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Research Article

Approaching the Discrete Dynamical Systems by means of Skew-Evolution Semiflows

Codruţa Stoica

Department of Mathematics and Computer Science, Aurel Vlaicu University of Arad, 2 Elena Drăgoi Street, 310330 Arad, Romania

Correspondence should be addressed to Codruţa Stoica; codruta.stoica@uav.ro

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The aim of this paper is to highlight current developments and new trends in the stability theory. Due to the outstanding role played in the study of stable, instable, and, respectively, central manifolds, the properties of exponential dichotomy and trichotomy for evolution equations represent two domains of the stability theory with an impressive development. Hence, we intend to construct a framework for an asymptotic approach of these properties for discrete dynamical systems using the associated skew-evolution semiflows. To this aim, we give definitions and characterizations for the properties of exponential stability and instability, and we extend these techniques to obtain a unified study of the properties of exponential dichotomy and trichotomy. The results are underlined by several examples.

1. Introduction

The phenomena of the real world, in domains as economics, biology, or environmental sciences, do not take place continuously, but at certain moments in time. Therefore, a discrete-time approach is required. By means of skew-evolution semiflows, we intend to construct a framework that deepens the analysis of discrete dynamical systems.

Playing an outstanding role in the study of stable and instable manifolds and in approaching several types of differential equations and difference equations, the exponential dichotomy for evolution equations is one of the domains of the stability theory with an impressive development. The dichotomy is a conditional stability, due to the fact that the asymptotic properties of the solutions of a given evolution equation depend on the location of the initial condition in a certain subspace of the phase space. Over the last decades, the classic techniques used to characterize asymptotic properties as stability and instability were generalized towards a natural generalization of the classic concept of dichotomy, the notion of trichotomy. The main idea in the study of trichotomy is to obtain, at any moment, a decomposition of the state space in three subspaces: a stable subspace, an instable one, and a third one called the central manifold. We intend to give several conditions in order to describe the behavior related to the third subspace.

A relevant step in the study of evolution equations is due to Henry, who, in [1], studied the property of dichotomy in the discrete setting, in the spirit of the classic theory initiated by Perron in [2].

A special interest is dedicated to the study of dynamic linear systems by means of associated difference equations, as emphasized by Chow and Leiva in [3] and Latushkin and Schnaubelt in [4].

In [5], the uniform exponential dichotomy of discrete-time linear systems given by difference equations is presented, and the results are applied at the study of dichotomy of evolution families generated by evolution equations. In [6], a characterization of exponential dichotomy for evolution families associated with linear difference systems in terms of admissibility is given.

In [7], characterizations for the uniform exponential stability of variational difference equations are obtained and, in [8], the uniform exponential dichotomy of semigroups of linear operators in terms of the solvability of discrete-time equations over $\mathbb{N}$ is characterized. In [9], new characterizations for the exponential dichotomy of evolution families in terms of solvability of associate difference and integral equations are deduced. In [10, 11], some dichotomous behaviors for variational difference equations are emphasized, such as a new method in the study of exponential dichotomy based on the convergence of some associated series of nonlinear trajectories or characterizations in terms of the admissibility of pairs of sequence spaces over $\mathbb{N}$ with respect to an associated control system.

The notion of trichotomy was introduced in 1976 by Sacker and Sell and studied for the case of linear differential equations in the finite dimensional setting in [12]. For the first time, a sufficient condition for the existence of the trichotomy, in fact a continuous invariant decomposition of the state space $\mathbb{R}^n$ into three subspaces, was given. In the same study, the case of skew-product semiflows was as well approached.

A stronger notion, but still in the finite dimensional case, was introduced by Elaydi and Hajek in [13], the exponential trichotomy for linear and nonlinear differential systems, by means of Lyapunov functions. They prove the fact that the property of exponential trichotomy of a differential system implies the Sacker-Sell type trichotomy. Meanwhile, the notion of invariance to perturbations of the property of trichotomy is given. Thus, in the case of a nonlinear perturbation of a linear exponential trichotomic system, the obtained system preserves the same qualitative behavior as the nonperturbed one.

In [14], a relation between Lyapunov function and exponential trichotomy for the linear equation on time scales is given and, as application, the roughness of exponential trichotomy on time scales is proved. Several asymptotic properties for difference equations were studied in [15–17] and, recently, in [18, 19]. Other asymptotic properties for discrete-time dynamical systems were considered in [20, 21]. A new concept of l^p-trichotomy for linear difference systems is given in [22], as an extension of the l^p-dichotomy and of the exponential trichotomy in l^p spaces.

The notion of skew-evolution semiflow considered in this paper and introduced by us in [23] generalizes the concepts of semigroups, evolution operators, and skew-product semiflows and seems to be more appropriate for the study of the asymptotic behavior of the solutions of evolution equations in the nonuniform case, as they depend on three variables. The applicability of the notion has been studied in [24–28].

The case of stability for skew-evolution semiflows is emphasized in [29], and various concepts for trichotomy are studied in [30]. Some asymptotic properties, as stability, instability, and trichotomy for difference equations in a uniform as well in a nonuniform setting, were studied by us in [31–33].

The following sections outline the structure of this paper. In Section 2, the definitions for evolution semiflows, evolution cocycles, and skew-evolution semiflows are given, featured by examples. In Section 3, we present definitions and characterizations for the properties of exponential growth and decay, respectively, for the exponential stability and instability. The main results are stated in Sections 4 and 5, where we give definitions and characterizations for these asymptotic properties in discrete time for skew-evolution semiflows. Finally, some conclusions are emphasized in Section 6. The list of references allows us to build the overall context in which the discussed problem is placed.

2. Preparatory Notions

Let us consider (X, d) a metric space, V a real or complex Banach space, and $\mathcal{B}(V)$ the family of linear V-valued bounded operators defined on V. The norm of vectors and operators is $\|\cdot\|$. In what follows, we will denote $Y = X \times V$, $T = \{(t, t_0) \in \mathbb{R}^2, t \geq t_0 \geq 0\}$, and $\Delta = \{(m, n) \in \mathbb{N}^2, m \geq n\}$. By I we denoted the identity operator on V.

Definition 1. The mapping $C : T \times Y \to Y$ defined by the relation

$$C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x) v), \tag{1}$$

where $\varphi : T \times X \to X$ has the properties

(s_1) $\varphi(t, t, x) = x$, $\forall (t, x) \in \mathbb{R}_+ \times X$,

(s_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x)$, $\forall (t, s, t_0) \in T$, $x \in X$

and $\Phi : T \times X \to \mathcal{B}(V)$ satisfies

(c_1) $\Phi(t, t, x) = I$, $\forall (t, x) \in \mathbb{R}_+ \times X$,

(c_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x)$, $\forall (t, s, t_0) \in T$, $x \in X$,

is called skew-evolution semiflow on Y.

Remark 2. φ is called *evolution semiflow* and Φ is an *evolution cocycle.*

The approach of asymptotic properties in discrete time is of an obvious importance because the results obtained in this setting can easily be extended in continuous time.

Example 3. Let $V = \mathbb{R}$ be a Banach space and let $X = \mathcal{C}(\mathbb{N}, V)$ be the set of $\mathbb{R}$-valued sequences $(x_n)_{n \geq 0}$. The mapping

$$\begin{gathered} \varphi : \Delta \times X \longrightarrow X, \\ \varphi(m, n, x) = x_n + m - n \end{gathered} \tag{2}$$

is an evolution semiflow on X. We consider the linear system in discrete time:

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}, \tag{3}$$

where $A : \mathbb{N} \to \mathcal{B}(V)$. If we denote

$$E(m+n, n) = \begin{cases} A_{m+n-1} \cdots A_{n+1} A_n, & m > 0 \\ I, & m = 0, \end{cases} \tag{4}$$

where $E : \mathbb{N}^2 \to \mathcal{B}(V)$, then every solution of system (3) satisfies the relation

$$x_{n+p} = E(n+p, n) x_n, \quad \forall n, p \in \mathbb{N}. \tag{5}$$

The pair $C_E = (\varphi, \Phi_E)$ is a skew-evolution semiflow, associated with system (3), where Φ_E is an evolution cocycle over the evolution semiflow φ, given by

$$\Phi_E(m,n,x) = E(m - n + [x_n], [x_n]), \quad \forall (m,n) \in \Delta, \ \forall x \in X, \tag{6}$$

where $[x_k]$ denotes the integer part of the term of rank k.

Example 4. To every skew-evolution semiflow $C = (\varphi, \Phi)$, one can associate the mapping $A_\Phi : \mathbb{N} \to \mathscr{B}(V)$ given by

$$A_\Phi(n) = \Phi(n+1, n, x), \quad n \in \mathbb{N}, \ x \in X, \tag{7}$$

such that $\Phi_{A_\Phi} = \Phi$.

3. Preliminary Results

This section aims to emphasize some asymptotic behaviors, as exponential growth and decay and exponential stability and instability, as a foundation for the main results. We give the definitions of these properties in continuous time and we underline the characterizations in discrete time, as results that play the role of equivalent definitions (see [33]).

Definition 5. A skew-evolution semiflow C has *exponential growth* if there exist mappings $M, \omega : \mathbb{R}_+ \to \mathbb{R}_+^*$, such that

$$\|\Phi(t, t_0, x) v\| \le M(s) e^{\omega(s)(t-s)} \|\Phi(s, t_0, x) v\|, \tag{8}$$

for all $(t,s), (s,t_0) \in T$ and all $(x,v) \in Y$.

Definition 6. A skew-evolution semiflow C is said to be *exponentially stable* if there exist a constant $\nu > 0$ and a mapping $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0, x) v\| \le N(s) e^{-\nu(t-s)} \|\Phi(s, t_0, x) v\|, \tag{9}$$

for all $(t,s), (s,t_0) \in T$ and all $(x,v) \in Y$.

Proposition 7. *A skew-evolution semiflow C with exponential growth is exponentially stable if and only if there exist a constant $\mu > 0$ and a sequence of real numbers $(a_n)_{n\ge 0}$ with the property $a_n \ge 1$, $\forall n \ge 0$, such that*

$$\|\Phi(n, m, x) v\| \le a_m e^{-\mu(n-m)} \|v\|, \tag{10}$$

for all $(n,m) \in \Delta$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. It is obtained immediately if we consider in relation (9) $t = n$ and $s = t_0 = m$, and if we define

$$a_m = N(m), \quad m \in \mathbb{N}, \qquad \mu = \nu > 0, \tag{11}$$

where the existence of $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and of ν is given by Definition 6.

Sufficiency. As a first step, if $t \ge t_0 + 1$, we denote $n = [t]$ and $n_0 = [t_0]$. The following relations hold:

$$\begin{gathered} n \le t < n+1, \\ n_0 \le t_0 < n_0 + 1, \\ n_0 + 1 \le n. \end{gathered} \tag{12}$$

We obtain

$$\begin{aligned} \|\Phi(t, t_0, x) v\| &\le M(n) \\ &\cdot e^{\omega(n)(t-n)} \|\Phi(n, n_0+1, \varphi(n_0+1, t_0, x)) \\ &\cdot \Phi(n_0+1, t_0, x) v\| \le a_n M^2(n) \\ &\cdot e^{2[\omega(n)+\mu]} e^{-\mu(t-t_0)} \|v\|, \end{aligned} \tag{13}$$

for all $(x,v) \in Y$, where functions M and ω are given by Definition 5.

As a second step, for $t \in [t_0, t_0+1)$, we have

$$\begin{aligned} \|\Phi(t, t_0, x) v\| &\le M(t_0) e^{\omega(t_0)(t-t_0)} \|v\| \\ &\le M(t_0) e^{\omega(t_0)+\mu} e^{-\mu(t-t_0)} \|v\|, \end{aligned} \tag{14}$$

for all $(x,v) \in Y$.

Hence, C is exponentially stable. □

Definition 8. A skew-evolution semiflow C has *exponential decay* if there exist mappings $M, \omega : \mathbb{R}_+ \to \mathbb{R}_+^*$, such that

$$\|\Phi(s, t_0, x) v\| \le M(t) e^{\omega(t)(t-s)} \|\Phi(t, t_0, x) v\|, \tag{15}$$

for all $(t,s), (s,t_0) \in T$ and all $(x,v) \in Y$.

Definition 9. A skew-evolution semiflow C is said to be *exponentially instable* if there exist a mapping $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$N(t) \|\Phi(t, t_0, x) v\| \ge e^{\nu(t-s)} \|\Phi(s, t_0, x) v\|, \tag{16}$$

for all $(t,s), (s,t_0) \in T$ and all $(x,v) \in Y$.

Proposition 10. *A skew-evolution semiflow with exponential decay C is exponentially instable if and only if there exist a constant $\mu > 0$ and a sequence of real numbers $(a_n)_{n\ge 0}$ with the property $a_n \ge 1$, $\forall n \ge 0$, such that*

$$\|\Phi(n, n_0, x) v\| \le a_m e^{-\mu(m-n_0)} \|\Phi(m, n_0, x) v\|, \tag{17}$$

for all $(m,n), (n,n_0) \in \Delta$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. We take in relation (16) $t = m$, $s = n$, and $t_0 = n_0$ and we define

$$a_m = N(m), \quad m \in \mathbb{N}, \qquad \mu = \nu > 0, \tag{18}$$

where the existence of function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and of constant ν is given by Definition 9.

Sufficiency. First step, let us take $t \geq t_0 + 1$ and we denote $n = [t]$ and, respectively, $n_0 = [t_0]$. We obtain

$$n \leq t < n + 1, \quad n_0 \leq t_0 < n_0 + 1, \quad n_0 + 1 \leq n. \tag{19}$$

It follows that

$$\begin{aligned} \|\Phi(t, t_0, x) v\| &= \|\Phi(t, n, \varphi(n, n_0 + 1, x)) \\ &\quad \cdot \Phi(n, n_0 + 1, \varphi(n_0 + 1, t_0, x)) \Phi(n_0 + 1, t_0, x) v\| \\ &\geq [M(t)]^{-1} e^{-\omega(t-n)} [M(t)]^{-1} \\ &\quad \cdot e^{-\omega(n_0+1-t_0)} \|\Phi(n, n_0 + 1, x) v\| \geq \frac{[M(t)]^{-2}}{\widetilde{N}} \\ &\quad \cdot e^{-2\omega} e^{\widetilde{\nu}(n-n_0+1)} \|v\| \geq \frac{e^{\widetilde{\nu}}}{[M(t)]^2 \widetilde{N} e^{2\omega}} e^{\widetilde{\nu}(t-t_0)} \|v\|, \end{aligned} \tag{20}$$

for all $(x, v) \in Y$, where the existence of function M and of constant ω is assured by Definition 8.

As a second step, if we consider $t \in [t_0, t_0 + 1)$, we obtain

$$M \|\Phi(t, t_0, x) v\| \geq e^{-\omega(t-t_0)} \|v\| \geq e^{-(\widetilde{\nu}+\omega)} e^{\widetilde{\nu}(t-t_0)} \|v\|, \tag{21}$$

for all $(x, v) \in Y$.

Hence,

$$N \|\Phi(t, t_0, x) v\| \geq e^{\nu(t-t_0)} \|v\|, \tag{22}$$

for all $(t, t_0, x, v) \in T \times Y$, where we have denoted

$$N = M e^{(\widetilde{\nu}+\omega)} + M^2 \widetilde{N} e^{(-\widetilde{\nu}+2\omega)}, \quad \nu = \widetilde{\nu}, \tag{23}$$

which proves the exponential instability of C. □

4. Nonuniform Discrete Dichotomic Behaviors

Definition 11. A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C = (\varphi, \Phi)$ if the following relation

$$P(\varphi(t, s, x)) \Phi(t, s, x) = \Phi(t, s, x) P(x) \tag{24}$$

holds for all $(t, s) \in T$ and all $x \in X$.

Definition 12. Two projectors P_1 and P_2 are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

(d_1) projectors P_1 and P_2 are invariant on Y;

(d_2) for all $x \in X$, the projections $P_1(x)$ and $P_2(x)$ verify the relations

$$P_1(x) + P_2(x) = I, \quad P_1(x) P_2(x) = P_2(x) P_1(x) = 0. \tag{25}$$

Definition 13. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *exponentially dichotomic* if there exist functions $N_1, N_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, constants $\nu_1, \nu_2 > 0$, and two projectors P_1 and P_2 compatible with C such that

$$e^{\nu_1(t-s)} \|\Phi(t, t_0, x) P_1(x) v\| \leq N_1(s) \|\Phi(s, t_0, x) P_1(x) v\| \tag{26}$$

$$e^{\nu_2(t-s)} \|\Phi(s, t_0, x) P_2(x) v\| \leq N_2(t) \|\Phi(t, t_0, x) P_2(x) v\| \tag{27}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Example 14. We denote by $\mathscr{C} = \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$ the set of all continuous functions $x : \mathbb{R}_+ \to \mathbb{R}_+$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$, metrizable relative to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \quad \text{where } d_n(x, y) = \sup_{t \in [0, n]} |x(t) - y(t)|. \tag{28}$$

If $x \in \mathscr{C}$, then for all $t \in \mathbb{R}_+$ we denote $x_t(s) = x(t + s)$, $x_t \in \mathscr{C}$. Let X be the closure in $\mathscr{C}$ of the set $\{f_t, t \in \mathbb{R}_+\}$, where $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a nondecreasing function with the property $\lim_{t\to\infty} f(t) = l > 0$. Then, (X, d) is a metric space and the mapping

$$\varphi : T \times X \longrightarrow X, \quad \varphi(t, s, x)(\tau) = x(t - s + \tau) \tag{29}$$

is an evolution semiflow on X.

Let $V = \mathbb{R}^2$ be endowed with the norm

$$\|(v_1, v_2)\| = |v_1| + |v_2|, \quad v = (v_1, v_2) \in V. \tag{30}$$

The mapping $\Phi : T \times X \to \mathscr{B}(V)$, given by

$$\Phi(t, s, x)(v_1, v_2) = \left(e^{t \sin t - s \sin s - 2t + 2s} v_1, e^{2t - 2s - 3t \cos t + 3s \cos s} v_2\right), \tag{31}$$

is an evolution cocycle over the evolution semiflow φ. We consider the projectors

$$P_1(x)(v_1, v_2) = (v_1, 0), \quad P_2(x)(v_1, v_2) = (0, v_2). \tag{32}$$

As

$$t \sin t - s \sin s - 2t + 2s \leq -t + 3s, \quad \forall (t, s) \in T, \tag{33}$$

we obtain that

$$\|\Phi(t, s, x) P_1(x) v\| \leq e^{2s} e^{-(t-s)} |v_1|, \quad \forall (t, s, x, v) \in T \times Y. \tag{34}$$

Similarly, as

$$2t - 2s - 3t\cos t + 3s\cos s \geq -t - 5s, \quad \forall (t,s) \in T \tag{35}$$

it follows that

$$e^{6t}\left\|\Phi(t,s,x)P_2(x)v\right\| \geq e^{5(t-s)}\left|v_2\right|, \quad \forall (t,s,x,v) \in T \times Y. \tag{36}$$

The skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic with characteristics

$$N(u) = e^{6u}, \quad \nu = 1. \tag{37}$$

In what follows, let us denote $C_k(t,s,x,v) = (\varphi(t,s,x), \Phi(t,t_0,x)P_k(x)v)$, where $(t,t_0,x,v) \in T \times Y$ and $k \in \{1,2\}$.

In discrete time, we will describe the property of exponential dichotomy as given in the next proposition.

Proposition 15. *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic if and only if there exist two projectors $\{P_k\}_{k\in\{1,2\}}$, compatible with C, constants $\nu_1 \leq 0 \leq \nu_2$, and a sequence of real positive numbers $(a_n)_{n\geq 0}$ such that*

(d_1')

$$\left\|\Phi(m,n,x)P_1(x)v\right\| \leq a_n\left\|P_1(x)v\right\|e^{\nu_1(m-n)} \tag{38}$$

(d_2')

$$\left\|P_2(x)v\right\| \leq a_m\left\|\Phi(m,n,x)P_2(x)v\right\|e^{-\nu_2(m-n)} \tag{39}$$

for all $(m,n) \in \Delta$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. If we consider for C_1 in relation (26) of Definition 13 $t = m$ and $s = t_0 = n$ and if we define

$$a_n = N(n), \quad n \in \mathbb{N}, \tag{40}$$

relation (d_1') is obtained.

Statement $(d_2)'$ results from Definition 13 for C_2 if we consider in relation (27) $t = m$ and $s = t_0 = n$ and

$$a_m = N(m), \quad m \in \mathbb{N}. \tag{41}$$

Sufficiency. It is obtained by means of Proposition 7 for C_1 and, respectively, of Proposition 10 for C_2.

Hence, C is exponentially dichotomic. □

Theorem 16. *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic if and only if there exist two projectors $\{P_k\}_{k\in\{1,2\}}$, compatible with C such that*

(ed_1') *there exist a constant $\rho_1 > 0$ and a sequence of real positive numbers $(\alpha_n)_{n\geq 0}$ such that*

$$\sum_{k=n}^{m} e^{\rho_1(k-n)}\left\|\Phi(k,n,x)P_1(x)v\right\| \leq \alpha_n\left\|P_1(x)v\right\| \tag{42}$$

(ed_2') *there exist a constant $\rho_2 < 0$ and a sequence of real positive numbers $(\beta_n)_{n\geq 0}$ such that*

$$\sum_{k=n}^{m} e^{-\rho_2(m-k)}\left\|\Phi(k,n,x)P_2(x)v\right\| \leq \beta_m\left\|\Phi(m,n,x)P_2(x)v\right\| \tag{43}$$

for all $(m,p), (p,n) \in \Delta$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. According to Proposition 7, there exist a constant $\nu > 0$ and a sequence of real numbers $(a_n)_{n\geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$. We obtain for $\rho = \nu/2 > 0$ and according to Proposition 7

$$\begin{aligned}\sum_{k=n}^{m} e^{\rho(k-n)}\left\|\Phi(k,n,x)v\right\| &\leq a_n\sum_{k=n}^{m} e^{\rho(k-n)}e^{-\nu(k-n)}\left\|\Phi(n,n,x)v\right\| \\ &= a_n\left\|v\right\|\sum_{k=n}^{m} e^{-(\nu/2)(k-n)} \leq \alpha_n\left\|v\right\|, \\ &\qquad \forall m,n \in \Delta,\ \forall (x,v) \in Y,\end{aligned} \tag{44}$$

where we have denoted

$$\alpha_n = a_n e^{\nu/2}, \quad n \in \mathbb{N}. \tag{45}$$

By Proposition 10, there exist a constant $\nu > 0$ and a sequence of real numbers $(a_n)_{n\geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$. We obtain for $\rho = -\nu/2 > 0$

$$\begin{aligned}\sum_{k=n}^{m} e^{-\rho(m-k)}\left\|\Phi(k,n,x)v\right\| &\leq a_m\sum_{k=n}^{m} e^{-\rho(m-k)}e^{-\nu(m-k)}\left\|\Phi(m,n,x)v\right\| \\ &= a_m\left\|\Phi(m,n,x)v\right\|\sum_{k=n}^{m} e^{-(\nu/2)(k-n)} \\ &\leq \alpha_m\left\|\Phi(m,n,x)v\right\|,\end{aligned} \tag{46}$$

for all $(m,n) \in \Delta$ and all $(x,v) \in Y$, where we have denoted

$$\alpha_m = \frac{1}{1-e^{\rho}}a_m, \quad m \in \mathbb{N}. \tag{47}$$

Sufficiency. Let $t \geq t_0 + 1$, $t_0 \geq 0$. We define $n = [t]$ and $n_0 = [t_0]$. We consider $C_{-\rho} = (\varphi, \Phi_{-\rho})$, where we define $\Phi_{-\rho}(t, t_0, x) = e^{\rho(t-t_0)}\Phi(t, t_0, x)$. We have that

$$\begin{aligned} &\left\|\Phi_{-\rho}(t,t_0,x)v\right\| \\ &\quad = \left\|\Phi_{-\rho}(t,n,\varphi(n,t_0,x))\Phi_{-\rho}(n,t_0,x)v\right\| \\ &\quad \leq M(n)e^{\omega(t-n)}\left\|\Phi_{-\rho}(n,t_0,x)v\right\| \\ &\quad \leq M(n)e^{\omega}\left\|\Phi_{-\rho}(n,t_0,x)v\right\|, \end{aligned} \tag{48}$$

for all $(x, v) \in Y$, where M and ω are given by Definition 5. We obtain further for $m \geq n$

$$\begin{aligned} &\int_{t_0+1}^{t}\left\|\Phi_{-\rho}(\tau,t_0,x)v\right\|d\tau \\ &\quad \leq \sum_{k=[t_0]+1}^{m} M(n)e^{\omega}\left\|\Phi_{-\rho}(k,t_0,x)v\right\| \leq \beta_n\|v\|, \end{aligned} \tag{49}$$

for all $(x, v) \in Y$, where $\beta_n = M(n)\alpha_n e^{\omega}$. Then, there exist $N \geq 1$ and $\nu > 0$ such that

$$\begin{aligned} &\left\|\Phi_{-\rho}(t,t_0,x)v\right\| \leq Ne^{-\nu(t-t_0)}\|v\|, \\ &\qquad\qquad \forall t \geq t_0 + 1,\ \forall (x,v) \in Y. \end{aligned} \tag{50}$$

On the other hand, for $t \in [t_0, t_0 + 1)$, we have

$$\begin{aligned} &\left\|\Phi_{-\rho}(t,t_0,x)v\right\| \leq Me^{\omega(t-t_0)}\|v\| \leq Me^{\omega}\|v\|, \\ &\qquad\qquad \forall (x,v) \in Y. \end{aligned} \tag{51}$$

We obtain that $C_{-\rho}$ is stable, where

$$\begin{aligned} &\Phi_{-\rho}(m,n,x) = e^{\rho(m-n)}\|\Phi(m,n,x)\|, \\ &\qquad\qquad (m,n,x) \in \Delta \times X. \end{aligned} \tag{52}$$

Hence, there exists a sequence $(a_n)_{n\geq 0}$ with the property $a_n \geq 1$, $\forall n \geq 0$, such that

$$\begin{aligned} &e^{\rho(m-n)}\|\Phi(m,n,x)v\| \leq a_n\|v\|, \\ &\qquad\qquad \forall (m,n,x,v) \in \Delta \times Y, \end{aligned} \tag{53}$$

which implies the exponential stability of C_1 and ends the proof.

According to the hypothesis, if we consider $k = n$ we obtain

$$e^{-\rho(m-n)}\|\Phi(n,n,x)v\| \leq a_m\|\Phi(m,n,x)v\|, \tag{54}$$

for all $(m, n, x, v) \in \Delta \times Y$, which implies the exponential instability of C_2 and ends the proof. □

5. Nonuniform Discrete Trichotomic Behaviors

Definition 17. Three projectors $\{P_k\}_{k\in\{1,2,3\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

(t_1) each projector P_k, $k \in \{1, 2, 3\}$ is invariant on Y;

(t_2) for all $x \in X$, the projections $P_0(x)$, $P_1(x)$, and $P_2(x)$ verify the relations

$$\begin{aligned} &P_1(x) + P_2(x) + P_3(x) = I, \\ &\qquad P_i(x)P_j(x) = 0, \quad \forall i, j \in \{1,2,3\},\ i \neq j. \end{aligned} \tag{55}$$

Definition 18. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *exponentially trichotomic* if there exist the functions N_1, N_2, N_3, $N_4 : \mathbb{R}_+ \to \mathbb{R}_+^*$, constants ν_1, ν_2, ν_3, and ν_4 with the properties

$$\nu_1 \leq \nu_2 \leq 0 \leq \nu_3 \leq \nu_4, \tag{56}$$

and three projectors P_1, P_2, and P_3 compatible with C such that

$$\begin{aligned} &\left\|\Phi(t,t_0,x)P_1(x)v\right\| \\ &\quad \leq N_1(s)\left\|\Phi(s,t_0,x)P_1(x)v\right\|e^{\nu_1(t-s)} \end{aligned} \tag{57}$$

$$\begin{aligned} &N_4(t)\left\|\Phi(t,t_0,x)P_2(x)v\right\| \\ &\quad \geq \left\|\Phi(s,t_0,x)P_2(x)v\right\|e^{\nu_4(t-s)} \end{aligned} \tag{58}$$

$$\begin{aligned} &N_2(t)\left\|\Phi(t,t_0,x)P_3(x)v\right\| \\ &\quad \geq \left\|\Phi(s,t_0,x)P_3(x)v\right\|e^{\nu_2(t-s)} \end{aligned} \tag{59}$$

$$\begin{aligned} &\left\|\Phi(t,t_0,x)P_3(x)v\right\| \\ &\quad \leq N_3(s)\left\|\Phi(s,t_0,x)P_3(x)v\right\|e^{\nu_3(t-s)} \end{aligned} \tag{60}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Example 19. We consider the evolution semiflow φ defined in Example 14. Let $V = \mathbb{R}^3$ be endowed with the norm

$$\left\|(v_1, v_2, v_3)\right\| = |v_1| + |v_2| + |v_3|. \tag{61}$$

The mapping $\Phi : T \times X \to \mathscr{B}(V)$, given by

$$\begin{aligned} \Phi(t,s,x)v = &\Big(e^{-2(t-s)x(0)+\int_0^t x(\tau)d\tau}v_1, e^{t-s+\int_0^t x(\tau)d\tau}v_2, \\ &e^{-(t-s)x(0)+2\int_0^t x(\tau)d\tau}v_3\Big), \end{aligned} \tag{62}$$

is an evolution cocycle. Then, $C = (\varphi, \Phi)$ is a skew-evolution semiflow.

We consider the projections

$$\begin{aligned} P_1(x)(v) &= (v_1, 0, 0), \\ P_2(x)(v) &= (0, v_2, 0), \\ P_3(x)(v) &= (0, 0, v_3). \end{aligned} \tag{63}$$

The skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic with characteristics

$$\begin{aligned} \nu_1 &= \nu_2 = -x(0), \\ \nu_3 &= x(0), \\ \nu_4 &= 1, \\ N_1(u) &= e^{ux(0)}, \\ N_2(u) &= e^{-2lu}, \\ N_3(u) &= e^{2ux(0)}, \\ N_4(u) &= e^{-lu}. \end{aligned} \tag{64}$$

As in the case of dichotomy, let $C_k(t,s,x,v) = (\varphi(t,s,x), \Phi(t,t_0,x)P_k(x)v)$, where $(t,t_0,x,v) \in T \times Y$ and $k \in \{1,2,3\}$.

In discrete time, the trichotomy of a skew-evolution semiflow can be described as in the next proposition.

Proposition 20. *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic if and only if there exist three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C, constants ν_1, ν_2, ν_3, and ν_4 with the property $\nu_1 \le \nu_2 \le 0 \le \nu_3 \le \nu_4$, and a sequence of positive real numbers $(a_n)_{n\ge 0}$ such that*

(t_1)

$$\begin{aligned} &\left\|\Phi(m,n,x)P_1(x)v\right\| \\ &\quad \le a_p \left\|\Phi(p,n,x)P_1(x)v\right\| e^{\nu_1(m-p)}, \end{aligned} \tag{65}$$

(t_2)

$$\begin{aligned} &\left\|\Phi(p,n,x)P_2(x)v\right\| \\ &\quad \le a_m \left\|\Phi(m,n,x)P_2(x)v\right\| e^{-\nu_4(m-p)}, \end{aligned} \tag{66}$$

(t_3)

$$\begin{aligned} &\left\|\Phi(p,n,x)P_3(x)v\right\| \\ &\quad \le a_m \left\|\Phi(m,n,x)P_3(x)v\right\| e^{-\nu_2(m-p)}, \end{aligned} \tag{67}$$

(t_4)

$$\begin{aligned} &a_p \left\|\Phi(p,n,x)P_3(x)v\right\| \\ &\quad \ge \left\|\Phi(m,n,x)P_3(x)v\right\| e^{-\nu_3(m-p)} \end{aligned} \tag{68}$$

for all $(m,p), (p,n) \in \Delta$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. (t_1) is obtained if we consider for C_1 in relation (9) of Definition 6 $t = n$ and $s = t_0 = m$ and if we define

$$\begin{aligned} a_p &= N(p), \quad p \in \mathbb{N}, \\ \nu_1 &= -\nu < 0. \end{aligned} \tag{69}$$

(t_2) follows according to Definition 9 for C_2 if we consider in relation (16) $t = m$, $s = n$, and $t_0 = n_0$ and

$$\begin{aligned} a_m &= N(m), \quad m \in \mathbb{N}, \\ \nu_4 &= \nu > 0. \end{aligned} \tag{70}$$

(t_3) is obtained for C_3 out of relation (8) of Definition 5 for $t = m$, $s = p$, and $t_0 = n$ and if we define

$$\begin{aligned} a_m &= M(m), \\ \nu_2 &= -\omega(m) < 0, \\ & m \in \mathbb{N}. \end{aligned} \tag{71}$$

(t_4) follows for C_3 from relation (15) of Definition 8 for $t = m$, $s = p$, and $t_0 = n$ and if we consider

$$\begin{aligned} a_p &= M(p), \\ \nu_3 &= \omega(p) > 0, \\ & p \in \mathbb{N}. \end{aligned} \tag{72}$$

Sufficiency. Let $t \ge t_0 + 1$. We denote $n = [t]$ and $n_0 = [t_0]$ and we obtained the relations

$$\begin{aligned} n &\le t < n+1, \\ n_0 &\le t_0 < n_0 + 1, \\ n_0 + 1 &\le n. \end{aligned} \tag{73}$$

According to (t_1), we have

$$\begin{aligned} &\left\|\Phi(t,t_0,x)P_1(x)v\right\| \le M(n) \\ &\quad \cdot e^{\omega(n)(t-n)} \left\|\Phi(n, n_0+1, \varphi(n_0+1, t_0, x)) \right. \\ &\quad \left. \cdot \Phi(n_0+1, t_0, x) P_1(x) v\right\| \le a_n M^2(n) \\ &\quad \cdot e^{2[\omega(n)+\mu]} e^{-\mu(t-t_0)} \left\|P_1(x)v\right\|, \end{aligned} \tag{74}$$

for all $(x,v) \in Y$, where functions M and ω are given as in Definition 5.

For $t \in [t_0, t_0+1)$, we have

$$\begin{aligned} &\left\|\Phi(t,t_0,x)P_1(x)v\right\| \le M(t_0) e^{\omega(t_0)(t-t_0)} \left\|P_1(x)v\right\| \\ &\quad \le M(t_0) e^{\omega(t_0)+\mu} e^{-\mu(t-t_0)} \left\|P_1(x)v\right\|, \end{aligned} \tag{75}$$

for all $(x,v) \in Y$. Hence, relation (57) is obtained.

Let $t \ge t_0 + 1$ and $n = [t]$ and, respectively, $n_0 = [t_0]$. It follows that

$$\begin{aligned} n &\le t < n+1, \\ n_0 &\le t_0 < n_0 + 1, \\ n_0 + 1 &\le n. \end{aligned} \tag{76}$$

From (t_2), it is obtained that

$$\begin{aligned}\|\Phi(t,t_0,x)P_2(x)v\| &= \|\Phi(t,n,\varphi(n,n_0+1,x)) \\ &\cdot \Phi(n,n_0+1,\varphi(n_0+1,t_0,x))\Phi(n_0+1,t_0,x) \\ &\cdot P_2(x)v\| \geq [M(t)]^{-1}e^{-\omega(t-n)}[M(t)]^{-1} \\ &\cdot e^{-\omega(n_0+1-t_0)}\|\Phi(n,n_0+1,x)P_2(x)v\| \\ &\geq \frac{[M(t)]^{-2}}{\widetilde{N}}e^{-2\omega}e^{\widetilde{\nu}(n-n_0+1)}\|P_2(x)v\| \\ &\geq \frac{e^{\widetilde{\nu}}}{[M(t)]^2\widetilde{N}e^{2\omega}}e^{\widetilde{\nu}(t-t_0)}\|P_2(x)v\|,\end{aligned} \tag{77}$$

for all $(x,v)\in Y$, where M and ω are given by Definition 8.

For $t\in[t_0,t_0+1)$, we have

$$\begin{aligned}M\|\Phi(t,t_0,x)P_2(x)v\| &\geq e^{-\omega(t-t_0)}\|P_2(x)v\| \\ &\geq e^{-(\widetilde{\nu}+\omega)}e^{\widetilde{\nu}(t-t_0)}\|P_2(x)v\|,\end{aligned} \tag{78}$$

for all $(x,v)\in Y$. It follows that

$$N\|\Phi(t,t_0,x)P_2(x)v\| \geq e^{\nu(t-t_0)}\|P_2(x)v\|, \tag{79}$$

for all $(t,t_0,x,v)\in T\times Y$, where we have denoted

$$N = Me^{(\widetilde{\nu}+\omega)} + M^2\widetilde{N}e^{(-\widetilde{\nu}+2\omega)}, \quad \nu=\widetilde{\nu} \tag{80}$$

and which implies relation (58).

By a similar reasoning, from (t_3) relation (59) is obtained, and from (t_4) relation (60) follows.

Hence, the skew-evolution semiflow C is exponentially trichotomic. □

Some characterizations in discrete time for the exponential trichotomy for skew-evolution semiflows are given in what follows.

Theorem 21. *A skew-evolution semiflow $C=(\varphi,\Phi)$ is exponentially trichotomic if and only if there exist three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that C_1 has exponential growth, and C_2 has exponential decay and such that the following relations hold:*

(t_1') *there exist a constant $\rho_1>0$ and a sequence of positive real numbers $(\alpha_n)_{n\geq 0}$ such that*

$$\sum_{k=n}^{m}e^{\rho_1(k-n)}\|\Phi(k,n,x)P_1(x)v\| \leq \alpha_n\|P_1(x)v\|, \tag{81}$$

(t_2') *there exist a constant $\rho_2>0$ and a sequence of positive real numbers $(\beta_n)_{n\geq 0}$ such that*

$$\begin{aligned}&\sum_{k=n}^{m}e^{-\rho_2(k-n)}\|\Phi(k,n,x)P_2(x)v\| \\ &\quad\leq \beta_m e^{-\rho_2(m-n)}\|\Phi(m,n,x)P_2(x)v\|,\end{aligned} \tag{82}$$

(t_3') *there exist a constant $\rho_3>0$ and a sequence of positive real numbers $(\gamma_n)_{n\geq 0}$ such that*

$$\begin{aligned}&\sum_{k=p}^{m}e^{-\rho_3(k-n)}\|\Phi(k,n,x)P_3(x)v\| \\ &\quad\leq \gamma_n e^{-\rho_3(p-n)}\|\Phi(p,n,x)P_3(x)v\|,\end{aligned} \tag{83}$$

(t_4') *there exist a constant $\rho_4>0$ and a sequence of positive real numbers $(\delta_n)_{n\geq 0}$ such that*

$$\begin{aligned}&\sum_{k=p}^{m}e^{\rho_4(k-n)}\|\Phi(k,n,x)P_3(x)v\| \\ &\quad\leq \delta_m e^{\rho_4(m-n)}\|\Phi(m,n,x)P_3(x)v\|\end{aligned} \tag{84}$$

for all $(m,p),(p,n)\in\Delta$ and all $(x,v)\in Y$.

Proof. We have the following.

$(t_1)\Leftrightarrow(t_1')$ *Necessity*. As C is exponentially trichotomic, Proposition 20 assures the existence of a projector P, of a constant $\nu_1\leq 0$, and of a sequence of positive real numbers $(a_n)_{n\geq 0}$ such (t_1) holds. Let $P_1=P$. If we consider

$$\rho_1 = -\frac{\nu_1}{2} > 0, \tag{85}$$

we obtain

$$\begin{aligned}&\sum_{k=n}^{m}e^{\rho_1(k-n)}\|\Phi(k,n,x)P_1(x)v\| \\ &\quad\leq a_n\sum_{k=n}^{m}e^{\rho_1(k-n)}e^{\nu_1(k-n)}\|\Phi(n,n,x)P_1(x)v\| \\ &\quad= a_n\|P_1(x)v\|\sum_{k=n}^{m}e^{(\rho_1+\nu_1)(k-n)} \leq \alpha_n\|P_1(x)v\|, \\ &\qquad\qquad \forall m,n\in\mathbb{N},\ \forall(x,v)\in Y,\end{aligned} \tag{86}$$

where we have denoted

$$\alpha_n = a_n e^{-\nu_1/2}, \quad n\in\mathbb{N}. \tag{87}$$

Sufficiency. As C_1 has exponential growth, there exist constants $M\geq 1$ and $r>1$ such that relation

$$\|\Phi(n+p,n,x)v\| \leq Mr^p\|v\|, \tag{88}$$

holds for all $n,p\in\mathbb{N}$ and all $(x,v)\in Y$. If we denote $\omega=\ln r>0$, the inequality can be written as follows:

$$\begin{aligned}&\|\Phi(m,n,x)\| \leq Me^{\omega(m-k)}\|\Phi(k,n,x)\|, \\ &\qquad\forall(m,k),(k,n)\in\Delta,\ x\in X.\end{aligned} \tag{89}$$

We consider successively the $m-n+1$ relations. By denoting $P=P_1$, we obtain

$$\begin{aligned}&\|\Phi(m,n,x)P(x)v\| \\ &\quad\leq \frac{Me^{\omega(m-n)}}{1+e^{\rho_1}+\cdots+e^{\rho_1(m-n)}}\alpha_n\|P(x)v\|,\end{aligned} \tag{90}$$

for all $(m,n) \in \Delta$ and all $(x,v) \in Y$. If we define the constant

$$\nu_1 = \omega - \rho_1 \quad \text{for } \omega < \rho_1 \tag{91}$$

and the sequence of nonnegative real numbers

$$a_n = M\alpha_n, \quad n \in \mathbb{N}, \tag{92}$$

relation (t_1) is obtained.

Similarly, the other equivalences can also be proved. □

In order to characterize the exponential trichotomy by means of four projectors, we give the next definition.

Definition 22. Four invariant projectors $\{R_k\}_{k\in\{1,2,3,4\}}$ that satisfy for all $(x,v) \in Y$ the following relations

(pc_1') $R_1(x) + R_3(x) = R_2(x) + R_4(x) = I$,

(pc_2') $R_1(x)R_2(x) = R_2(x)R_1(x) = 0$ and $R_3(x)R_4(x) = R_4(x)R_3(x)$,

(pc_3') $\|[R_1(x) + R_2(x)]v\|^2 = \|R_1(x)v\|^2 + \|R_2(x)v\|^2$,

(pc_4') $\|[R_1(x) + R_3(x)R_4(x)]v\|^2 = \|R_1(x)v\|^2 + \|R_3(x)R_4(x)v\|^2$,

(pc_5') $\|[R_2(x) + R_3(x)R_4(x)]v\|^2 = \|R_2(x)v\|^2 + \|R_3(x)R_4(x)v\|^2$,

are called *compatible* with the skew-evolution semiflow C.

Theorem 23. *A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially trichotomic if and only if there exist four projectors $\{R_k\}_{k\in\{1,2,3,4\}}$ compatible with C, constants $\mu > \nu > 0$, and a sequence of positive real numbers $(\alpha_n)_{n\geq 0}$ such that*

(t_1'')

$$\|\Phi(m+p,m,x)R_1(x)v\| \leq \alpha_m \|R_1(x)v\| e^{-\nu p}, \tag{93}$$

(t_2'')

$$\|R_2(x)v\| \leq \alpha_p \|\Phi(m+p,m,x)R_2(x)v\| e^{-\mu p}, \tag{94}$$

(t_3'')

$$\|R_3(x)v\| \leq \alpha_p \|\Phi(m+p,m,x)R_3(x)v\| e^{\nu p}, \tag{95}$$

(t_4'')

$$\|\Phi(m+p,m,x)R_4(x)v\| \leq \alpha_m \|R_4(x)v\| e^{\mu p} \tag{96}$$

for all $m, p \in \mathbb{N}$ and all $(x,v) \in Y$.

Proof. We have the following.

Necessity. As C is exponentially trichotomic, according to Proposition 20 there exist three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C, constants $\nu_1 \leq \nu_2 \leq 0 \leq \nu_3 \leq \nu_4$, and a sequence of positive real numbers $(a_n)_{n\geq 0}$ such that relations (65)–(68) hold.

We will define the projectors

$$\begin{aligned} R_1 &= P_1, \\ R_2 &= P_2, \\ R_3 &= I - P_1, \\ R_4 &= I - P_2, \end{aligned} \tag{97}$$

such that

$$R_3R_4 = R_4R_3 = P_3. \tag{98}$$

Projectors R_1, R_2, R_3, and R_4 are compatible with C. Let us define

$$\begin{aligned} \mu &= \nu_3 = \nu_4 > 0, \\ \nu &= -\nu_1 = -\nu_2 > 0, \\ \alpha_n &= a_n, \quad n \in \mathbb{N}. \end{aligned} \tag{99}$$

Hence, relations (93)–(96) hold.

Sufficiency. We consider the projectors

$$\begin{aligned} P_1 &= R_1, \\ P_2 &= R_2, \\ P_3 &= R_3R_4. \end{aligned} \tag{100}$$

These are compatible with C.

The statements of Proposition 20 follow if we consider

$$\begin{aligned} \nu_1 &= \nu_2 = -\nu < 0, \\ \nu_3 &= \nu_4 = \mu > 0, \\ a_n &= \alpha_n, \quad n \in \mathbb{N}. \end{aligned} \tag{101}$$

Hence, C is exponentially trichotomic, which ends the proof. □

6. Conclusions

The paper emphasizes a way to unify the analysis of continuous and discrete asymptotic properties for skew-evolution semiflows, such as the exponential dichotomy and trichotomy. Thus, we give characterizations for the asymptotic behaviors in discrete time, in order to gain necessary instruments in punctuating the properties of the solutions of difference equations.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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