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Research Article

On the Limit Cycles for Continuous and Discontinuous Cubic Differential Systems

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We study the number of limit cycles for the quadratic polynomial differential systems $\dot{x} = -y+x^2$, $\dot{y} = x+xy$ having an isochronous center with continuous and discontinuous cubic polynomial perturbations. Using the averaging theory of first order, we obtain that 3 limit cycles bifurcate from the periodic orbits of the isochronous center with continuous perturbations and at least 7 limit cycles bifurcate from the periodic orbits of the isochronous center with discontinuous perturbations. Moreover, this work shows that the discontinuous systems have at least 4 more limit cycles surrounding the origin than the continuous ones.

1. Introduction

In 1900, Hilbert [1] proposed 23 famous mathematical problems in the second International Congress of Mathematicians, and the second part of the sixteenth problem asks for the maximum $H(n)$ of the number of limit cycles and the relative positions for all planar polynomial differential systems of degree n. Mathematicians have done a lot of effective works to research on the numbers of limit cycles for the continuous polynomial system (especially the quadratic polynomial system); see for instance the books [2, 3] and the hundreds of references quoted therein. One of the methods for studying on the number of the limit cycles is averaging method. In [4], Buică and Llibre introduced the averaging method for finding limit cycles of continuous differential systems via Brouwer degree. Further, in [5], Llibre et al. used the averaging method for studying the periodic orbits of discontinuous differential systems. Since most phenomena in real life are discontinuous, this created a great interest for the mathematicians to study the limit cycle of discontinuous differential system, especially for the numbers of limit cycles of the discontinuous quadratic polynomial system; see, for instance, [5–9]. For the discontinuous Liénard equations, in [10], Martins and Mereu showed that for any $n \geq 1$ there are differential equations of the form $\ddot{x} + f(x)\dot{x} + \operatorname{sgn}(\dot{x})g(x) = 0$, with f and g being polynomials of degree n and 1, respectively, having $[n/2] + 1$ limit cycles. In [11], Libre and Teixeira provided lower bounds for the maximum number of limit cycles for m-piecewise discontinuous polynomial differential equations. For the switching systems, Han and Sheng [12] discussed the bifurcation of limit cycles in piecewise smooth systems via Melnikov functions. Ten limit cycles are found around a center in switching quadratic systems in [13].

Consider a quadratic polynomial system:

$$\begin{aligned} \dot{x} &= -y + x^2, \\ \dot{y} &= x + xy. \end{aligned} \tag{1}$$

Chicone and Jacobs proved in [14] that at most 2 limit cycles bifurcate from the periodic orbits of the isochronous center of system (1). Their study is based on the displacement function using some results of Bautin [15]. In [4], Buică and Llibre easily reproved, using the averaging method, by continuous quadratic polynomial perturbing system (1), the existence of at least 2 limit cycles bifurcating from the periodic orbit of the center of system (1) when this is perturbed inside the class of all quadratic polynomial differential systems. Recently, Llibre and Mereu [7] proved, using the averaging method, by discontinuous quadratic polynomial perturbing system (1), the existence of at least 5 limit cycles bifurcating from the periodic orbit of the center of system (1) when this is perturbed inside the class of all quadratic polynomial differential systems, and the discontinuous systems have at least 3 more limit cycles surrounding the origin than the continuous systems.

In this paper, consider the following systems:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + x^2 + \varepsilon p_1(x, y) \\ x + xy + \varepsilon q_1(x, y) \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + x^2 + \varepsilon p_1(x, y) \\ x + xy + \varepsilon q_1(x, y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} -y + x^2 + \varepsilon p_2(x, y) \\ x + xy + \varepsilon q_2(x, y) \end{pmatrix}, & y < 0, \end{cases} \tag{3}$$

where ε is a small parameter, and

$$\begin{aligned} p_1(x, y) &= a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 \\ &\quad + a_7 x^2 y + a_8 xy^2 + a_9 y^3, \\ q_1(x, y) &= b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 \\ &\quad + b_7 x^2 y + b_8 xy^2 + b_9 y^3, \\ p_2(x, y) &= c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 \\ &\quad + c_7 x^2 y + c_8 xy^2 + c_9 y^3, \\ q_2(x, y) &= d_1 x + d_2 y + d_3 x^2 + d_4 xy + d_5 y^2 + d_6 x^3 \\ &\quad + d_7 x^2 y + d_8 xy^2 + d_9 y^3. \end{aligned} \tag{4}$$

Using averaging method and some results established in [12], we obtain our main results as follows.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small there are continuous cubic polynomial differential systems (2) having exactly 3 limit cycles bifurcating from the periodic orbits of isochronous center (1).*

Theorem 2. *For $|\varepsilon| \neq 0$ sufficiently small there are discontinuous cubic polynomial differential systems (3) having at least 7 limit cycles bifurcating from the periodic orbits of isochronous center (1).*

By Theorems 1 and 2, we can get the following.

Corollary 3. *Using the averaging method of first order, the discontinuous systems have at least 4 more limit cycles surrounding the origin than the continuous systems when we perturbed the center (1).*

In some sense, we extend the work by Llibre and Mereu with the difference of number of limit cycles of discontinuous and continuous differential systems.

2. Preliminary Results

In this section, we introduce some preliminary results on the averaging theory which will be applied to study the cubic continuous and discontinuous polynomial systems (2) and (3).

The following theorems are the first-order averaging theory for continuous and discontinuous differential systems. For the proof, we refer the reader to [4, 5].

Theorem 4 (see [4]). *Consider the following differential system:*

$$x'(t) = \varepsilon F(t, x) + \varepsilon^2 G(t, x, \varepsilon), \tag{5}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$, $G : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Define $f : D \to \mathbb{R}^n$:

$$f(x) = \frac{1}{T} \int_0^T F(t, x)\, dt \tag{6}$$

and assume that

(i) *F and G are locally Lipschitz with respect to x.*

(ii) *For $a \in D$ with $f(a) = 0$, there exists a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(f, V, a) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $x(\cdot, \varepsilon)$ of system (5) such that $x(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem 5 (see [5]). *Consider the following discontinuous differential system:*

$$x'(t) = \varepsilon F(t, x) + \varepsilon^2 G(t, x, \varepsilon) \tag{7}$$

with

$$\begin{aligned} F(t, x) &= F_1(t, x) + \operatorname{sign}(h(t, x)) F_2(t, x), \\ G(t, x) &= G_1(t, x) + \operatorname{sign}(h(t, x)) G_2(t, x), \end{aligned} \tag{8}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $G_1, G_2 : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote by $\mathscr{M} = h^{-1}(0)$, $\Sigma = 0 \times D \not\subseteq \mathscr{M}$, $\Sigma_0 = \Sigma \setminus \mathscr{M} \neq \varnothing$, and its elements by $z \equiv (0, z) \notin \mathscr{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$:

$$f(x) = \frac{1}{T} \int_0^T F(t, x)\, dt. \tag{9}$$

Assume the following three conditions:

(i) *F_1, F_2, G_1, G_2, and h are locally L-Lipschitz with respect to x.*

(ii) *For $a \in \Sigma_0$ with $f(a) = 0$, there exists a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(f, V, a) \neq 0$ (i.e., the Brouwer degree of f at a is not zero).*

(iii) *If* $\partial_t h(t_0, z_0) = 0$ *for some* $(t_0, z_0) \in \mathcal{M}$, *then* $(\langle \nabla_x h, F_1 \rangle^2 - \langle \nabla_x h, F_2 \rangle^2)(t_0, z_0) > 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a T-periodic solution* $x(\cdot, \varepsilon)$ *of system (7) such that* $x(\cdot, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

Remark 6 (see [4]). Let $g : D \to \mathbb{R}^n$ be a C^1 function, with $g(a) = 0$, where D is an open subset of $\mathbb{R}^n$ and $a \in D$. Whenever $J_g(a) \neq 0$, there exists a neighborhood V of a such that $g(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then $d_B(g, V, a) \neq 0$.

Consider a integrable planar differential system:

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{10}$$

where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions under the assumption

(A1) System (10) has a period annulus around the singular point (0, 0):

$$\Gamma_h : \left\{(x, y) \in \mathbb{R}^2 : H(x, y) = h,\ h_c < h < h_s\right\}, \tag{11}$$

where H is a first integral of (10), h_c is the critical level of H corresponding to the center $(0, 0)$, and h_s denotes the value of H for which the period annulus terminates at a separatrix polycycle. Without loss of generality we can assume that $h_s > h_c \geq 0$. We denote by $\mu = \mu(x, y)$ the integrating factor of system (10) corresponding to the first integral H.

Perturbed systems (10) are as follows:

$$\begin{aligned} \dot{x} &= P(x, y) + \varepsilon p(x, y), \\ \dot{y} &= Q(x, y) + \varepsilon q(x, y), \end{aligned} \tag{12}$$

where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In order to apply the averaging method for studying limit cycles of (12) for ε sufficiently small, we need write system (12) into the standard forms (5) and (7). The following result by [4] provides a way to fulfill it.

Theorem 7 (see [4]). *Consider system (10) and its first integral H. Assume that (A1) holds for system (10) and that* $xQ(x, y) - yP(x, y) \neq 0$ *for all* (x, y) *in the period annulus formed by the ovals* $\{\Gamma_h\}$. *Let* $\rho : (\sqrt{h_c}, \sqrt{h_s}) \times [0, 2\pi) \to [0, +\infty)$ *be a continuous function such that*

$$H(\rho(R, \varphi)\cos\varphi, \rho(R, \varphi)\sin\varphi) = R^2 \tag{13}$$

for all $R \in (\sqrt{h_c}, \sqrt{h_s})$ *and* $\varphi \in [0, 2\pi)$. *Then the differential equation which describes the dependence between the square root of energy,* $R = \sqrt{h}$, *and the angle* φ *for system (12) is*

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py) + 2R\varepsilon(qx - py)}, \tag{14}$$

where $x = \rho(R, \varphi)\cos\varphi$, $y = \rho(R, \varphi)\sin\varphi$.

Remark 8. Further, system (14) can become

$$\frac{dR}{d\varphi} = \varepsilon F(\varphi, R) + \varepsilon^2 G(t, x, \varepsilon), \tag{15}$$

where $F(\varphi, R) = \mu(x^2 + y^2)(Qp - Pq)/2R(Qx - Py)$ and $G(t, x, \varepsilon) = \mu(x^2 + y^2)(Qp - Pq)2R(py - qx)/2R(Qx - Py)[2R(Qx - Py) + 2R\varepsilon(qx - py)]$.

The following lemma presents the version of the formula of the first-order Melnikov function associated with system (12) in the polar coordinates [4].

Lemma 9 (see [4]). *Under the conditions of Theorem 7, define*

$$d(R, \varepsilon) = \varepsilon \int_0^{2\pi} \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py) + 2R\varepsilon(qx - py)} d\varphi, \tag{16}$$

$$M_1(R) = \int_0^{2\pi} \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} d\varphi, \tag{17}$$

for system (12), where $\mu = \mu(x, y)$ *is the integrating factor of system (10) corresponding to the first integral H and* $x = \rho(R, \varphi)\cos\varphi$; $y = \rho(R, \varphi)\sin\varphi$. *Then* $d(R, \varepsilon)$ *and* $M_1(R)$ *expressed by (16) are the displacement function and the first-order Melnikov function of system (12), respectively.*

Under the assumption (A1), the assumptions (I) and (II) hold as in [12]. So, we can establish the function $M(h)$ as follows:

$$\begin{aligned} M(R^2) &= 2\int_{\Gamma_{R^2}} q(x, y)\,dx - p(x, y)\,dy, \\ &\qquad R = \sqrt{h} \in \left(\sqrt{h_c}, \sqrt{h_s}\right), \end{aligned} \tag{18}$$

since $\int_{\Gamma_{R^2}} q(x, y)dx - p(x, y)dy$ is the first-order Melnikov function of system (12). Based on Theorems 4 and 7, Lemma 9, Theorem 1.1 in [12], and (18), we have the following.

Lemma 10. *Under the assumption (A1), let* $f(R)$ *be the averaged function of system (14); then the following relation holds:*

$$4\pi f(R) = M(R^2), \tag{19}$$

where $f(R)$ *is defined by (6) and* $M(R^2)$ *is defined by (18).*

In order to study the number of zeros of the averaged functions (6) and (9), we will use the following result proved in [16].

Let A be a set and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$. We say that $f_1, f_2, \ldots, f_n$ are *linearly independent* functions if and only if we have that

$$\sum_{i=1}^{n} \alpha_i f_i(a) = 0, \quad \forall a \in A \Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0. \tag{20}$$

Proposition 11 (see [16]). *If $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ are linearly independent, then there exist $a_1, a_2, \ldots, a_{n-1} \in A$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that for every $i \in \{1, 2, \ldots, n-1\}$*

$$\sum_{k=1}^{n} \alpha_k f_k (a_i) = 0. \tag{21}$$

3. Proof of Theorem 1

In this section, we will prove Theorem 1 by using Theorem 4 for the continuous case. We recall that the period annulus of a center is the topological annulus formed by all the periodic orbits surrounding the center which is the only singular point of the system.

A first integral H and an integrating factor μ in the period annulus of the center of the quadratic differential system (1) have the expressions $H(x, y) = (x^2 + y^2)/(1 + y)^2$ and $\mu(x, y) = 2/(1 + y)^3$, respectively. For this system we note that $h_c = 0$, $h_s = 1$. By (13), the function ρ satisfying the hypotheses of Theorem 7 is given by $\rho(R, \varphi) = R/(1-R\sin\varphi)$, $0 < R < 1$, $\varphi \in [0, 2\pi)$. It is easy to know that assumption (A1) holds.

Using Theorem 7, we transform system (2) into the form

$$\begin{aligned} &\frac{\mathrm{d}R}{\mathrm{d}\varphi} \\ &= \varepsilon \frac{A(\varphi; a, b) R + B(\varphi; a, b) R^2 + C(\varphi; a, b) R^3 + D(\varphi; a, b) R^4}{(1 - R\sin(\varphi))^2} \\ &\quad + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &A(\varphi; a, b) \\ &\quad = a_1 \cos^2\varphi + (a_2 + b_1) \sin\varphi\cos\varphi + b_2 \sin^2\varphi, \\ &B(\varphi; a, b) \\ &\quad = (a_4 - b_1)\cos^3\varphi - (2a_1 - a_3 + b_2 - b_4)\sin\varphi\cos^2\varphi \\ &\qquad - (2a_2 - a_5 + 3b_1 - b_3)\cos\varphi\sin^2\varphi \\ &\qquad + (b_5 - 3b_2)\sin^3\varphi, \\ &C(\varphi; a, b) \\ &\quad = (a_6 - b_4)\cos^4\varphi \\ &\qquad - (a_4 - a_7 - 2b_1 + b_3 - b_6)\cos^3\varphi\sin\varphi \\ &\qquad + (a_1 - a_3 + a_8 + 2b_2 - 2b_4 - b_5 + b_7)\cos^2\varphi\sin^2\varphi \\ &\qquad + (a_2 - a_5 + a_9 + 3b_1 - 2b_3 + b_8)\cos\varphi\sin^3\varphi \\ &\qquad + (3b_2 - 2b_5 + b_9)\sin^4\varphi, \\ &D(\varphi; a, b) \\ &\quad = -b_6\cos^5\varphi + (b_4 - b_7)\cos^4\varphi\sin\varphi \\ &\qquad - (b_1 - b_3 + b_6 + b_8)\cos^3\varphi\sin^2\varphi \\ &\qquad - (b_2 - b_4 - b_5 + b_7 + b_9)\cos^2\varphi\sin^3\varphi \\ &\qquad - (b_1 - b_3 + b_8)\cos\varphi\sin^4\varphi \\ &\qquad - (b_2 - b_5 + b_9)\sin^5\varphi, \end{aligned} \tag{23}$$

and $a = (a_1, a_2, \ldots, a_9)$, $b = (b_1, b_2, \ldots, b_9)$.

The continuous differential system (22) is under the assumptions of Theorem 4. So we just need to study the zeros of the averaged function $f : (0, 1) \to \mathbb{R}$:

$$f(R) = \frac{1}{2\pi} \int_0^{2\pi} \frac{A(\varphi; a, b) R + B(\varphi; a, b) R^2 + C(\varphi; a, b) R^3 + D(\varphi; a, b) R^4}{2(1 - R\sin\varphi)^2} \mathrm{d}\varphi. \tag{24}$$

Using Mathematica (Mathematica software), we compute the above integral and obtain

$$\begin{aligned} f(R) = \frac{1}{2} &[(a_1 + b_2) g_1 + a_3 g_2 + a_6 g_3 + a_8 g_4 + b_4 g_5 \\ &+ b_5 g_6 + b_7 g_7 + b_9 g_8], \end{aligned} \tag{25}$$

where

$$\begin{aligned} g_1 &= R, \\ g_2 &= \frac{2\left(\sqrt{1-R^2} - 1\right) - R^2\left(\sqrt{1-R^2} - 2\right)}{R\sqrt{1-R^2}}, \\ g_3 &= \frac{6\left(\sqrt{1-R^2} - 1\right) - 3R^2\left(\sqrt{1-R^2} - 2\right)}{R\sqrt{1-R^2}}, \\ g_4 &= \frac{\sqrt{1-R^2}R^2 - 4R^2 - 6\sqrt{1-R^2} + 6}{R\sqrt{1-R^2}}, \\ g_5 &= \frac{-2R^4 - 3\sqrt{1-R^2}R^2 + 4R^2 + 2\sqrt{1-R^2} - 2}{R\sqrt{1-R^2}}, \\ g_6 &= \frac{\sqrt{1-R^2}R^2 - 2R^2 - 2\sqrt{1-R^2} + 2}{R\sqrt{1-R^2}}, \\ g_7 &= \frac{2R^4 + 5\sqrt{1-R^2}R^2 - 8R^2 - 6\sqrt{1-R^2} + 6}{R\sqrt{1-R^2}}, \\ g_8 &= \frac{-3\sqrt{1-R^2}R^2 + 6R^2 + 6\sqrt{1-R^2} - 6}{R\sqrt{1-R^2}}. \end{aligned} \tag{26}$$

We have the equalities

$$\begin{aligned} g_3 &= 3g_2, \\ g_6 &= -g_2, \\ g_7 &= -2g_2 - g_5, \\ g_8 &= 3g_2. \end{aligned} \tag{27}$$

Thus the function f can be written as

$$\begin{aligned} f(R) = \frac{1}{2} & [(a_1 + b_2) g_1 \\ & + (a_3 + 3a_6 - b_5 - 2b_7 + 3b_9) g_2 + a_8 g_4 \\ & + (b_4 - b_7) g_5]. \end{aligned} \tag{28}$$

The Wronskian of the functions g_1, g_2, g_4, g_5 in variable R is

$$\begin{aligned} & W[g_1, g_2, g_4, g_5](R) \\ & = \frac{96\pi^4 R^2 \left(R^4 + 4\left(\sqrt{1-R^2} - 2\right) R^2 - 8\sqrt{1-R^2} + 8\right)}{\left(R^2-1\right)^5}. \end{aligned} \tag{29}$$

We have $W[g_1, g_2, g_4, g_5](R) \neq 0$ for all $R \in (0,1)$. In fact, if there exist $R_0 \in (0,1)$ such that $W[g_1, g_2, g_4, g_5](R_0) = 0$, then

$$R_0^4 - 8\sqrt{1-R_0^2} + 8 = 4R_0^2\left(2 - \sqrt{1-R_0^2}\right), \tag{30}$$

and it is not difficult to obtain that

$$-R_0^8 = 0. \tag{31}$$

Obviously, it is impossible for all $R_0 \in (0,1)$. Then, the functions g_1, g_2, g_4, g_5 in R are linearly independent. By Proposition 11, there exist $\alpha_i \in \mathbb{R}$, $i = 1,2,3,4$, such that the linear combination $\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_4 + \alpha_4 g_5$ of four functions g_1, g_2, g_4, g_5 has at least 3 zeros R_1, R_2, R_3; that is, for all $i = 1,2,3$, the following equations hold:

$$\alpha_1 g_1(R_i) + \alpha_2 g_2(R_i) + \alpha_3 g_4(R_i) + \alpha_4 g_5(R_i) = 0. \tag{32}$$

Comparing (28) and (32), we get a set of equations about variables $a_1, a_3, a_6, a_8, b_4, b_5, b_7, b_9$:

$$\begin{aligned} \alpha_1 &= a_1 + b_2, \\ \alpha_2 &= a_3 + 3a_6 - b_5 - 2b_7 + 3b_9, \\ \alpha_3 &= a_8, \\ \alpha_4 &= b_4 - b_7. \end{aligned} \tag{33}$$

The rank of coefficients matrix of (33) is 4; then the solutions of these equations exist. Thus there exist $a_j, b_j \in \mathbb{R}$, $j = 1,2,\ldots,9$ such that the averaged function $f(R)$ has at least 3 zeros R_1, R_2, R_3, where $a_1, a_3, a_6, a_8, b_4, b_5, b_7, b_9$ are the solutions of (33); other ones are 0.

On the other hand, let $R = (1-w^2)/(1+w^2)$, $0 < w < 1$; the averaged function $f(R)$ becomes

$$f(R) = \frac{(1-w)}{2w(1+w)\left(1+w^2\right)^2} g(w), \tag{34}$$

where

$$\begin{aligned} g(w) = {} & N_1 + N_2 w + N_3 w^2 + N_4 w^3 + N_3 w^4 + N_2 w^5 \\ & + N_1 w^6, \\ N_1 = {} & a_8, \\ N_2 = {} & a_1 + a_3 + 3a_6 - 3a_8 + b_2 - b_4 - b_5 - b_7 + 3b_9, \\ N_3 = {} & 2a_1 - 2a_3 - 6a_6 + 5a_8 + 2b_2 - 2b_4 + 2b_5 + 6b_7 \\ & - 6b_9, \\ N_4 = {} & 2a_1 + 2a_3 + 6a_6 - 6a_8 + 2b_2 + 6b_4 - 2b_5 \\ & - 10b_7 + 6b_9. \end{aligned} \tag{35}$$

As a result of the symmetry of coefficients of $g(w)$, we know that if $w_0 \neq 0$ is one root of $g(w) = 0$, so is $1/w_0$, but only one of w_0 and $1/w_0$ is in interval $(0,1)$. Hence, the fact that $g(w)$ has at most 3 zeros in $w \in (0,1)$ implies that there exist at most 3 zeros for $f(R)$ in $R \in (0,1)$.

Thus, the averaged function $f(R)$ has exactly 3 zeros. By Theorem 4, Lemmas 9 and 10, and Theorem 1.1 in [12], we get that there are exactly 3 limit cycles bifurcate from the period annulus around the isochronous center for (2) with sufficiently small $|\varepsilon|$. This completes the proof of Theorem 1.

4. Example

In this section, we not only provide some examples satisfying the property of Theorem 1, but also introduce a method to construct such systems.

Suppose that

$$\begin{aligned} \widetilde{g}(w) = {} & \left(w - \frac{1}{10}\right)\left(w - \frac{1}{5}\right)\left(w - \frac{1}{2}\right)(w-10)(w-5) \\ & \cdot (w-2) = 1 - \frac{89}{5}w + \frac{9377}{100}w^2 - \frac{1669}{10}w^3 \\ & + \frac{9377}{100}w^4 - \frac{89}{5}w^5 + w^6. \end{aligned} \tag{36}$$

Take the constants

$$\begin{aligned} N_1 &= 1, \\ N_2 &= -\frac{89}{5}, \\ N_3 &= \frac{9377}{100}, \\ N_4 &= -\frac{1669}{10}. \end{aligned} \tag{37}$$

From (35) and (37), we have

$$a_1 = -\frac{81}{50} - b_2,$$
$$a_3 = -\frac{11837}{400} - 3a_6 + b_5 + 2b_7 - 3b_9,$$
$$a_8 = 1,$$
$$b_4 = -\frac{1313}{80} + b_7, \tag{38}$$

where a_6, b_2, b_5, b_7, b_9 and other ones are any real constants.

Hence, in system (2), for the sufficiently small $|\varepsilon|$, we obtain a family of systems:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + x^2 + \varepsilon\left[\left(-\frac{81}{50} - b_2\right)x + a_2 y + \left(-\frac{11837}{400} - 3a_6 + b_5 + 2b_7 - 3b_9\right)x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + xy^2 + a_9 y^3\right] \\ x + xy + \varepsilon\left[b_1 x + b_2 y + b_3 x^2 + \left(-\frac{1313}{80} + b_7\right)xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3\right] \end{pmatrix}. \tag{39}$$

The averaged function of systems (39) is

$$f(R) = \frac{(1-w)}{2w(1+w)(1+w^2)^2}\left(w - \frac{1}{10}\right)\left(w - \frac{1}{5}\right) \cdot \left(w - \frac{1}{2}\right)(w-10)(w-5)(w-2). \tag{40}$$

Apparently, $f(R)$ has exactly 3 positive zeros in $(0,1)$, denoted by

$$R_1 = \frac{99}{101},$$
$$R_2 = \frac{12}{13},$$
$$R_3 = \frac{3}{5}, \tag{41}$$

corresponding to $w_1 = 1/10$, $w_2 = 1/5$, and $w_3 = 12$ in $R \in (0,1)$. Moreover, we have

$$\frac{df(R_1)}{dR} = \frac{75411\pi}{1000},$$
$$\frac{df(R_2)}{dR} = -\frac{1323\pi}{125},$$
$$\frac{df(R_3)}{dR} = \frac{513\pi}{200}. \tag{42}$$

That is to say, $f'(R_i) \neq 0$, $i \in \{1,2,3\}$; it follows from Theorem 4 and Lemmas 9 and 10 that, for the sufficiently small $|\varepsilon|$, system (39) has exactly 3 limit cycles emerging from the period annulus of the corresponding unperturbed system.

5. Proof of Theorem 2

In this section, we will prove Theorem 2 by using the first-order averaging method for the continuous case. The unperturbed system of system (3) is also system (1); then using Theorem 7 and Remark 8 we transform system (3) into the form

$$\frac{dR}{d\varphi} = \begin{cases} \varepsilon\dfrac{A(\varphi;a,b)R + B(\varphi;a,b)R^2 + C(\varphi;a,b)R^3 + D(\varphi;a,b)R^4}{2(1-R\sin\varphi)^2} + \mathcal{O}(\varepsilon^2), & y > 0, \\ \varepsilon\dfrac{A(\varphi;c,d)R + B(\varphi;c,d)R^2 + C(\varphi;c,d)R^3 + D(\varphi;c,d)R^4}{2(1-R\sin\varphi)^2} + \mathcal{O}(\varepsilon^2), & y < 0, \end{cases} \tag{43}$$

where

$$\begin{aligned}
&A(\varphi;a,b)\\
&= a_1\cos^2\varphi + (a_2+b_1)\sin\varphi\cos\varphi + b_2\sin^2\varphi,\\
&B(\varphi;a,b)\\
&= (a_4-b_1)\cos^3\varphi - (2a_1-a_3+b_2-b_4)\sin\varphi\cos^2\varphi\\
&\quad - (2a_2-a_5+3b_1-b_3)\cos\varphi\sin^2\varphi\\
&\quad + (b_5-3b_2)\sin^3\varphi,\\
&C(\varphi;a,b)\\
&= (a_6-b_4)\cos^4\varphi\\
&\quad - (a_4-a_7-2b_1+b_3-b_6)\cos^3\varphi\sin\varphi\\
&\quad + (a_1-a_3+a_8+2b_2-2b_4-b_5+b_7)\cos^2\varphi\sin^2\varphi\\
&\quad + (a_2-a_5+a_9+3b_1-2b_3+b_8)\cos\varphi\sin^3\varphi\\
&\quad + (3b_2-2b_5+b_9)\sin^4\varphi,\\
&D(\varphi;a,b)\\
&= -b_6\cos^5\varphi + (b_4-b_7)\cos^4\varphi\sin\varphi\\
&\quad - (b_1-b_3+b_6+b_8)\cos^3\varphi\sin^2\varphi\\
&\quad - (b_2-b_4-b_5+b_7+b_9)\cos^2\varphi\sin^3\varphi\\
&\quad - (b_1-b_3+b_8)\cos\varphi\sin^4\varphi\\
&\quad - (b_2-b_5+b_9)\sin^5\varphi,
\end{aligned} \tag{44}$$

and $a = (a_1, a_2, \ldots, a_9)$, $b = (b_1, b_2, \ldots, b_9)$, $c = (c_1, c_2, \ldots, c_9)$, and $d = (d_1, d_2, \ldots, d_9)$.

The discontinuous differential system (43) is under the assumptions of Theorem 5. So we just need to study the zeros of the averaged function $f : (0,1) \to \mathbb{R}$:

$$\begin{aligned}
f(R) &= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{A(\varphi;a,b)R + B(\varphi;a,b)R^2 + C(\varphi;a,b)R^3 + D(\varphi;a,b)R^4}{2(1-R\sin\varphi)^2} d\varphi\right.\\
&\quad\left. + \int_{\pi}^{2\pi} \frac{A(\varphi;c,d)R + B(\varphi;c,d)R^2 + C(\varphi;c,d)R^3 + D(\varphi;c,d)R^4}{2(1-R\sin\varphi)^2} d\varphi\right].
\end{aligned} \tag{45}$$

The above integral is calculated by the mathematical software Mathematica obtaining

$$\begin{aligned}
f(R) &= \frac{1}{2\pi}\left[(a_1+c_1)g_1 + a_3g_2 + a_6g_3 + a_8g_4 + b_2g_5\right.\\
&\quad + b_4g_6 + b_5g_7 + b_7g_8 + b_9g_9 + c_3g_{10} + c_6g_{11}\\
&\quad \left.+ c_8g_{12} + d_2g_{13} + d_4g_{14} + d_5g_{15} + d_7g_{16} + d_9g_{17}\right],
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
g_1 &= \frac{\pi R}{2},\\
g_2 &\\
&= \frac{4\sqrt{1-R^2}R + \pi\left(2\left(\sqrt{1-R^2}-1\right) - R^2\left(\sqrt{1-R^2}-2\right)\right)}{2R\sqrt{1-R^2}}\\
&\quad + \frac{2\left(R^2-1\right)\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}},\\
g_3 &\\
&= -2R^2 + \frac{3\pi R}{\sqrt{1-R^2}} - \frac{3\pi}{R\sqrt{1-R^2}}\\
&\quad + \frac{6\left(R^2-1\right)\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{3\pi}{R} + 6,\\
g_4 &\\
&= -\frac{2\pi R}{\sqrt{1-R^2}} + \frac{3\pi}{R\sqrt{1-R^2}} - \frac{4R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}}\\
&\quad + \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R}{2} - \frac{3\pi}{R} - 6,\\
g_5 &= \frac{\pi R}{2} - 2R^2,\\
g_6 &\\
&= -2R^2 + \frac{2\pi R}{\sqrt{1-R^2}} - \frac{\pi}{R\sqrt{1-R^2}}\\
&\quad + \frac{4R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{2\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}}\\
&\quad - \frac{\pi R^3}{\sqrt{1-R^2}} - \frac{2R^3\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{\pi}{R}\\
&\quad + 2,\\
g_7 &\\
&= 2R^2 - \frac{\pi R}{\sqrt{1-R^2}} + \frac{\pi}{R\sqrt{1-R^2}}
\end{aligned}$$

$$- \frac{2R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{2\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R}{2} - \frac{\pi}{R} - 2,$$

$$g_8 = 4R^2 - \frac{4\pi R}{\sqrt{1-R^2}} + \frac{3\pi}{R\sqrt{1-R^2}} - \frac{8R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R^3}{\sqrt{1-R^2}} + \frac{2R^3\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{5\pi R}{2} - \frac{3\pi}{R} - 6,$$

$$g_9 = -2R^2 + \frac{3\pi R}{\sqrt{1-R^2}} - \frac{3\pi}{R\sqrt{1-R^2}} + \frac{6R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{3\pi}{R} + 6,$$

$$g_{10} = -\frac{4\sqrt{1-R^2}R + \pi\left(\left(\sqrt{1-R^2}-2\right)R^2 - 2\sqrt{1-R^2} + 2\right)}{2R\sqrt{1-R^2}} + \frac{2\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{2R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}},$$

$$g_{11} = 2R^2 + \frac{3\pi R}{\sqrt{1-R^2}} - \frac{3\pi}{R\sqrt{1-R^2}} - \frac{6\left(R^2-1\right)\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{3\pi}{R} - 6,$$

$$g_{12} = -\frac{2\pi R}{\sqrt{1-R^2}} + \frac{3\pi}{R\sqrt{1-R^2}} + \frac{4R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R}{2} - \frac{3\pi}{R} + 6,$$

$$g_{13} = 2R^2 + \frac{\pi R}{2},$$

$$g_{14} = 2R^2 + \frac{2\pi R}{\sqrt{1-R^2}} - \frac{\pi}{R\sqrt{1-R^2}} - \frac{4R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{2\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{\pi R^3}{\sqrt{1-R^2}} + \frac{2R^3\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{\pi}{R} - 2,$$

$$g_{15} = -2R^2 - \frac{\pi R}{\sqrt{1-R^2}} + \frac{\pi}{R\sqrt{1-R^2}} + \frac{2R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{2\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R}{2} - \frac{\pi}{R} + 2,$$

$$g_{16} = -4R^2 - \frac{4\pi R}{\sqrt{1-R^2}} + \frac{3\pi}{R\sqrt{1-R^2}} + \frac{8R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} - \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} + \frac{\pi R^3}{\sqrt{1-R^2}} - \frac{2R^3\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{5\pi R}{2} - \frac{3\pi}{R} + 6,$$

$$g_{17} = 2R^2 + \frac{3\pi R}{\sqrt{1-R^2}} - \frac{3\pi}{R\sqrt{1-R^2}} - \frac{6R\arctan\left(R/\sqrt{1-R^2}\right)}{\sqrt{1-R^2}} + \frac{6\arctan\left(R/\sqrt{1-R^2}\right)}{R\sqrt{1-R^2}} - \frac{3\pi R}{2} + \frac{3\pi}{R} - 6. \tag{47}$$

The following equalities hold:

$$\begin{aligned} g_5 &= g_1 - 3g_2 + g_3, \\ g_7 &= 2g_2 - g_3, \\ g_8 &= g_2 - g_3 - g_6, \\ g_9 &= g_3, \\ g_{11} &= 3g_2 - g_3 + 3g_{10}, \\ g_{13} &= g_1 + 3g_2 - g_3, \\ g_{15} &= -3g_2 + g_3 - g_{10}, \\ g_{16} &= -3g_2 + g_3 - 2g_{10} - g_{14}, \\ g_{17} &= 3g_2 - g_3 + 3g_{10}. \end{aligned} \tag{48}$$

So we can rewrite the function f as

$$\begin{aligned} f(R) = \frac{1}{2\pi} & \left[(a_1 + b_2 + c_1 + d_2) g_1 + (a_3 - 3b_2 + 2b_5 \right. \\ & + b_7 + 3c_6 + 3d_2 - 3d_5 - 5d_7 + 3d_9) g_2 + (a_6 + b_2 \\ & - b_5 - b_7 + b_9 - c_6 - d_2 + d_5 + d_7 - d_9) g_3 + a_8 g_4 \\ & + (b_4 - b_7) g_6 + (c_3 + 3c_6 - d_5 + 3d_9) g_{10} + c_8 g_{12} \\ & \left. + (d_4 - d_7) g_{14} \right]. \end{aligned} \tag{49}$$

Through calculation, the Wronskian of the eight functions g_i, $i = 1, 2, 3, 4, 6, 10, 12, 14$ in variable R is valued as $W[g_1, g_2, g_3, g_4, g_6, g_{10}, g_{12}, g_{14}](0.1) = 4.66942 \neq 0$ at $R = 0.1$, so the eight functions are linearly independent. By Proposition 11, there exist $\alpha_j \in \mathbb{R}$, $j = 1, 2, \ldots, 8$, and at least seven $R_i \in (0, 1)$, $i = 1, 2, \ldots, 7$, such that the equality

$$\begin{aligned} & \alpha_1 g_1(R_i) + \alpha_2 g_2(R_i) + \alpha_3 g_3(R_i) + \alpha_4 g_4(R_i) \\ & \quad + \alpha_5 g_6(R_i) + \alpha_6 g_{10}(R_i) + \alpha_7 g_{12}(R_i) \\ & \quad + \alpha_8 g_{14}(R_i) = 0 \end{aligned} \tag{50}$$

holds, for all $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

From (49) and (50), we obtain linear equations about variables $a_1, a_3, a_6, a_8, b_2, b_4, b_5, b_7, b_9, c_1, c_3, c_6, c_8, d_2, d_4, d_5, d_7, d_9$, and it is easy to know that these linear equations have solutions. Hence, there exist $a_j, b_j, c_j, d_j \in \mathbb{R}$, $j \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (where $a_1, a_3, a_6, a_8, b_2, b_4, b_5, b_7, b_9, c_1, c_3, c_6, c_8, d_2, d_4, d_5, d_7, d_9$ are the solutions of these equations, and other ones are 0), such that the averaged function $f(R)$ has at least 7 zeros $R_i \in (0, 1)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

In short, there are discontinuous cubic polynomial systems (3) having at least 7 limit cycles bifurcating from the periodic orbits of the isochronous center of system (1) by using Theorem 5. This completes the proof of Theorem 2.

6. Example

In this section, we will introduce a method to construct some examples satisfying the property of Theorem 2; moreover, we will provide such systems.

Firstly, we take $R_i = i/10$, $i = 1, 2, \ldots, 7$, in (50), respectively. Thus, we get linear equations about variable $\alpha_j \in \mathbb{R}$, $j = 1, 2, \ldots, 8$. Solve these equations, and we choose one of the solutions of these equations:

$$\begin{aligned} \alpha_1 &= -0.00217973183257539, \\ \alpha_2 &= 492.047754227194, \\ \alpha_3 &= -171.09394768524, \\ \alpha_4 &= 1.53874330716414, \\ \alpha_5 &= -19.6427452695672, \\ \alpha_6 &= 372.273977923452, \\ \alpha_7 &= -113.700137927805, \\ \alpha_8 &= 100. \end{aligned} \tag{51}$$

From (49) and (50), we obtain the following linear equations about variables $a_1, a_3, a_6, a_8, b_2, b_4, b_5, b_7, b_9, c_1, c_3, c_6, c_8, d_2, d_4, d_5, d_7, d_9$:

$$\begin{aligned} & a_1 + b_2 + c_1 + d_2 = \alpha_1, \\ & a_3 - 3b_2 + 2b_5 + b_7 + 3c_6 + 3d_2 - 3d_5 - 5d_7 + 3d_9 \\ & \quad = \alpha_2, \\ & a_6 + b_2 - b_5 - b_7 + b_9 - c_6 - d_2 + d_5 + d_7 - d_9 = \alpha_3, \\ & a_8 = \alpha_4, \\ & b_4 - b_7 = \alpha_5, \\ & c_3 + 3c_6 - d_5 + 3d_9 = \alpha_6, \\ & c_8 = \alpha_7, \\ & d_4 - d_7 = \alpha_8. \end{aligned} \tag{52}$$

Second, solve (52), in (3), and we choose

$$\begin{aligned} a_1 &= -0.00217973183298289 - b_2 - c_1 - d_2, \\ a_3 &= 492.047754229033 + 3b_2 - 2b_5 - b_7 - 3c_6 - 3d_2 \\ & \quad + 3d_5 + 5d_7 - 3d_9, \\ a_6 &= -171.093947685661 - b_2 + b_5 + b_7 - b_9 + c_6 \\ & \quad + d_2 - d_5 - d_7 + d_9, \\ a_8 &= 1.5387433071157, \\ b_4 &= -19.642745269155 + b_7, \\ c_3 &= 372.273977925687 - 3c_6 + d_5 - 3d_9, \\ c_8 &= -113.700137929543, \\ d_4 &= 100 + d_7, \end{aligned} \tag{53}$$

and all other coefficients are any real constants; then system (3) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + x^2 + \varepsilon X_1(x, y) \\ x + xy + \varepsilon Y_1(x, y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} -y + x^2 + \varepsilon X_2(x, y) \\ x + xy + \varepsilon Y_2(x, y) \end{pmatrix}, & y < 0, \end{cases} \tag{54}$$

where

$$\begin{aligned} X_1(x, y) &= (-0.00217973183298289 - b_2 - c_1 - d_2) \\ &\quad \cdot x + a_2 y + (492.047754229033 + 3b_2 - 2b_5 - b_7 \\ &\quad - 3c_6 - 3d_2 + 3d_5 + 5d_7 - 3d_9) x^2 + a_4 xy + a_5 y^2 \\ &\quad + (-171.093947685661 - b_2 + b_5 + b_7 - b_9 + c_6 \\ &\quad + d_2 - d_5 - d_7 + d_9) x^3 + a_7 x^2 y \\ &\quad + 1.5387433071157 xy^2 + a_9 y^3, \\ Y_1(x, y) &= b_1 x + b_2 y + b_3 x^2 + (-19.642745269155 \\ &\quad + b_7) xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3, \\ X_2(x, y) &= c_1 x + c_2 y + (372.273977925687 - 3c_6 \\ &\quad + d_5 - 3d_9) x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y \\ &\quad - 113.700137929543 xy^2 + c_9 y^3, \\ Y_2(x, y) &= d_1 x + d_2 y + d_3 x^2 + (100 + d_7) xy + d_5 y^2 \\ &\quad + d_6 x^3 + d_7 x^2 y + d_8 xy^2 + d_9 y^3. \end{aligned} \tag{55}$$

The averaged function of system (3) is

$$\begin{aligned} & f(R) \\ &= \frac{1}{2\pi R\sqrt{1-R^2}} \left[R^2 \left(2312.45 - 1106.27\sqrt{1-R^2} \right) \right. \\ &\quad - 252.45 R^4 - 1717.73\sqrt{1-R^2} R \\ &\quad + 2412.37\sqrt{1-R^2} \\ &\quad + \left(239.285 R^4 - 1726.54 R^2 + 1717.73 \right) \\ &\quad \cdot \arctan\left(\frac{R}{\sqrt{1-R^2}} \right) + 581.473\sqrt{1-R^2} R^3 \\ &\quad \left. - 2412.37 \right]. \end{aligned} \tag{56}$$

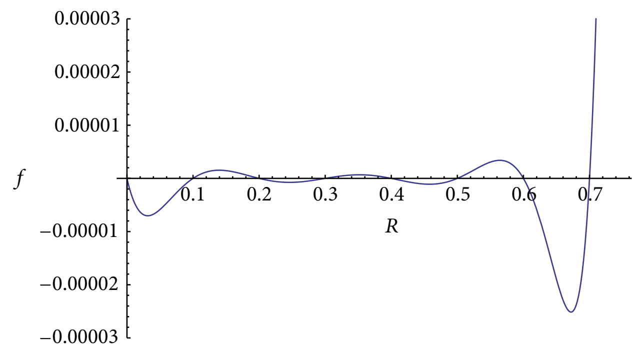

Figure 1

The function $f(R)$ has at lest 7 zeros:

$$\begin{aligned} R_1 &= 0.1, \\ R_2 &= 0.2, \\ R_3 &= 0.3, \\ R_4 &= 0.4, \\ R_5 &= 0.5, \\ R_6 &= 0.6, \\ R_7 &= 0.7, \\ f'(0.1) &= 0.0000825499574205021, \\ f'(0.2) &= -0.0000302356277760415, \\ f'(0.3) &= 0.0000207317994750498, \\ f'(0.4) &= -0.0000247975117812018, \\ f'(0.5) &= 0.0000522877306220519, \\ f'(0.6) &= -0.000213685595035765, \\ f'(0.7) &= 0.00226678597503152; \end{aligned} \tag{57}$$

that is, $f'(R_i) \neq 0$, $i = 1, 2, \ldots, 7$. $f(R)$ is shown in Figure 1. Hence, by Theorem 5, it follows that, for $|\varepsilon| \neq 0$ sufficiently small, system (54) has at least 7 limit cycles surround the origin.

Competing Interests

The author declares that there are no competing interests.

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