$. Then we have

$$\Phi^{-1}((-\infty, r_k)) \subseteq \left\{ u \in X;\ \sum_{i=1}^n |u_i(x)| \le \xi_k \right\}. \tag{48}$$

Note that $\Phi(0, \dots, 0) = 0$, $\Psi(0, \dots, 0) \ge 0$; then

$$\begin{aligned}
\varphi(r_k) &= \inf_{u \in \Phi^{-1}((-\infty, r_k))} \frac{\sup_{v \in \Phi^{-1}((-\infty, r_k))} \Psi(v) - \Psi(u)}{r_k - \Phi(u)} \\
&\le \frac{\sup_{v \in \Phi^{-1}((-\infty, r_k))} \Psi(v)}{r_k} \le C^{\underline{p}} \left(\sum_{i=1}^n (p_i^+)^{1/p_i^-} \right)^{\underline{p}} \\
&\quad \cdot \frac{\int_\Omega \sup_{(t_1, t_2, \dots, t_n) \in Q(\xi_k)} F(x, t_1, t_2, \dots, t_n) dx}{\xi_k^{\underline{p}}}.
\end{aligned} \tag{49}$$

Therefore, from (A5), we have

$$\begin{aligned}
\gamma &\le \liminf_{k \to +\infty} \varphi(r_k) \le C^{\underline{p}} \left(\sum_{i=1}^n (p_i^+)^{1/p_i^-} \right)^{\underline{p}} \\
&\quad \cdot \lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1, t_2, \dots, t_n) \in Q(\xi_k)} F(x, t_1, \dots, t_n) dx}{\xi_k^{\underline{p}}} \\
&< +\infty.
\end{aligned} \tag{50}$$

It is clear that $\Lambda \subseteq (0, 1/\gamma)$.

For the fixed $\lambda \in \Lambda$, the other step is to show that the functional I_λ has no global minimum. Arguing as in [15], since $1/\lambda < C^{\underline{p}} (\sum_{i=1}^n (p_i^+)^{1/p_i^-})^{\underline{p}} L\beta$, we can consider n positive real sequences $\{\eta_{i,k}\}_{i=1}^n$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^n \eta_{i,k}^2} \to +\infty$ as $k \to +\infty$ and

$$\begin{aligned}
\frac{1}{\lambda} &< \theta < LC^{\underline{p}} \left(\sum_{i=1}^n (p_i^+)^{1/p_i^-} \right)^{\underline{p}} \\
&\quad \cdot \frac{\int_{B(x_1, R_3)} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx}{\sum_{i=1}^n \left(\eta_{i,k}^{p_i^+} / p_i^- \right)}.
\end{aligned} \tag{51}$$

Let $\{u_k(x) = (u_{1k}, u_{2k}, \dots, u_{nk})\}$ be a sequence in X defined by

$$u_{ik}(x) = \begin{cases} 0, & x \in \Omega \setminus B(x_1, R_4), \\ \eta_{i,k}, & x \in B(x_1, R_3), \\ \dfrac{\eta_{i,k}}{R_4^2 - R_3^2} \left(R_4^2 - \sum_{i=1}^N (x_i - x_i^1)^2 \right), & x \in B(x_1, R_4) \setminus B(x_1, R_3), \end{cases} \tag{52}$$

for $1 \le i \le n$. For any fixed $k \in \mathbb{N}$, it is to see that $u_k \in X$, and

$$\begin{aligned}
&\sum_{i=1}^N \frac{\partial^2 u_{ik}(x)}{\partial x_i^2} \\
&\quad = \begin{cases} 0, & x \in \Omega \setminus B(x_1, R_4) \cup B(x_1, R_3), \\ -\dfrac{2\eta_{i,k} N}{R_4^2 - R_3^2}, & x \in B(x_1, R_4) \setminus B(x_1, R_3). \end{cases}
\end{aligned} \tag{53}$$

Then

$$\begin{aligned}
\Phi(u_k) &= \sum_{i=1}^n \int_\Omega \frac{1}{p_i(x)} |\Delta u_{ik}(x)|^{p_i(x)} dx \\
&\le \sum_{i=1}^n \frac{1}{p_i^-} \int_\Omega |\Delta u_{ik}(x)|^{p_i(x)} dx
\end{aligned}$$

$$\leq \sum_{i=1}^{n} \frac{1}{p_i^-} \int_{B(x_1,R_4)\setminus B(x_1,R_3)} \left|\Delta u_{ik}(x)\right|^{p_i(x)} dx$$

$$\leq \sum_{i=1}^{n} \frac{1}{p_i^-} \left(\frac{2\eta_{i,k} N}{R_4^2 - R_3^2}\right)^{p_i^+}$$

$$\cdot \frac{\pi^{N/2}}{\Gamma(1+N/2)} \left(R_4^N - R_3^N\right)$$

$$= \sum_{i=1}^{n} \frac{1}{C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}}} \frac{\eta_{i,k}^{p_i^+}}{p_i^- L_{p_i^+}}. \tag{54}$$

By (A1), we have

$$\Psi(u_k) = \int_\Omega F(x, u_{1k}, \ldots, u_{nk})\, dx$$
$$\geq \int_{B(x_1,R_3)} F(x, \eta_{1,k}, \ldots, \eta_{n,k})\, dx, \tag{55}$$

and combining (51), (54), and (55), we obtain

$$I_\lambda(u_k) = \Phi(u_k) - \lambda \Psi(u_k)$$

$$\leq \frac{1}{C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}}} \sum_{i=1}^{n} \frac{\eta_{i,k}^{p_i^+}}{p_i^- L_{p_i^+}}$$

$$- \lambda \int_{B(x_1,R_3)} F(x, \eta_{1,k}, \ldots, \eta_{n,k})\, dx$$

$$\leq \frac{1}{L C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}}} \sum_{i=1}^{n} \frac{\eta_{i,k}^{p_i^+}}{p_i^-} \tag{56}$$

$$- \lambda \int_{B(x_1,R_3)} F(x, \eta_{1,k}, \ldots, \eta_{n,k})\, dx$$

$$< \frac{1 - \lambda\theta}{L C^{\underline{p}} \left(\sum_{i=1}^{n} (p_i^+)^{1/p_i^-}\right)^{\underline{p}}} \sum_{i=1}^{n} \frac{\eta_{i,k}^{p_i^+}}{p_i^-},$$

for k large enough, so

$$I_\lambda(u_k) = -\infty. \tag{57}$$

Hence, our claim is proved. Since all assumptions of Theorem B case (b) are satisfied, the functional I_λ admits an unbounded sequence $\{u_k = (u_{1k}, \ldots, u_{nk})\} \subset X$ of critical points. This completes the proof of Theorem 9. □

Theorem 10. *Assume that (A1), (A4) hold and consider the following:*

(A6) *There exist $x_1 \in \Omega$, $0 < R_3 < R_4$ such that, if one puts*

$$\alpha^0 := \liminf_{\xi \to 0^+} \frac{\int_\Omega \sup_{(t_1,t_2,\ldots,t_n) \in Q(\xi)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\xi^{\widehat{p}}}, \tag{58}$$

$$\beta^0 := \limsup_{(t_1,\ldots,t_n) \to (0^+,\ldots,0^+)} \frac{\int_{B(x_1,R_3)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\sum_{i=1}^{n} \left(t_i^{p_i^-} / p_i^-\right)},$$

one has

$$\alpha^0 < L_1 \beta^0, \tag{59}$$

where $L_1 := \min\{L_{p_i^-},\ i = 1, 2, \ldots, n\}$,

$$L_{p_i^-} = \frac{\Gamma(1+N/2)}{\left(C \sum_{i=1}^{n} (p_i^+)^{1/p_i^+}\right)^{\widehat{p}} \pi^{N/2}} \frac{1}{R_4^N - R_3^N} \left(\frac{R_4^2 - R_3^2}{2N}\right)^{p_i^-}. \tag{60}$$

Then, for every

$$\lambda \in \Lambda := \frac{1}{\left(C \sum_{i=1}^{n} (p_i^+)^{1/p_i^+}\right)^{\widehat{p}}} \left(\frac{1}{L_1 \beta^0}, \frac{1}{\alpha^0}\right) \tag{61}$$

problem (1) admits a sequence of weak solutions which converges to 0.

Proof. From condition (A1), we have $\min_X \Phi = \Phi(0, \ldots, 0) = 0$, $\Psi(0, \ldots, 0) = 0$.

Let $\{\xi_k\}$ be a real sequence such that $\xi_k \to 0^+$ as $k \to +\infty$ and

$$\lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1,t_2,\ldots,t_n) \in Q(\xi_k)} F(x, t_1, t_2, \ldots, t_n)\, dx}{\xi_k^{\widehat{p}}} = \alpha^0 < \infty. \tag{62}$$

Put $r_k = \xi_k^{\widehat{p}} / (C \sum_{i=1}^{n} (p_i^+)^{1/p_i^+})^{\widehat{p}}$ for all $k \in \mathbb{N}$. Therefore, from (A6), we have

$$\delta \leq \liminf_{k \to +\infty} \varphi(r_k) \leq \left(C \sum_{i=1}^{n} (p_i^+)^{1/p_i^+}\right)^{\widehat{p}} \alpha^0 < +\infty. \tag{63}$$

It is clear that $\Lambda \subseteq (0, 1/\delta)$.

For the fixed $\lambda \in \Lambda$, the other step is to show that the functional I_λ has not a local minimum at zero. Arguing as in [15], since $1/\lambda < (C \sum_{i=1}^{n} (p_i^+)^{1/p_i^+})^{\widehat{p}} L_1 \beta^0$, we can consider

n positive real sequences $\{\eta_{i,k}\}_{i=1}^{n}$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^{n}\eta_{i,k}^2} \to 0$ as $k \to +\infty$ and

$$\frac{1}{\lambda} < \theta < L_1\left(C\sum_{i=1}^{n}\left(p_i^+\right)^{1/p_i^+}\right)^{\widehat{p}} \cdot \frac{\int_{B(x_1,R_3)} F\left(x,\eta_{1,k},\dots,\eta_{n,k}\right)dx}{\sum_{i=1}^{n}\left(\eta_{i,k}^{p_i^-}/p_i^-\right)}. \quad (64)$$

Let $\{u_k(x) = (u_{1k}, u_{2k}, \dots, u_{nk})\}$ be a sequence in X defined by (52):

$$\begin{aligned}\Phi(u_k) &\le \sum_{i=1}^{n}\frac{1}{p_i^-}\int_{\Omega}\left|\Delta u_{ik}(x)\right|^{p_i(x)}dx \\ &\le \sum_{i=1}^{n}\frac{1}{p_i^-}\int_{B(x_1,R_4)\setminus B(x_1,R_3)}\left|\Delta u_{ik}(x)\right|^{p_i(x)}dx \\ &\le \sum_{i=1}^{n}\frac{1}{p_i^-}\left(\frac{2\eta_{i,k}N}{R_4^2-R_3^2}\right)^{p_i^-} \cdot \frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(R_4^N-R_3^N\right) \\ &= \sum_{i=1}^{n}\frac{1}{\left(C\sum_{i=1}^{n}\left(p_i^+\right)^{1/p_i^+}\right)^{\widehat{p}}}\frac{\eta_{i,k}^{p_i^-}}{p_i^-L_{p_i^-}}.\end{aligned} \quad (65)$$

Combining (55), (64), and (65), for k large enough, we have

$$\begin{aligned}I_\lambda(u_k) &= \Phi(u_k) - \lambda\Psi(u_k) \\ &< \frac{1-\lambda\theta}{L_1\left(C\sum_{i=1}^{n}\left(p_i^+\right)^{1/p_i^+}\right)^{\widehat{p}}}\sum_{i=1}^{n}\frac{\eta_{i,k}^{p_i^-}}{p_i^-} < 0 \\ &= I_\lambda(0,\dots,0).\end{aligned} \quad (66)$$

The alternative of Theorem B case (c) ensures the existence of sequence (u_k) of pairwise distinct critical points (local minima) of I_λ which weakly converges to 0. This completes the proof of Theorem 10. □

Example 11. Let $\Omega = ((-1,1))^3$, with p, q, r being three functions defined on Ω by $p(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+3$, $q(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+4$, and $r(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+5$, and consider the increasing sequence of positive real numbers given by

$$\begin{aligned}a_1 &= 2, \\ a_{n+1} &= n!\left(a_n\right)^3 + 2 \qquad (n \ge 1).\end{aligned} \quad (67)$$

Define the function $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x_1,x_2,x_3,t_1,t_2,t_3) = \begin{cases}\left(a_{n+1}\right)^9 e^{1-1/(1-\sum_{i=1}^{3}(t_i-a_{n+1})^2)+x_1^2+x_2^2+x_3^2}, & \text{if } (x_1,x_2,x_3,t_1,t_2,t_3) \in \Omega \times \bigcup_{n\ge1} S\left(\left(a_{n+1},a_{n+1},a_{n+1}\right),1\right), \\ 0, & \text{otherwise,}\end{cases} \quad (68)$$

where $S((a_{n+1},a_{n+1},a_{n+1}),1)$ denotes the open unit ball with center at $(a_{n+1},a_{n+1},a_{n+1})$. It is easy to verify that F is nonnegative function such that $F(\cdot,\cdot,\cdot,t_1,t_2,t_3)$ is continuous in Ω for all $(t_1,t_2,t_3) \in \mathbb{R}^3$. $F(x_1,x_2,x_3,\cdot,\cdot,\cdot)$ is C^1 in $\mathbb{R}^3$ for every $(x_1,x_2,x_3) \in \Omega$. $F(x_1,x_2,x_3,0,0,0) = 0$ for all $(x_1,x_2,x_3) \in \Omega$, for every $\rho > 0$:

$$\begin{aligned}\sup_{|(t_1,t_2,t_3)|<\rho}&\left(\left|F_{t_1}(x_1,x_2,x_3,t_1,t_2,t_3)\right|\right. \\ &+ \left|F_{t_2}(x_1,x_2,x_3,t_1,t_2,t_3)\right| \\ &\left.+ \left|F_{t_3}(x_1,x_2,x_3,t_1,t_2,t_3)\right|\right) \in L^1(\Omega).\end{aligned} \quad (69)$$

The restriction of F on $S((a_{n+1},a_{n+1},a_{n+1}),1)$ attains its maximum in $S((a_{n+1},a_{n+1},a_{n+1}),1)$ and $F(x_1,x_2,x_3,a_{n+1},a_{n+1},a_{n+1}) = (a_{n+1})^9 e^{x_1^2+x_2^2+x_3^2}$.

Hence

$$\limsup_{n\to\infty}\frac{F\left(x_1,x_2,x_3,a_{n+1},a_{n+1},a_{n+1}\right)}{a_{n+1}^6/3+a_{n+1}^7/4+a_{n+1}^8/5} = +\infty. \quad (70)$$

Therefore

$$\begin{aligned}\beta &= \limsup_{t_1\to+\infty,t_2\to+\infty,t_3\to+\infty}\frac{\int_{B(x_1,R_3)}F(x_1,x_2,x_3,t_1,t_2,t_3)\,dx_1dx_2dx_3}{t_1^6/3+t_2^7/4+t_3^8/5} \\ &= \left|B(x_1,R_3)\right|\limsup_{t_1\to+\infty,t_2\to+\infty,t_3\to+\infty}\frac{F(x_1,x_2,x_3,t_1,t_2,t_3)}{t_1^6/3+t_2^7/4+t_3^8/5} = +\infty.\end{aligned} \quad (71)$$

Moreover, by choosing $\xi_n = a_{n+1}-1$, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\sup_{|t_1|+|t_2|+|t_3|\le a_{n+1}-1}&F(x_1,x_2,x_3,t_1,t_2,t_3) \\ &= \left(a_n\right)^9 e^{x_1^2+x_2^2+x_3^2},\end{aligned} \quad (72)$$

then

$$\lim_{n\to+\infty} \frac{\sup_{|t_1|+|t_2|+|t_3|\le a_{n+1}-1} F(x_1,x_2,x_3,t_1,t_2,t_3)}{(a_{n+1}-1)^3} = 0, \quad (73)$$

and so

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t_1|+|t_2|+|t_3|\le \xi} F(x_1,x_2,x_3,t_1,t_2,t_3)}{\xi^3} = 0. \quad (74)$$

Then,

$$\begin{aligned}\alpha &= \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t_1|+|t_2|+|t_3|\le \xi} F(x_1,x_2,x_3,t_1,t_2,t_3)\, dx_1 dx_2 dx_3}{\xi^3} \\ &= |\Omega| \liminf_{\xi\to+\infty} \frac{\sup_{|t_1|+|t_2|+|t_3|\le \xi} F(x_1,x_2,x_3,t_1,t_2,t_3)}{\xi^3} = 0 < L\beta \\ &= +\infty.\end{aligned} \quad (75)$$

Hence, from Theorem 9, for each $\lambda > 0$, the problem

$$\begin{aligned}&\Delta\left(|\Delta u|^{x_1^2+x_2^2+x_3^2+1}\,\Delta u\right) = \lambda F_u(x_1,x_2,x_3,u,v,w) \quad \text{in } \Omega,\\ &\Delta\left(|\Delta v|^{x_1^2+x_2^2+x_3^2+2}\,\Delta v\right) = \lambda F_v(x_1,x_2,x_3,u,v,w) \quad \text{in } \Omega,\\ &\Delta\left(|\Delta w|^{x_1^2+x_2^2+x_3^2+3}\,\Delta w\right) = \lambda F_w(x_1,x_2,x_3,u,v,w) \quad \text{in } \Omega,\\ &u = v = w = \Delta u = \Delta v = \Delta w = 0 \quad \text{on } \partial\Omega,\end{aligned} \quad (76)$$

admits an unbounded sequence of weak solutions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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