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Research Article

Dynamical Analysis of a Computer Virus Model with Delays

Juan Liu,[1] Carlo Bianca,[2] and Luca Guerrini[3]

[1]*Department of Mathematics and Physics, Bengbu University, Bengbu 233030, China*
[2]*Laboratoire de Physique Statistique, Ecole Normale Supérieure, PSL Research University, Université Paris Diderot Sorbonne Paris-Cité, Sorbonne Universités, UPMC Univ Paris 06, CNRS, 24 rue Lhomond, 75005 Paris, France*
[3]*Department of Management, Polytechnic University of Marche, 60121 Ancona, Italy*

Correspondence should be addressed to Juan Liu; liujuan7216@163.com

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An SIQR computer virus model with two delays is investigated in the present paper. The linear stability conditions are obtained by using characteristic root method and the developed asymptotic analysis shows the onset of a Hopf bifurcation occurs when the delay parameter reaches a critical value. Moreover the direction of the Hopf bifurcation and stability of the bifurcating period solutions are investigated by using the normal form theory and the center manifold theorem. Finally, numerical investigations are carried out to show the feasibility of the theoretical results.

1. Introduction

The advances of information technology and the widespread popularity of computer networks have increased the interest in the computer viruses. In the past decades, many epidemic models such as SIR model [1, 2], SIRS model [3, 4], SEIR model [5–7], SEIRS model [8] and SEIQRS model [9] characterizing the spread of computer viruses in networks were investigated by many scholars. It is worthwhile to note that the dynamical models above neglect the time delay in the spreading process of computer viruses. To our knowledge, there have been some computer virus models with time delay proposed to depict the spread of a computer virus. In [10, 11], Muroya et al. studied the global stability of a delayed SIRS computer virus propagation model, respectively. In [12], Feng et al. investigated the Hopf bifurcation of a delayed SIRS viral infection model in computer networks by regarding the delay due to temporary immune period of the recovered computers as a bifurcation parameter. In [13], Dong et al. analyzed the Hopf bifurcation of a delayed SEIR computer virus model with multistate antivirus by regarding the time delay due to the period that the computers use antivirus software to clean the viruses as the bifurcation parameter. In [14], Zhang and Yang studied the Hopf bifurcation of the following SIQR computer virus model with time delay:

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)b - \beta S(t-\tau)I(t-\tau) - dS(t),\\
\frac{dI(t)}{dt} &= \beta S(t-\tau)I(t-\tau) - (\delta + d + \alpha_1 + \gamma)I(t),\\
\frac{dQ(t)}{dt} &= \delta I(t) - (\varepsilon + d + \alpha_2)Q(t),\\
\frac{dR(t)}{dt} &= \gamma I(t) + pb + \varepsilon Q(t) - dR(t),
\end{aligned} \tag{1}$$

where $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ denote the numbers of nodes in states susceptible, infectious, quarantined, and recovered at time t, respectively. b, p, α_1, α_2, β, γ, δ, and ε are the parameters of system (1) and τ is the time delay due to the latent period of the computer virus. Zhang and Yang obtained the sufficient conditions for the local stability and existence of local Hopf bifurcation by regarding the delay as a bifurcation parameter and investigated the properties of the Hopf bifurcation by using the normal form method and center manifold theory. As stated in [13], it usually takes

a period to clean the viruses in the computers infected by viruses for antivirus software. Therefore, it is reasonable to incorporate the time delay due to the period that antivirus software uses to clean viruses in the infectious and the quarantined computers into system (1). Bearing all above in mind, this paper deals with the analysis of the Hopf bifurcation of the following system with two delays:

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)b - \beta S(t-\tau_1) I(t-\tau_1) - dS(t), \\
\frac{dI(t)}{dt} &= \beta S(t-\tau_1) I(t-\tau_1) - (\delta + d + \alpha_1) I(t) \\
&\quad - \gamma I(t-\tau_2), \\
\frac{dQ(t)}{dt} &= \delta I(t) - (d+\alpha_2) Q(t) - \varepsilon Q(t-\tau_2), \\
\frac{dR(t)}{dt} &= \gamma I(t-\tau_2) + pb + \varepsilon Q(t-\tau_2) - dR(t),
\end{aligned} \tag{2}$$

where τ_1 is the time delay due to the latent period of the computer virus and τ_2 is the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes.

This paper mainly investigates the effect of the two delays on system (2). The remainder of this paper is organized as follows. The local stability of the positive equilibrium and existence of local Hopf bifurcation are analyzed in Section 2 by choosing different combination of the two delays as a bifurcation parameter. Direction of the Hopf bifurcation, stability, and period of the bifurcating periodic solutions on the center manifold are determined in Section 3. Some numerical simulations are presented to illustrate the validity of the main results in Section 4. Finally a critical analysis and further research directions are contained in Section 5.

2. Local Stability of the Positive Equilibrium and Existence of Local Hopf Bifurcation

According to the analysis in [14] we know that if $R_0 = (1-p)b\beta/d(\delta + d\alpha_1 + \gamma) > 1$, system (2) has a unique positive equilibrium $D_*(S_*, I_*, Q_*, R_*)$, where

$$\begin{aligned}
S_* &= \frac{\delta + d + \alpha_1 + \gamma}{\beta}, \\
I_* &= \frac{(1-p)b\beta - d(\delta + d + \alpha_1 + \gamma)}{\beta(\delta + d + \alpha_1 + \gamma)}, \\
Q_* &= \frac{\delta I_*}{\varepsilon + d + \alpha_2}, \\
R_* &= \frac{pb + \gamma I_* + \varepsilon Q_*}{d}.
\end{aligned} \tag{3}$$

The characteristic equation of system (2) at D_* is

$$\begin{vmatrix}
\lambda - a_{11} - b_{11}e^{-\lambda\tau_1} & -b_{12}e^{-\lambda\tau_1} & 0 & 0 \\
-b_{21}e^{-\lambda\tau_1} & \lambda - a_{22} - b_{22}e^{-\lambda\tau_1} - c_{22}e^{-\lambda\tau_2} & 0 & 0 \\
0 & -a_{32} & \lambda - a_{33} - c_{33}e^{-\lambda\tau_2} & 0 \\
0 & -c_{42}e^{-\lambda\tau_2} & -c_{43}e^{-\lambda\tau_2} & \lambda - a_{44}
\end{vmatrix} = 0, \tag{4}$$

from which we obtain

$$\begin{aligned}
&\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 \\
&\quad + \left(B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau_1} \\
&\quad + \left(C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0\right)e^{-\lambda\tau_2} \\
&\quad + \left(D_2\lambda^2 + D_1\lambda + D_0\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + \left(E_2\lambda^2 + E_1\lambda + E_0\right)e^{-2\lambda\tau_2} \\
&\quad + \left(F_1\lambda + F_0\right)e^{-\lambda(\tau_1+2\tau_2)} = 0,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
A_0 &= a_{11}a_{22}a_{33}a_{44}, \\
A_1 &= -\left(a_{11}a_{22}(a_{33}+a_{44}) + a_{33}a_{44}(a_{11}+a_{22})\right), \\
A_2 &= a_{11}a_{22} + a_{33}a_{44} + (a_{11}+a_{22})(a_{33}+a_{44}), \\
A_3 &= -(a_{11}+a_{22}+a_{33}+a_{44}), \\
B_0 &= a_{33}a_{44}(a_{11}b_{22} + a_{22}b_{11}), \\
B_2 &= a_{11}b_{22} + a_{22}b_{11} + (a_{33}+a_{44})(b_{11}+b_{22}), \\
B_1 &= -\left(a_{33}a_{44}(b_{11}+b_{22}) \right. \\
&\quad \left. + (a_{33}+a_{44})(a_{11}b_{22}+a_{22}b_{11})\right), \\
B_3 &= -(b_{11}+b_{22}), \\
C_0 &= a_{11}a_{44}(a_{22}c_{33} + a_{33}c_{22}), \\
C_1 &= -\left(c_{22}(a_{11}a_{33} + a_{11}a_{44} + a_{33}a_{44}) \right. \\
&\quad \left. + c_{33}(a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44})\right), \\
C_2 &= c_{22}(a_{11}+a_{33}+a_{44}) + c_{33}(a_{11}+a_{22}+a_{44}), \\
C_3 &= -(c_{22}+c_{33}), \\
D_0 &= a_{33}a_{44}b_{11}c_{22} + a_{44}c_{33}(a_{11}b_{22} + a_{22}b_{11}),
\end{aligned}$$

$$\begin{aligned}
D_1 &= -(a_{33}+a_{44})b_{11}b_{22} - (a_{11}+a_{44})b_{22}c_{33} - (a_{22} \\
&\quad + a_{44})b_{11}c_{33}, \\
D_2 &= b_{11}c_{22} + b_{11}c_{33} + b_{22}c_{33}, \\
E_0 &= a_{11}a_{44}c_{22}c_{33}, \\
E_1 &= -(a_{11}+a_{44})c_{22}c_{33}, \\
E_2 &= c_{22}c_{33}, \\
F_0 &= a_{44}b_{11}c_{22}c_{33}, \\
F_1 &= -b_{11}c_{22}c_{33},
\end{aligned} \tag{6}$$

with

$$\begin{aligned}
a_{11} &= -d, \\
a_{22} &= -(d+\alpha_1+\delta), \\
a_{32} &= \delta, \\
a_{33} &= -(d+\alpha_2), \\
a_{44} &= -d, \\
b_{11} &= -\beta I_*, \\
b_{12} &= -\beta S_*, \\
b_{21} &= \beta I_*, \\
b_{22} &= \beta S_*, \\
c_{22} &= -\gamma, \\
c_{33} &= -\varepsilon, \\
c_{42} &= \gamma, \\
c_{43} &= \varepsilon.
\end{aligned} \tag{7}$$

Case 1 ($\tau_1 = \tau_2 = 0$). When $\tau_1 = \tau_2 = 0$, (5) becomes

$$\begin{aligned}
&\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \\
&A_{10} = A_0 + B_0 + C_0 + D_0 + E_0 + F_0, \\
&A_{11} = A_1 + B_1 + C_1 + D_1 + E_1 + F_1, \\
&A_{12} = A_2 + B_2 + C_2 + D_2 + E_2, \\
&A_{13} = A_3 + B_3 + C_3 = 3d + \alpha_2 + \varepsilon + \beta I_*.
\end{aligned} \tag{8}$$

Thus, if the condition (H_{11}) (9) holds, then the positive equilibrium D_* of system (2) is locally asymptotically stable when $\tau_1 = \tau_2 = 0$.

$$\begin{aligned}
\mathrm{Det}_2 &= \begin{vmatrix} A_{13} & 1 \\ A_{11} & A_{12} \end{vmatrix} > 0, \\
\mathrm{Det}_3 &= \begin{vmatrix} A_{13} & 1 & 0 \\ A_{11} & A_{12} & A_{13} \\ 0 & A_{10} & A_{11} \end{vmatrix} > 0, \\
\mathrm{Det}_4 &= \begin{vmatrix} A_{13} & 1 & 0 & 0 \\ A_{11} & A_{12} & A_{13} & 1 \\ 0 & A_{10} & A_{11} & A_{12} \\ 0 & 0 & 0 & A_{10} \end{vmatrix} > 0.
\end{aligned} \tag{9}$$

Case 2 ($\tau_1 > 0$, $\tau_2 = 0$). When $\tau_1 > 0$, $\tau_2 = 0$, (5) becomes

$$\begin{aligned}
&\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} \\
&\quad + \left(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)e^{-\lambda\tau_1} = 0,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
A_{20} &= A_0 + C_0 + E_0, \\
A_{21} &= A_1 + C_1 + E_1, \\
A_{22} &= A_2 + C_2 + E_2, \\
A_{23} &= A_3 + C_3, \\
B_{20} &= B_0 + D_0 + F_0, \\
B_{21} &= B_1 + D_1 + F_1, \\
B_{22} &= B_2 + D_2.
\end{aligned} \tag{11}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be the root of (10); we get

$$\begin{aligned}
&\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\sin\tau_1\omega_1 + \left(B_{20} - B_{22}\omega_1^2\right)\cos\tau_1\omega_1 \\
&\quad = A_{22}\omega_1^2 - \omega_1^4 - B_{20}, \\
&\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\cos\tau_1\omega_1 - \left(B_{20} - B_{22}\omega_1^2\right)\sin\tau_1\omega_1 \\
&\quad = A_{23}\omega_1^3 - A_{21}\omega_1,
\end{aligned} \tag{12}$$

from which one can obtain

$$\omega_1^8 + e_{23}\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0, \tag{13}$$

where

$$\begin{aligned}
e_{20} &= A_{20}^2 - B_{20}^2, \\
e_{21} &= A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \\
e_{22} &= A_{22}^2 - B_{22}^2 + 2A_{20} - 2A_{21}A_{23} + 2B_{21}B_{23}, \\
c_{23} &= A_{23}^2 - B_{23}^2 - 2A_{22}.
\end{aligned} \tag{14}$$

Let $\omega_1^2 = v_1$; then (13) becomes

$$v_1^4 + e_{23}v_1^3 + e_{22}^2v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{15}$$

Obviously, if all the coefficients of system (2) are given, then we can obtain all the roots of (15) by Matlab software package easily. Thus, we make the following assumption.

(H_{21}) (15) has at least one positive root.

If the condition (H_{21}) holds, then there exists a positive root v_{10} for (15) such that (10) has a pair of purely imaginary roots $\pm i\omega_{10} = \pm i\sqrt{v_{10}}$. For ω_{10},

$$\begin{aligned}
\tau_{10} &= \frac{1}{\omega_{10}} \\
&\quad \times \arccos\frac{n_{26}\omega_{10}^6 + n_{24}\omega_{10}^4 + n_{22}\omega_{10}^2 + n_{20}}{m_{26}\omega_{10}^6 + m_{24}\omega_{10}^4 + m_{22}\omega_{10}^2 + m_{20}},
\end{aligned} \tag{16}$$

where

$$\begin{aligned} m_{20} &= B_{20}^2, \\ m_{22} &= B_{21}^2 - 2B_{20}B_{22}, \\ m_{24} &= B_{22}^2 - 2B_{21}B_{23}, \\ m_{26} &= B_{23}^2, \\ n_{20} &= -A_{20}B_{20}, \\ n_{22} &= A_{20}B_{22} - A_{21}B_{21} + A_{22}B_{20}, \\ n_{24} &= A_{21}B_{23} - A_{22}B_{22} + A_{23}B_{21} - B_{20}, \\ n_{26} &= B_{22} - A_{23}B_{23}. \end{aligned} \tag{17}$$

Substituting $\lambda(\tau_1)$ into the left side of (10) and taking the derivative with respect to τ_1, one can obtain

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1}\right]^{-1} &= -\frac{4\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda\left(\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20}\right)} \\ &\quad + \frac{3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21}}{\lambda\left(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)} - \frac{\tau_1}{\lambda}. \end{aligned} \tag{18}$$

Thus,

$$\begin{aligned} &\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} \\ &= \frac{f_1'(v_{1*})}{B_{23}^2\omega_{10}^6 + \left(B_{22}^2 - 2B_{21}B_{23}\right)\omega_{10}^4 + \left(B_{21}^2 - 2B_{20}B_{22}\right)\omega_{10}^2 + B_{20}^2}, \end{aligned} \tag{19}$$

where $v_{1*} = \omega_{10}^2$ and $f_1(v_1) = v_1^4 + e_{23}v_1^3 + e_{22}^2v_1^2 + e_{21}v_1 + e_{20}$.

Thus, if the condition (H_{22}) $f_1'(v_{1*}) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega_{10}} \neq 0$. According to the Hopf bifurcation theorem in [15], we have the following results.

Theorem 1. *If the conditions (H_{21})-(H_{22}) hold, then the positive equilibrium $D_*(S_*, I_*, Q_*, R_*)$ is asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and system (2) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, I_*, Q_*, R_*)$ when $\tau_1 = \tau_{10}$.*

Case 3 ($\tau_1 = 0, \tau_2 > 0$). When $\tau_1 = 0, \tau_2 > 0$, (5) becomes

$$\begin{aligned} &\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} \\ &\quad + \left(C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right)e^{-\lambda\tau_2} \\ &\quad + \left(E_{32}\lambda^2 + E_{31}\lambda + E_{30}\right)e^{-2\lambda\tau_2} = 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} A_{30} &= A_0 + B_0, \\ A_{31} &= A_1 + B_1, \\ A_{32} &= A_2 + B_2, \\ A_{33} &= A_3 + B_3, \\ C_{30} &= C_0 + D_0, \\ C_{31} &= C_1 + D_1, \\ C_{32} &= C_2 + D_2, \\ C_{33} &= C_3, \\ E_{30} &= E_0 + F_0, \\ E_{31} &= E_1 + F_1, \\ E_{32} &= E_2. \end{aligned} \tag{21}$$

Multiplying by $e^{\lambda\tau_2}$, (20) becomes

$$\begin{aligned} &C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30} \\ &\quad + \left(\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30}\right)e^{\lambda\tau_2} \\ &\quad + \left(E_{32}\lambda^2 + E_{31}\lambda + E_{30}\right)e^{-\lambda\tau_2} = 0. \end{aligned} \tag{22}$$

Let $\lambda = i\omega_2$ ($\omega_2 > 0$) be the root of (22); then

$$\begin{aligned} &\left(\omega_2^4 - \left(A_{32} + E_{32}\right)\omega_2^2 + A_{30} - E_{30}\right)\sin\tau_2\omega_2 \\ &\quad + \left(\left(A_{31} + E_{31}\right)\omega_2 - A_{33}\omega_2^3\right)\cos\tau_2\omega_2 = C_{33}\omega_2^3 \\ &\quad - C_{31}\omega_2, \\ &\left(\omega_2^4 - \left(A_{32} + E_{32}\right)\omega_2^2 + A_{30} + E_{30}\right)\cos\tau_2\omega_2 \\ &\quad - \left(\left(A_{31} - E_{31}\right)\omega_2 - A_{33}\omega_2^3\right)\sin\tau_2\omega_2 = C_{32}\omega_2^2 \\ &\quad - A_{30}. \end{aligned} \tag{23}$$

It follows that

$$\begin{aligned} \cos\tau_2\omega_2 &= \frac{n_{36}\omega_2^6 + n_{34}\omega_2^4 + n_{32}\omega_2^2 + n_{30}}{\omega_2^8 + m_{36}\omega_2^6 + m_{34}\omega_2^4 + m_{32}\omega_2^2 + m_{30}}, \\ \sin\tau_2\omega_2 &= \frac{n_{37}\omega_2^7 + n_{35}\omega_2^5 + n_{33}\omega_2^3 + n_{31}\omega_2}{\omega_2^8 + m_{36}\omega_2^6 + m_{34}\omega_2^4 + m_{32}\omega_2^2 + m_{30}}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} m_{30} &= A_{30}^2 - E_{30}^2, \\ m_{32} &= A_{31}^2 - E_{31}^2 - 2A_{30}\left(A_{32} + E_{32}\right), \\ m_{34} &= \left(A_{32} + E_{32}\right)^2 - 2A_{31}A_{33} + 2A_{30}, \\ m_{36} &= A_{33}^2 - 2\left(A_{32} + E_{32}\right), \\ n_{30} &= \left(E_{30} - A_{30}\right)C_{30}, \\ n_{31} &= \left(A_{31} + E_{31}\right)C_{30} - \left(A_{30} + E_{30}\right)C_{31}, \\ n_{32} &= \left(A_{32} + E_{32}\right)C_{30} - \left(A_{31} - E_{31}\right)C_{31} \\ &\quad + \left(A_{30} - E_{30}\right)C_{32}, \end{aligned}$$

$$\begin{aligned}
n_{33} &= (A_{30}+E_{30})C_{33}-(A_{31}+E_{31})C_{32}\\
&\quad+(A_{32}+E_{32})C_{31}-A_{33}C_{30},\\
n_{34} &= (A_{31}-E_{31})C_{33}-(A_{32}+E_{32})C_{32}-A_{33}C_{31}\\
&\quad-C_{30},\\
n_{35} &= A_{33}C_{32}-C_{31}-(A_{32}+E_{32})C_{33},\\
n_{36} &= C_{32}-A_{33}C_{33},\\
n_{37} &= C_{33}.
\end{aligned} \tag{25}$$

Then, we can get

$$\begin{aligned}
&\omega_2^{16}+e_{37}\omega_2^{14}+e_{36}\omega_2^{12}+e_{35}\omega_2^{10}+e_{34}\omega_2^{8}+e_{33}\omega_2^{6}\\
&\quad+e_{32}\omega_2^{4}+e_{31}\omega_2^{2}+e_{30}=0,
\end{aligned} \tag{26}$$

with

$$\begin{aligned}
e_{30} &= m_{30}^2-n_{30}^2,\\
e_{31} &= 2m_{30}m_{32}-2n_{30}n_{32}-n_{31}^2,\\
e_{32} &= m_{32}^2-n_{32}^2+2m_{30}m_{34}-2n_{30}n_{34}-2n_{31}n_{33},\\
e_{33} &= 2m_{30}m_{36}+2m_{32}m_{34}-2n_{30}n_{36}-2n_{31}n_{35}\\
&\quad-2n_{32}n_{34}-n_{33}^2,\\
e_{34} &= m_{34}^2-n_{34}^2+2m_{30}+2m_{32}m_{36}-2n_{31}n_{37}\\
&\quad-2n_{32}n_{36}-2n_{33}n_{35},\\
e_{35} &= 2m_{32}+2m_{34}m_{36}-2n_{33}n_{37}-2n_{34}n_{36}-n_{35}^2,\\
e_{36} &= m_{36}^2-n_{36}^2+2m_{34}-2n_{35}n_{37},\\
e_{37} &= 2m_{36}-n_{37}^2.
\end{aligned} \tag{27}$$

Let $\omega_2^2=v_2$; (26) becomes

$$\begin{aligned}
&v_2^8+e_{37}v_2^7+e_{36}v_2^6+e_{35}v_2^5+e_{34}v_2^4+e_{33}v_2^3+e_{32}v_2^2\\
&\quad+e_{31}v_2+e_{30}=0.
\end{aligned} \tag{28}$$

Similar as in Case 2, we make the following assumption.

(H_{31}) (28) has at least one positive root.

If condition (H_{31}) holds, there exists a positive root v_{20} for (22) such that (22) has a pair of purely imaginary root $\pm i\omega_{20}=\pm i\sqrt{v_{20}}$. For ω_{20},

$$\begin{aligned}
&\tau_{20}\\
&=\frac{1}{\omega_{20}}\\
&\quad\times\arccos\frac{n_{36}\omega_{20}^6+n_{34}\omega_{20}^4+n_{32}\omega_{20}^2+n_{30}}{\omega_{20}^8+m_{36}\omega_{20}^6+m_{34}\omega_{20}^4+m_{32}\omega_{20}^2+m_{30}}.
\end{aligned} \tag{29}$$

Differentiating the two sides of (22) regarding τ_2, we obtain

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1}=\frac{P_{30}(\lambda)}{Q_{30}(\lambda)}-\frac{\tau_2}{\lambda}, \tag{30}$$

where

$$\begin{aligned}
&P_{30}(\lambda)\\
&\quad=3C_{33}\lambda^2+2C_{32}\lambda+C_{31}+(2E_{32}+E_{31})e^{-\lambda\tau_2}\\
&\qquad+\left(4\lambda^3+3A_{33}\lambda^2+2A_{32}\lambda+A_{31}\right)e^{\lambda\tau_2},\\
&Q_{30}(\lambda)\\
&\quad=\left(E_{32}\lambda^3+E_{31}\lambda^2+E_{30}\lambda\right)e^{-\lambda\tau_2}\\
&\qquad-\left(\lambda^5+A_{33}\lambda^4+A_{32}\lambda^3+A_{31}\lambda^2+A_{30}\lambda\right)e^{\lambda\tau_2},
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
P_{3R} &= \left(A_{31}+E_{31}-3A_{33}\omega_{20}^2\right)\cos\tau_{20}\omega_{20}\\
&\quad-2\left((A_{32}-E_{32})\omega_{20}-2\omega_{20}^3\right)\sin\tau_{20}\omega_{20}+C_{31}\\
&\quad-3C_{33}\omega_{20}^2,\\
P_{3I} &= \left(A_{31}-E_{31}-3A_{33}\omega_{20}^2\right)\sin\tau_{20}\omega_{20}\\
&\quad-2\left((A_{32}+E_{32})\omega_{20}-2\omega_{20}^3\right)\cos\tau_{20}\omega_{20}\\
&\quad+2C_{32}\omega_{20},\\
Q_{3R} &= \left((A_{31}-E_{31})\omega_{20}^2-A_{33}\omega_{20}^4\right)\cos\tau_{20}\omega_{20}\\
&\quad+\left(\omega_{20}^5-(A_{32}+E_{32})\omega_{20}^3+(A_{30}+E_{30})\omega_{20}\right)\\
&\quad\cdot\sin\tau_{20}\omega_{20},\\
Q_{3I} &= \left((A_{31}+E_{31})\omega_{20}^2-A_{33}\omega_{20}^4\right)\sin\tau_{20}\omega_{20}\\
&\quad-\left(\omega_{20}^5+(A_{32}-E_{32})\omega_{20}^3+(A_{30}-E_{30})\omega_{20}\right)\\
&\quad\cdot\cos\tau_{20}\omega_{20}.
\end{aligned} \tag{32}$$

Therefore, if condition (H_{32}) $P_{3R}Q_{3R}+P_{3I}Q_{3I}\neq 0$ holds, then $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\lambda=i\omega_{20}}$. Thus, we have the following results according to the Hopf bifurcation theorem in [15].

Theorem 2. *If conditions (H_{31})-(H_{32}) hold, then the positive equilibrium $D_*(S_*,I_*,Q_*,R_*)$ is asymptotically stable for $\tau_2\in[0,\tau_{20})$ and system (2) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*,I_*,Q_*,R_*)$ when $\tau_2=\tau_{20}$.*

Case 4 ($\tau_1>0$, $\tau_2>0$ and $\tau_2\in(0,\tau_{20})$). Let $\lambda=i\omega_1'$ $(\omega_1'>0)$ be the root of (5); then we obtain

$$\begin{aligned}
G_{41}(\omega_1')\sin\tau_1\omega_1'+G_{42}(\omega_1')\cos\tau_1\omega_1' &= G_{43}(\omega_1'),\\
G_{41}(\omega_1')\cos\tau_1\omega_1'-G_{42}(\omega_1')\sin\tau_1\omega_1' &= G_{44}(\omega_1'),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
G_{41}(\omega_1') &= B_1\omega_1' - B_3(\omega_1')^3 + D_1\omega_1'\cos\tau_2\omega_1' \\
&\quad - \left(D_0 - D_2(\omega_1')^2\right)\sin\tau_2\omega_1' \\
&\quad + F_1\omega_1'\cos 2\tau_2\omega_1' - F_0\sin 2\tau_2\tau_2\omega_1', \\
G_{42}(\omega_1') &= B_0 - B_2(\omega_1')^2 + D_1\omega_1'\sin\tau_2\omega_1' \\
&\quad + \left(D_0 - D_2(\omega_1')^2\right)\cos\tau_2\omega_1' \\
&\quad + F_1\omega_1'\sin 2\tau_2\omega_1' - F_0\cos 2\tau_2\tau_2\omega_1', \\
G_{43}(\omega_1') &= A_2(\omega_1')^2 - (\omega_1')^4 - A_0 \\
&\quad - \left(C_1\omega_1' - C_3(\omega_1')^3\right)\sin\tau_2\omega_1' \\
&\quad - \left(C_0 - C_2(\omega_1')^2\right)\cos\tau_2\omega_1' \\
&\quad - E_1\omega_1'\sin 2\tau_2\omega_1' \\
&\quad - \left(E_0 - E_2(\omega_1')^2\right)\cos 2\tau_2\omega_1', \\
G_{44}(\omega_1') &= A_3(\omega_1')^3 - A_1\omega_1' \\
&\quad - \left(C_1\omega_1' - C_3(\omega_1')^3\right)\cos\tau_2\omega_1' \\
&\quad + \left(C_0 - C_2(\omega_1')^2\right)\sin\tau_2\omega_1' \\
&\quad - E_1\omega_1'\cos 2\tau_2\omega_1' \\
&\quad + \left(E_0 - E_2(\omega_1')^2\right)\sin 2\tau_2\omega_1'.
\end{aligned} \tag{34}$$

Then, we can obtain the following equation with respect to ω_1':

$$\begin{aligned}
&f_{40}(\omega_1') + 2f_{41}(\omega_1')\cos\tau_2\omega_1' + 2f_{42}(\omega_1')\sin\tau_2\omega_1' \\
&\quad + 2f_{43}(\omega_1')\cos 2\tau_2\omega_1' + 2f_{44}(\omega_1')\sin 2\tau_2\omega_1' \\
&= 0,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
f_{40}(\omega_1') &= (\omega_1')^8 + \left(A_3^2 - B_3^2 + C_3^2 - 2A_2\right)(\omega_1')^6 \\
&\quad + \left(A_2^2 - B_2^2 + C_2^2 - D_2^2 + E_2^2 - 2A_1A_3 + 2B_1B_3\right. \\
&\quad \left. - 2C_1C_3 + 2A_0\right)(\omega_1')^4 + \left(A_1^2 - B_1^2 + C_1^2 - D_1^2\right. \\
&\quad \left. + E_1^2 - F_1^2 - 2A_0A_2 + 2B_0B_2 - 2C_0C_2 + 2D_0D_2\right. \\
&\quad \left. - 2E_0E_2\right)(\omega_1')^2 + A_0^2 - B_0^2 + C_0^2 - D_0^2 + E_0^2 - F_0^2, \\
f_{41}(\omega_1') &= (A_3C_3 - C_2)(\omega_1')^6 + (A_2C_2 - A_1C_3 \\
&\quad - A_3C_1 - B_2D_2 + B_3D_1 + C_2E_2 - C_3E_1 + C_0) \\
&\quad \cdot (\omega_1')^4 + (A_1C_1 - A_0C_2 - A_2C_0 + B_0D_2 - B_1D_1 \\
&\quad + B_2D_0 - C_0E_2 + C_1E_1 - C_2E_0 - D_1F_1 + D_2F_0) \\
&\quad \cdot (\omega_1')^2 + A_0C_0 - B_0F_0 - D_0F_0, \\
f_{42}(\omega_1') &= -C_3(\omega_1')^7 + (A_2C_3 - A_3C_2 + B_3D_2 \\
&\quad - C_3E_2 + C_1)(\omega_1')^5 + (A_1C_2 - A_0C_3 - A_2C_1 \\
&\quad + A_3C_0 - B_1D_2 + B_2D_1 - B_3D_0 + C_1E_2 - C_2E_1 \\
&\quad + C_3E_0 + D_2F_1)(\omega_1')^3 + (A_0E_1 - A_1E_0 - B_0D_1 \\
&\quad + B_1D_0 - C_0E_1 + C_1E_0 - D_0F_1 + D_1F_0)\omega_1', \\
f_{43}(\omega_1') &= -E_2(\omega_1')^6 + (A_2E_2 + B_3F_1 + E_0)(\omega_1')^4 \\
&\quad - (A_0E_2 + A_2E_0 + B_1F_1 - B_2F_0)(\omega_1')^2 + B_0F_0 \\
&\quad - A_0E_0, \\
f_{44}(\omega_1') &= (E_1 - A_3E_2)(\omega_1')^5 + (A_1E_2 - A_2E_1 \\
&\quad + A_3E_0 + B_2F_1 - B_3F_0)(\omega_1')^3 (A_0E_1 - A_1E_0 \\
&\quad - B_0F_1 + B_1F_0)\omega_1'.
\end{aligned} \tag{36}$$

In order to give the main results in this paper, we make the following assumption.

(H_{41}) (35) has at least one positive root.

If the condition (H_{41}) holds, then there exists a positive root ω_{10}' for (35) such that (5) has a pair of purely imaginary root $\pm i\omega_{10}'$. For ω_{10}',

$$\begin{aligned}
\tau_{10}' &= \frac{1}{\omega_{10}'} \\
&\quad \times \arccos \frac{G_{42}(\omega_{10}') \times G_{43}(\omega_{10}') + G_{41}(\omega_{10}') \times G_{44}(\omega_{10}')}{G_{41}^2(\omega_{10}') + G_{42}^2(\omega_{10}')}.
\end{aligned} \tag{37}$$

Substituting $\lambda(\tau_1)$ into (5) and differentiating both sides of it with respect to τ_1, then

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{P_{40}(\lambda)}{Q_{40}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{38}$$

where

$$\begin{aligned}
P_{40}(\lambda) &= 4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1 + \left(3B_3\lambda^2\right. \\
&\quad \left. + 2B_2\lambda + B_1\right)e^{-\lambda\tau_1} - \left(\tau_2C_3\lambda^3 + (\tau_2C_2 - 3C_3)\lambda^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\tau_2 C_1 - 2C_2\right)\lambda + \tau_2 C_0 - C_1\Big) e^{-\lambda\tau_2} - \Big(\tau_2 D_2 \lambda^2 \\
&+ \left(\tau_2 D_1 - 2D_2\right)\lambda + \tau_2 D_0 - D_1\Big) e^{-\lambda(\tau_1+\tau_2)} \\
&- \Big(2\tau_2 E_2 \lambda^2 + 2\left(\tau_2 E_1 - E_2\right)\lambda + 2\tau_2 E_0 - E_1\Big) e^{-2\lambda\tau_2} \\
&- \left(2\tau_2 F_1 \lambda + 2\tau_2 F_0 - F_1\right) e^{-\lambda(\tau_1+2\tau_2)}, \\
Q_{40}(\lambda) &= \Big(B_3\lambda^4 + B_2\lambda^3 + B_1\lambda^2 + B_0\lambda\Big) e^{-\lambda\tau_1} + \Big(D_2\lambda^3 \\
&+ D_1\lambda^2 + D_0\Big) e^{-\lambda(\tau_1+\tau_2)} + \Big(F_1\lambda^2 + F_0\lambda\Big) e^{-\lambda(\tau_1+2\tau_2)}.
\end{aligned} \tag{39}
$$

We therefore derive that

$$
\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega'_{10}} = \frac{P_{4R}Q_{4R} + P_{4I}Q_{4I}}{Q_{4R}^2 + Q_{4I}^2}, \tag{40}
$$

where

$$
\begin{aligned}
P_{4R} &= A_1 - 3A_3\left(\omega'_{10}\right)^2 + \Big[\left(2C_2 - \tau_2 C_1\right)\omega'_{10} \\
&+ \tau_2 C_3\left(\omega'_{10}\right)^3\Big] \sin\tau_2\omega'_{10}\Big[\left(\tau_2 C_2 - 3C_3\right)\left(\omega'_{10}\right)^2 \\
&+ C_1 - \tau_2 C_0\Big] \cos\tau_2\omega'_{10} + 2\left(E_2 - \tau_2 E_1\right)\omega'_{10} \\
&\cdot \sin 2\tau_2\omega'_{10} + \Big(2\tau_2 E_2\left(\omega'_{10}\right)^2 + E_1 - 2\tau_2 E_0\Big) \\
&\cdot \cos 2\tau_2\omega'_{10} + \Big[\left(2D_2 - \tau_2 D_1\right)\omega'_{10}\cos\tau_2\omega'_{10} \\
&- \Big(\tau_2 D_2\left(\omega'_{10}\right)^2 + D_1 - \tau_2 D_0\Big)\sin\tau_2\omega'_{10} \\
&- \left(F_1 - 2\tau_2 F_0\right)\sin 2\tau_2\omega'_{10} + 2\tau_2 F_1\omega'_{10}\cos 2\tau_2\omega'_{10} \\
&+ 2B_2\omega'_{10}\Big]\sin\tau'_{10}\omega'_{10} \\
&+ \Big[\left(2D_2 - \tau_2 D_1\right)\omega'_{10}\sin\tau_2\omega'_{10} \\
&+ \Big(\tau_2 D_2\left(\omega'_{10}\right)^2 + D_1 - \tau_2 D_0\Big)\cos\tau_2\omega'_{10} \\
&+ \left(F_1 - 2\tau_2 F_0\right)\cos 2\tau_2\omega'_{10} - 2\tau_2 F_1\omega'_{10}\sin 2\tau_2\omega'_{10} \\
&+ B_1 - 3B_3\left(\omega'_{10}\right)^2\Big]\cos\tau'_{10}\omega'_{10}, \\
P_{4I} &= 2A_2\omega'_{10} - 4\left(\omega'_{10}\right)^3 + \Big[\left(2C_2 - \tau_2 C_1\omega'_{10}\right)\omega'_{10} \\
&+ \tau_2 C_3\left(\omega'_{10}\right)^3\Big]\cos\tau_2\omega'_{10} - \Big[\left(\tau_2 C_2 - 3C_3\right)\left(\omega'_{10}\right)^2 \\
&+ C_1 - \tau_2 C_0\Big]\sin\tau_2\omega'_{10} + 2\left(E_2 - \tau_2 E_1\right)\omega'_{10} \\
&\cdot \cos 2\tau_2\omega'_{10} - \Big(2\tau_2 E_2\left(\omega'_{10}\right)^2 + E_1 - 2\tau_2 E_0\Big) \\
&\cdot \sin 2\tau_2\omega'_{10}\Big[\left(2D_2 - \tau_2 D_1\right)\omega'_{10}\cos\tau_2\omega'_{10} \\
&- \Big(\tau_2 D_2\left(\omega'_{10}\right)^2 + D_1 - \tau_2 D_0\Big)\sin\tau_2\omega'_{10} \\
&- \left(F_1 - 2\tau_2 F_0\right)\sin 2\tau_2\omega'_{10} + 2\tau_2 F_1\omega'_{10}\cos 2\tau_2\omega'_{10} \\
&+ 2B_2\omega'_{10}\Big]\cos\tau'_{10}\omega'_{10} \\
&- \Big[\left(2D_2 - \tau_2 D_1\right)\omega'_{10}\sin\tau_2\omega'_{10} \\
&+ \Big(\tau_2 D_2\left(\omega'_{10}\right)^2 + D_1 - \tau_2 D_0\Big)\cos\tau_2\omega'_{10} \\
&+ \left(F_1 - 2\tau_2 F_0\right)\cos 2\tau_2\omega'_{10} 2\tau_2 F_1\omega'_{10}\sin 2\tau_2\omega'_{10} + B_1 \\
&- 3B_3\left(\omega'_{10}\right)^2\Big]\sin\tau'_{10}\omega'_{10}, \\
Q_{4R} &= \Big[\Big(D_0\omega'_{10} - D_2\left(\omega'_{10}\right)^3\Big)\sin\tau_2\omega'_{10} \\
&- D_1\left(\omega'_{10}\right)^2\cos\tau_2\omega'_{10} + F_0\omega'_{10}\sin 2\tau_2\omega'_{10} \\
&- F_1\left(\omega'_{10}\right)^2\cos 2\tau_2\omega'_{10} + B_3\left(\omega'_{10}\right)^4 - B_1\left(\omega'_{10}\right)^2\Big] \\
&\cdot \cos\tau'_{10}\omega'_{10} + \Big[\Big(D_0\omega'_{10} - D_2\left(\omega'_{10}\right)^3\Big)\cos\tau_2\omega'_{10} \\
&+ D_1\left(\omega'_{10}\right)^2\sin\tau_2\omega'_{10} + F_0\omega'_{10}\cos 2\tau_2\omega'_{10} \\
&+ F_1\left(\omega'_{10}\right)^2\sin 2\tau_2\omega'_{10} + B_0\omega'_{10} - B_2\left(\omega'_{10}\right)^3\Big] \\
&\cdot \sin\tau'_{10}\omega'_{10}, \\
Q_{4I} &= \Big[\Big(D_0\omega'_{10} - D_2\left(\omega'_{10}\right)^3\Big)\cos\tau_2\omega'_{10} \\
&+ D_1\left(\omega'_{10}\right)^2\sin\tau_2\omega'_{10} + F_0\omega'_{10}\cos 2\tau_2\omega'_{10} \\
&+ F_1\left(\omega'_{10}\right)^2\sin 2\tau_2\omega'_{10} + B_0\omega'_{10} - B_2\left(\omega'_{10}\right)^3\Big] \\
&\cdot \cos\tau'_{10}\omega'_{10} - \Big[\Big(D_0\omega'_{10} - D_2\left(\omega'_{10}\right)^3\Big)\sin\tau_2\omega'_{10} \\
&- D_1\left(\omega'_{10}\right)^2\cos\tau_2\omega'_{10} + F_0\omega'_{10}\sin 2\tau_2\omega'_{10} \\
&- F_1\left(\omega'_{10}\right)^2\cos 2\tau_2\omega'_{10} + B_3\left(\omega'_{10}\right)^4 - B_1\left(\omega'_{10}\right)^2\Big] \\
&\cdot \sin\tau'_{10}\omega'_{10}.
\end{aligned} \tag{41}
$$

Therefore, if condition (H_{42}) $P_{4R}Q_{4R} + P_{4I}Q_{4I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega'_{10}} \neq 0$. Thus, we have the following results according to the Hopf bifurcation theorem in [15].

Theorem 3. *For system (2), if conditions (H_{41})-(H_{42}) hold and $\tau_2 \in (0, \tau_{20})$, then the positive equilibrium $D_*(S_*, I_*, Q_*, R_*)$ is asymptotically stable for $\tau_1 \in [0, \tau'_{10})$ and system (2) undergoes a Hopf bifurcation at the positive equilibrium $D_*(S_*, I_*, Q_*, R_*)$ when $\tau_1 = \tau'_{10}$ and a family of periodic solutions bifurcate from $D_*(S_*, I_*, Q_*, R_*)$ near $\tau_1 = \tau'_{10}$.*

Case 5 ($\tau_1 > 0$, $\tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$). Multiplying by $e^{\lambda\tau_2}$, (5) becomes

$$\begin{aligned} &C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 \\ &\quad + \left(\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right)e^{\lambda\tau_2} \\ &\quad + \left(B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0\right)e^{\lambda(\tau_2-\tau_1)} \\ &\quad + \left(D_2\lambda^2 + D_1\lambda + D_0\right)e^{-\lambda\tau_1} \\ &\quad + \left(E_2\lambda^2 + E_1\lambda + E_0\right)e^{-\lambda\tau_2} \\ &\quad + \left(F_1\lambda + F_0\right)e^{-\lambda(\tau_1+\tau_2)} = 0. \end{aligned} \tag{42}$$

Let $\lambda = i\omega_2'$ ($\omega_2' > 0$) be the root of (42); then we get

$$\begin{aligned} G_{51}\left(\omega_2'\right)\cos\tau_2\omega_2' - G_{52}\left(\omega_2'\right)\sin\tau_2\omega_2' &= G_{55}\left(\omega_2'\right), \\ G_{53}\left(\omega_2'\right)\sin\tau_2\omega_2' + G_{54}\left(\omega_2'\right)\cos\tau_2\omega_2' &= G_{56}\left(\omega_2'\right), \end{aligned} \tag{43}$$

where

$$\begin{aligned} G_{51}\left(\omega_2'\right) &= \left(\omega_2'\right)^4 - \left(A_2 + E_2\right)\left(\omega_2'\right)^2 + A_0 + E_0 \\ &\quad + \left(B_0 + F_0 - B_2\left(\omega_2'\right)^2\right)\cos\tau_1\omega_2' \\ &\quad + \left(\left(B_1 + F_1\right)\omega_2' - B_3\left(\omega_2'\right)^3\right)\sin\tau_1\omega_2', \\ G_{52}\left(\omega_2'\right) &= \left(A_1 - E_1\right)\omega_2' - A_3\left(\omega_2'\right)^3 \\ &\quad - \left(B_0 - F_0 - B_2\left(\omega_2'\right)^2\right)\sin\tau_1\omega_2' \\ &\quad + \left(\left(B_1 - F_1\right)\omega_2' - B_3\left(\omega_2'\right)^3\right)\cos\tau_1\omega_2', \\ G_{53}\left(\omega_2'\right) &= \left(\omega_2'\right)^4 - \left(A_2 - E_2\right)\left(\omega_2'\right)^2 + A_0 - E_0 \\ &\quad + \left(B_0 - F_0 - B_2\left(\omega_2'\right)^2\right)\cos\tau_1\omega_2' \\ &\quad + \left(\left(B_1 - F_1\right)\omega_2' - B_3\left(\omega_2'\right)^3\right)\sin\tau_1\omega_2', \\ G_{54}\left(\omega_2'\right) &= \left(A_1 + E_1\right)\omega_2' - A_3\left(\omega_2'\right)^3 \\ &\quad - \left(B_0 + F_0 - B_2\left(\omega_2'\right)^2\right)\sin\tau_1\omega_2' \\ &\quad + \left(\left(B_1 + F_1\right)\omega_2' - B_3\left(\omega_2'\right)^3\right)\cos\tau_1\omega_2', \\ G_{55}\left(\omega_2'\right) &= C_2\left(\omega_2'\right)^2 - C_0 - D_1\omega_2\sin\tau_1\omega_2' \\ &\quad - \left(D_0 - D_2\left(\omega_2'\right)^2\right)\cos\tau_1\omega_2', \\ G_{56}\left(\omega_2'\right) &= C_3\left(\omega_2'\right)^3 - C_1\omega_2' - D_1\omega_2'\cos\tau_1\omega_2' \\ &\quad + \left(D_0 - D_2\left(\omega_2'\right)^2\right)\sin\tau_1\omega_2'. \end{aligned} \tag{44}$$

Then, we get

$$\begin{aligned} &\cos\tau_2\omega_2' \\ &\quad = \frac{G_{52}\left(\omega_2'\right)\times G_{56}\left(\omega_2'\right) + G_{53}\left(\omega_2'\right)\times G_{55}\left(\omega_2'\right)}{G_{51}\left(\omega_2'\right)\times G_{53}\left(\omega_2'\right) + G_{52}\left(\omega_2'\right)\times G_{54}\left(\omega_2'\right)}, \\ &\sin\tau_2\omega_2' \\ &\quad = \frac{G_{51}\left(\omega_2'\right)\times G_{56}\left(\omega_2'\right) - G_{54}\left(\omega_2'\right)\times G_{55}\left(\omega_2'\right)}{G_{51}\left(\omega_2'\right)\times G_{53}\left(\omega_2'\right) + G_{52}\left(\omega_2'\right)\times G_{54}\left(\omega_2'\right)}. \end{aligned} \tag{45}$$

Thus, we can obtain the following function with respect to ω_2':

$$\cos^2\tau_2\omega_2' + \sin^2\tau_2\omega_2' = 1. \tag{46}$$

Suppose that (H_{51}) (46) has at least one positive root.

If condition (H_{51}) holds, there exists a positive root $\omega_2' > 0$ of (46) such that (46) has a pair of purely imaginary roots $\pm i\omega_2'$. For ω_2',

$$\begin{aligned} \tau_{20}' &= \frac{1}{\omega_{20}'} \\ &\quad \times \arccos\frac{G_{52}\left(\omega_{20}'\right)\times G_{56}\left(\omega_{20}'\right) + G_{53}\left(\omega_{20}'\right)\times G_{55}\left(\omega_{20}'\right)}{G_{51}\left(\omega_{20}'\right)\times G_{53}\left(\omega_{20}'\right) + G_{52}\left(\omega_{20}'\right)\times G_{54}\left(\omega_{20}'\right)}. \end{aligned} \tag{47}$$

Taking the derivative with respect to τ_2 in (42), we can obtain

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{P_{50}(\lambda)}{Q_{50}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{48}$$

where

$$\begin{aligned} P_{50}(\lambda) &= 3C_3\lambda^2 + 2C_2\lambda + C_1 + \left(4\lambda^3 + 3A_3\lambda^2\right. \\ &\quad \left. + 2A_2\lambda + A_1\right)e^{\lambda\tau_2} + \left(2E_2\lambda + E_1\right)e^{-\lambda\tau_2} - \left(\tau_1D_2\lambda^2\right. \\ &\quad \left. + \left(\tau_1D_1 - 2D_2\right)\lambda + \tau_1D_0 - D_1\right)e^{-\lambda\tau_1} - \left(\tau_1F_1\lambda\right. \\ &\quad \left. + \tau_1F_0 - F_1\right)e^{-\lambda(\tau_1+\tau_2)} - \left(\tau_1B_3\lambda^3 + \left(\tau_1B_2 - 3B_3\right)\lambda^2\right. \\ &\quad \left. + \left(\tau_1B_1 - 2B_2\right)\lambda + \tau_1B_0 - B_1\right)e^{\lambda(\tau_2-\tau_1)}, \\ Q_{50}(\lambda) &= \left(E_2\lambda^3 + E_1\lambda^2 + E_0\lambda\right)e^{-\lambda\tau_2} + \left(F_1\lambda^2 + F_0\lambda\right) \\ &\quad \cdot e^{-\lambda(\tau_1+\tau_2)} - \left(\lambda^5 + A_3\lambda^4 + A_2\lambda^3 + A_1\lambda^2 + A_0\lambda\right) \\ &\quad \cdot e^{\lambda\tau_2} - \left(B_3\lambda^4 + B_2\lambda^3 + B_1\lambda^2 + B_0\lambda\right)e^{\lambda(\tau_2-\tau_1)}. \end{aligned} \tag{49}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}'} = \frac{P_{5R}Q_{5R} + P_{5I}Q_{5I}}{Q_{5R}^2 + Q_{5I}^2}. \tag{50}$$

Obviously, if the condition (H_{52}) $P_{5R}Q_{5R} + P_{5I}Q_{5I} \neq 0$, then $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\lambda=i\omega_{20}'} \neq 0$. Thus, we have the following results according to the Hopf bifurcation theorem in [15].

Theorem 4. *For system (2), if the conditions* (H_{51})-(H_{52}) *hold and* $\tau_1 \in (0, \tau_{10})$, *then the positive equilibrium* $D_*(S_*, I_*, Q_*, R_*)$ *is asymptotically stable for* $\tau_2 \in [0, \tau'_{20})$ *and system (2) undergoes a Hopf bifurcation at the positive equilibrium* $D_*(S_*, I_*, Q_*, R_*)$ *when* $\tau_2 = \tau'_{20}$ *and a family of periodic solutions bifurcate from* $D_*(S_*, I_*, Q_*, R_*)$ *near* $\tau_2 = \tau'_{20}$.

3. Properties of the Hopf Bifurcation

In this section, we investigate the direction of the Hopf bifurcation and the stability of the Hopf bifurcation of system (2) when $\tau_2 > 0$ and $\tau_1 = \tau_{1*} \in (0, \tau_{10})$ by using the normal form theory and the center manifold theorem in [15]. Throughout this section, we assume that $\tau_{1*} < \tau'_{20}$.

Let $u_1(t) = S(t) - S_*$, $u_2(t) = I(t) - I_*$, $u_3(t) = Q(t) - Q_*$, $u_4(t) = R(t) - R_*$, and $\tau_2 = \tau'_{20} + \mu$, $\mu \in R$ and normalize the time delay by $t \to (t/\tau_2)$. Then system (1) can be transformed into as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{51}$$

where

$$\begin{aligned}
&u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in R^4,\\
&L_\mu\phi = (\tau'_{20} + \mu)\\
&\quad\cdot\left(A_{\text{trix}}\phi(0) + B_{\text{trix}}\phi\left(-\frac{\tau_{1*}}{\tau^*_{20}}\right) + C_{\text{trix}}\phi(-1)\right),\\
&F(\mu, \phi) = (\tau^*_{10} + \mu)\\
&\quad\cdot\begin{pmatrix} -\beta\phi_1\left(-\frac{\tau_{1*}}{\tau^*_{20}}\right)\phi_2\left(-\frac{\tau_{1*}}{\tau^*_{20}}\right)\\ \beta\phi_1\left(-\frac{\tau_{1*}}{\tau^*_{20}}\right)\phi_2\left(-\frac{\tau_{1*}}{\tau^*_{20}}\right)\\ 0\\ 0\end{pmatrix},\\
&A_{\text{trix}} = \begin{pmatrix} a_{11} & 0 & 0 & 0\\ 0 & a_{12} & 0 & 0\\ 0 & a_{32} & a_{33} & 0\\ 0 & 0 & 0 & a_{44}\end{pmatrix},\\
&B_{\text{trix}} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0\\ b_{21} & b_{22} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\\
&C_{\text{trix}} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & c_{22} & 0 & 0\\ 0 & 0 & c_{33} & 0\\ 0 & c_{42} & c_{43} & 0\end{pmatrix}.
\end{aligned} \tag{52}$$

By the Riesz representation theorem, there is a 4×4 matrix function with bounded variation components $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C([-1, 0], R^4). \tag{53}$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau'_{20} + \mu)(A_{\text{trix}} + B_{\text{trix}} + C_{\text{trix}}), & \theta = 0,\\ (\tau'_{20} + \mu)(B_{\text{trix}} + C_{\text{trix}}), & \theta \in \left[-\frac{\tau_{1*}}{\tau'_{20}}, 0\right),\\ (\tau'_{20} + \mu)C_{\text{trix}}, & \theta \in \left(-1, -\frac{\tau_{1*}}{\tau'_{20}}\right),\\ 0, & \theta = -1. \end{cases} \tag{54}$$

For $\phi \in C([-1, 0], R^4)$, we define

$$\begin{aligned}
A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0,\\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases}\\
R(\mu)\phi &= \begin{cases} 0, & -1 \le \theta < 0,\\ F(\mu, \phi), & \theta = 0. \end{cases}
\end{aligned} \tag{55}$$

Then system (51) is equivalent to the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{56}$$

Next, we define the adjoint operator A^* of A

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1,\\ \displaystyle\int_{-1}^{0} d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases} \tag{57}$$

and a bilinear inner product

$$\begin{aligned}
\langle\varphi(s), \phi(\theta)\rangle &= \overline{\varphi}(0)\phi(0)\\
&\quad - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi - \theta)\, d\eta(\theta)\phi(\xi)\, d\xi,
\end{aligned} \tag{58}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $\rho(\theta) = (1, \rho_2, \rho_3, \rho_4)^T e^{i\omega'_{20}\tau'_{20}\theta}$ be the eigenvectors of $A(0)$ corresponding to $+i\omega'_{20}\tau'_{20}$ and $\rho'(s) = (1/K)(1, \rho'_2, \rho'_3, \rho'_4)e^{i\omega'_{20}\tau'_{20}s}$ be the eigenvectors of A^* corresponding to $-i\omega'_{20}\tau'_{20}$. Then, we obtain

$$\begin{aligned}
\rho_2 &= \frac{i\omega'_{20} - a_{11} - b_{11}e^{-i\tau_{1*}\omega'_{20}}}{b_{12}e^{-i\tau_{1*}\omega'_{20}}},\\
\rho_3 &= \frac{a_{32}\left(i\omega'_{20} - a_{11} - b_{11}e^{-i\tau_{1*}\omega'_{20}}\right)}{\left(i\omega'_{20} - a_{33} - c_{33}e^{-i\tau'_{20}\omega'_{20}}\right)b_{12}e^{-i\tau_{1*}\omega'_{20}}},\\
\rho_4 &= \frac{c_{42}\rho_2 + c_{43}\rho_3}{\left(i\omega'_{20} - a_{44}\right)e^{i\tau'_{20}\omega'_{20}}},\\
\rho'_2 &= -\frac{i\omega'_{20} + a_{11} + b_{11}e^{i\tau_{1*}\omega'_{20}}}{b_{21}e^{i\tau_{1*}\omega'_{20}}},\\
\rho'_4 &= -\frac{i\omega'_{20} + a_{33} + c_{33}e^{i\tau'_{20}\omega'_{20}}}{c_{43}e^{i\tau'_{20}\omega'_{20}}},\\
\rho'_3 &= -\left[\left(i\omega'_{20} + a_{22} + b_{22}e^{i\tau_{1*}\omega'_{20}}\right)\rho'_2 + b_{12}e^{i\tau_{1*}\omega'_{20}}\right]\\
&\quad\times\left(c_{42}e^{i\tau'_{20}\omega'_{20}} - \frac{a_{32}c_{43}e^{i\tau'_{20}\omega'_{20}}}{i\omega'_{20} + c_{33}e^{i\tau'_{20}\omega'_{20}}}\right)^{-1}.
\end{aligned} \tag{59}$$

From (58), we can obtain

$$\begin{aligned}
\overline{K} &= 1 + \rho_2\overline{\rho}'_2 + \rho_3\overline{\rho}'_3 + \rho_4\overline{\rho}'_4 + \tau_{1*}e^{-i\omega'_{20}\tau_{1*}}\left(b_{11}\right.\\
&\quad\left.+ b_{21}\overline{\rho}'_2 + \rho_2\left(b_{12} + b_{22}\overline{\rho}'_2\right)\right)\\
&\quad+ \tau'_{20}e^{-i\omega'_{20}\tau'_{20}}\left(\rho_2\left(c_{22}\overline{\rho}'_2 + c_{42}\overline{\rho}'_4\right)\right.\\
&\quad\left.+ \rho_3\left(c_{33}\overline{\rho}'_3 + c_{43}\overline{\rho}'_4\right)\right)
\end{aligned} \tag{60}$$

such that $\langle\rho^*, \rho\rangle = 1$, $\langle\rho^*, \overline{\rho}\rangle = 0$.

Following the algorithms given in [15] and using similar computation process in [16], we can get the coefficients which determine the properties of the Hopf bifurcation:

$$\begin{aligned}
g_{20} &= 2\beta\tau'_{20}\overline{K}\left(\overline{\rho}'_2 - 1\right)\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right),\\
g_{11} &= \beta\tau'_{20}\overline{K}\left(\overline{\rho}'_2 - 1\right)\left(\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right.\\
&\quad\left.+ \overline{\rho}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right),\\
g_{02} &= 2\beta\tau'_{20}\overline{K}\left(\overline{\rho}'_2 - 1\right)\overline{\rho}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right),\\
g_{21} &= 2\beta\tau'_{20}\overline{K}\left(\overline{\rho}'_2 - 1\right)\left(W_{11}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right.\\
&\quad+ \frac{1}{2}W_{20}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\\
&\quad+ W_{11}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\\
&\quad\left.+ \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right),
\end{aligned} \tag{61}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}\rho(0)}{\tau'_{20}\omega'_{20}}e^{i\tau'_{20}\omega'_{20}\theta} + \frac{i\overline{g}_{02}\overline{\rho}(0)}{3\tau'_{20}\omega'_{20}}e^{-i\tau'_{20}\omega'_{20}\theta}\\
&\quad+ E_1e^{2i\tau'_{20}\omega'_{20}\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}\rho(0)}{\tau'_{20}\omega'_{20}}e^{i\tau'_{20}\omega'_{20}} + \frac{i\overline{g}_{11}\overline{\rho}(0)}{\tau'_{20}\omega'_{20}}e^{-i\tau'_{20}\omega'_{20}\theta}\\
&\quad+ E_2,
\end{aligned} \tag{62}$$

where E_1 and E_2 can be determined by the following equations, respectively,

$$\begin{aligned}
&\begin{pmatrix} a'_{11} & -b_{12}e^{-2i\tau_{1*}\omega'_{20}} & 0 & 0\\ -b_{21}e^{-2i\tau_{1*}\omega'_{20}} & a'_{22} & 0 & 0\\ 0 & -a_{32} & a'_{33} & 0\\ 0 & -c_{42}e^{-2i\tau'_{20}\omega'_{20}} & -c_{43}e^{-2i\tau'_{20}\omega'_{20}} & 2i\omega'_{20} - a_{44}\end{pmatrix}E_1 = 2\begin{pmatrix} E_{11}\\ E_{12}\\ 0\\ 0\end{pmatrix},\\
&\begin{pmatrix} a_{11} + b_{11} & b_{21} & 0 & 0\\ b_{21} & a_{22} + b_{22} + c_{22} & 0 & 0\\ 0 & a_{32} & a_{33} + a_{44} & 0\\ 0 & c_{42} & c_{43} & a_{44}\end{pmatrix}E_2 = \begin{pmatrix} E_{21}\\ E_{22}\\ 0\\ 0\end{pmatrix},
\end{aligned} \tag{63}$$

with

$$
\begin{aligned}
a'_{11} &= 2i\omega'_{20} - a_{11} - b_{11}e^{-2i\tau_{1*}\omega'_{20}},\\
a'_{22} &= 2i\omega'_{20} - a_{22} - b_{22}e^{-2i\tau_{1*}\omega'_{20}} - c_{22}e^{-2i\tau'_{20}\omega'_{20}},\\
a'_{33} &= 2i\omega'_{20} - a_{33} - c_{33}e^{-2i\tau'_{20}\omega'_{20}},\\
E_{11} &= -\beta\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right),\\
E_{12} &= \beta\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right),\\
E_{21} &= -\beta\left(\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right.\\
&\quad\left.+\overline{\rho}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right),\\
E_{22} &= \beta\left(\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\overline{\rho}^{(2)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right.\\
&\quad\left.+\overline{\rho}^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\rho^{(1)}\left(-\frac{\tau_{1*}}{\tau'_{20}}\right)\right).
\end{aligned}
\tag{64}
$$

Then, we can get the following coefficients:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\tau'_{20}\omega'_{20}}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right)\\
&\quad + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau'_{20})\}},\\
\beta_2 &= 2\operatorname{Re}\{C_1(0)\},\\
T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau'_{20})\}}{\tau'_{20}\omega'_{20}},
\end{aligned}
\tag{65}
$$

in which μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcation periodic solutions and T_2 determines the period of the bifurcation periodic solutions. From the conclusion of [15], we have the main results in this section.

Theorem 5. *For system (2), μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical). β_2 determines the stability of the bifurcating periodic solution: if $\beta_2 < 0$ ($\beta_2 > 0$) the bifurcating periodic solutions are stable (unstable). T_2 determines the period of the bifurcating periodic solution: if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Investigations

In this section, a numerical example is given to support the theoretical results in Sections 2 and 3. We choose the same values of the parameters as in [14] and let $p = 0.2$, $\beta = 0.1$, $d = 0.01$, $\alpha_1 = 0.01$, $\gamma = 0.5$, $\delta = 0.1$, $\varepsilon = 0.1$, $\alpha_2 = 0.02$, and $b = 10$. Then, we get the following system:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= 8 - 0.1S(t-\tau_1)I(t-\tau_1) - 0.01S(t),\\
\frac{dI(t)}{dt} &= 0.1S(t-\tau_1)I(t-\tau_1) - 0.12I(t)\\
&\quad - 0.5I(t-\tau_2),\\
\frac{dQ(t)}{dt} &= 0.1I(t) - 0.03Q(t) - 0.1 * Q(t-\tau_2),\\
\frac{dR(t)}{dt} &= 0.5I(t-\tau_2) + 2 + 0.1Q(t-\tau_2)\\
&\quad - 0.01R(t).
\end{aligned}
\tag{66}
$$

Then, we can get $R_0 = 129.0323 > 1$ and the unique positive equilibrium $D_*(6.2, 12.8032, 9.8486, 938.646)$ of system (66) and condition (H_{11}) holds.

For $\tau_1 > 0$, $\tau_2 = 0$, we obtain $\omega_{10} = 0.4107$, $\tau_{10} = 1.3265$. By Theorem 1, we can conclude that the positive equilibrium D_* is locally asymptotically stable when $\tau_1 \in [0, 1.3265)$. In this case, the propagation of computer viruses can be predicted and controlled easily. This property can be illustrated by Figures 1–3. However, if we choose $\tau_1 = 1.3625 > \tau_{10} = 1.3265$, the positive equilibrium D_* becomes unstable and a Hopf bifurcation occurs, which can be illustrated by Figures 4–6. When a Hopf bifurcation occurs, the state of computer viruses propagation changes from an equilibrium point to a limit cycle and it means that the propagation of computer viruses is out of control. Similarly, we have $\omega_{20} = 1.6022$, $\tau_{20} = 1.7852$ when $\tau_1 = 0$, $\tau_2 > 0$. The corresponding waveforms and phase plots are shown as in Figures 7–12.

For $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_2 = 1.05 \in (0, \tau_{20})$. We can obtain $\omega'_1 = 1.9403$ and then we obtain $\tau'_{10} = 0.8903$. Figures 13–15 show that the positive equilibrium D_* of system (66) is asymptotically stable when $\tau_1 = 0.785 < 0.8903 = \tau'_{10}$ and Figures 16–18 show that there is a Hopf bifurcation occurs at the positive equilibrium D_* of system (66) and a family of periodic solutions bifurcate from D_* when $\tau_1 = 0.9755 > \tau'_{10} = 0.8903$. Similarly, we have $\omega'_2 = 2.0411$ and $\tau'_{20} = 1.3485$ when $\tau_2 > 0$, $\tau_1 = 0.65 \in (0, \tau_{10})$. The corresponding waveforms and plots are shown in Figures 19–24. In addition, we can obtain $\mu_2 = 1.3987$, $\beta_2 = -2.2318 < 0$, $T_2 = 8.4012 > 0$. Thus, we can conclude that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the periodic solutions increases according to Theorem 5.

5. Critical Analysis and Research Perspectives

In this paper, an SIQR computer virus model with two delays is investigated. Compared with the system considered in [14],

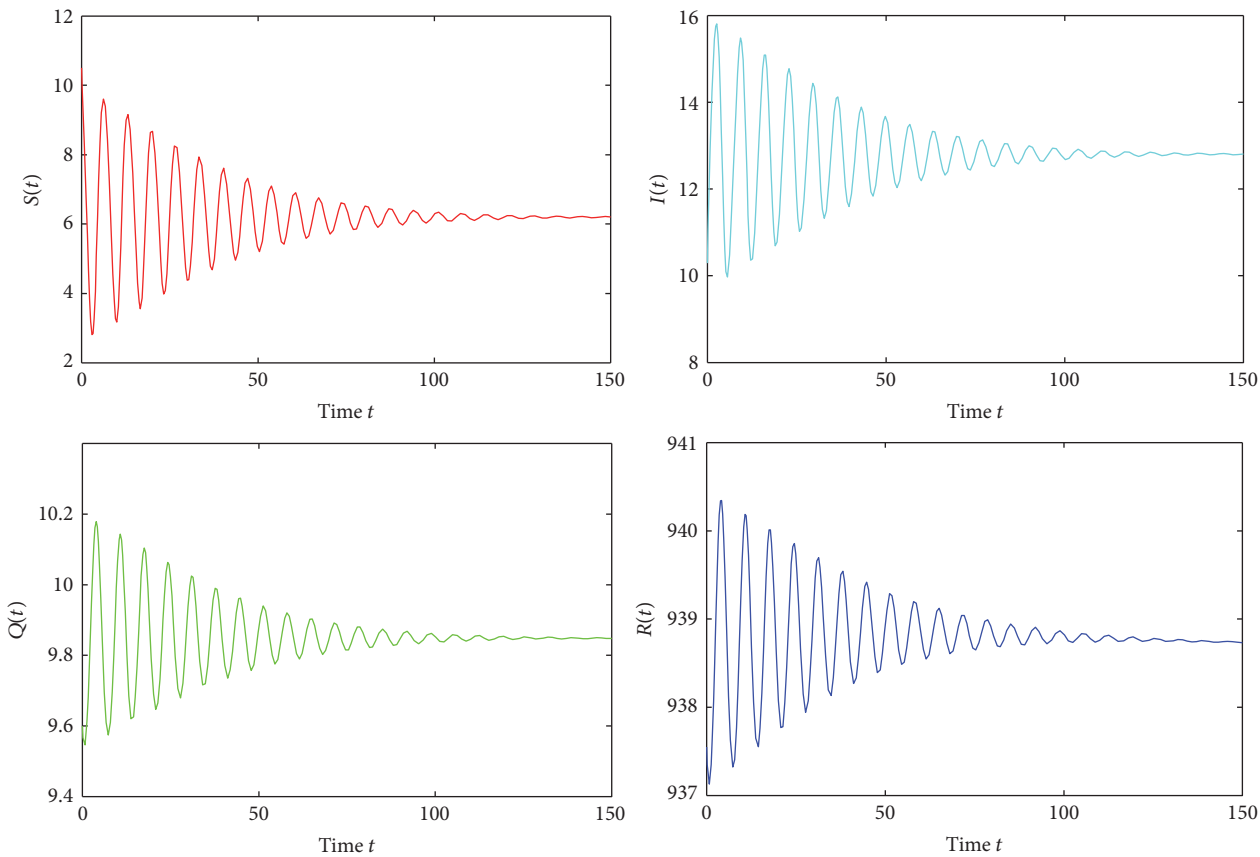

Figure 1: The trajectory of S, I, Q, and R when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.

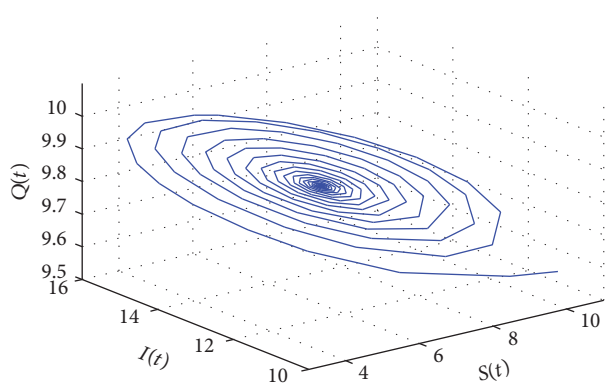

Figure 2: The phase plot of S, I, and Q when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.

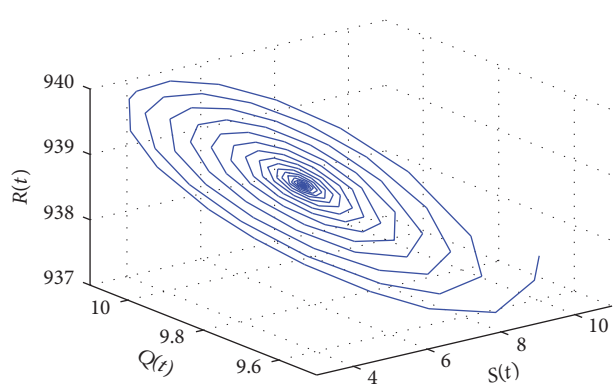

Figure 3: The phase plot of S, Q, and R when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.

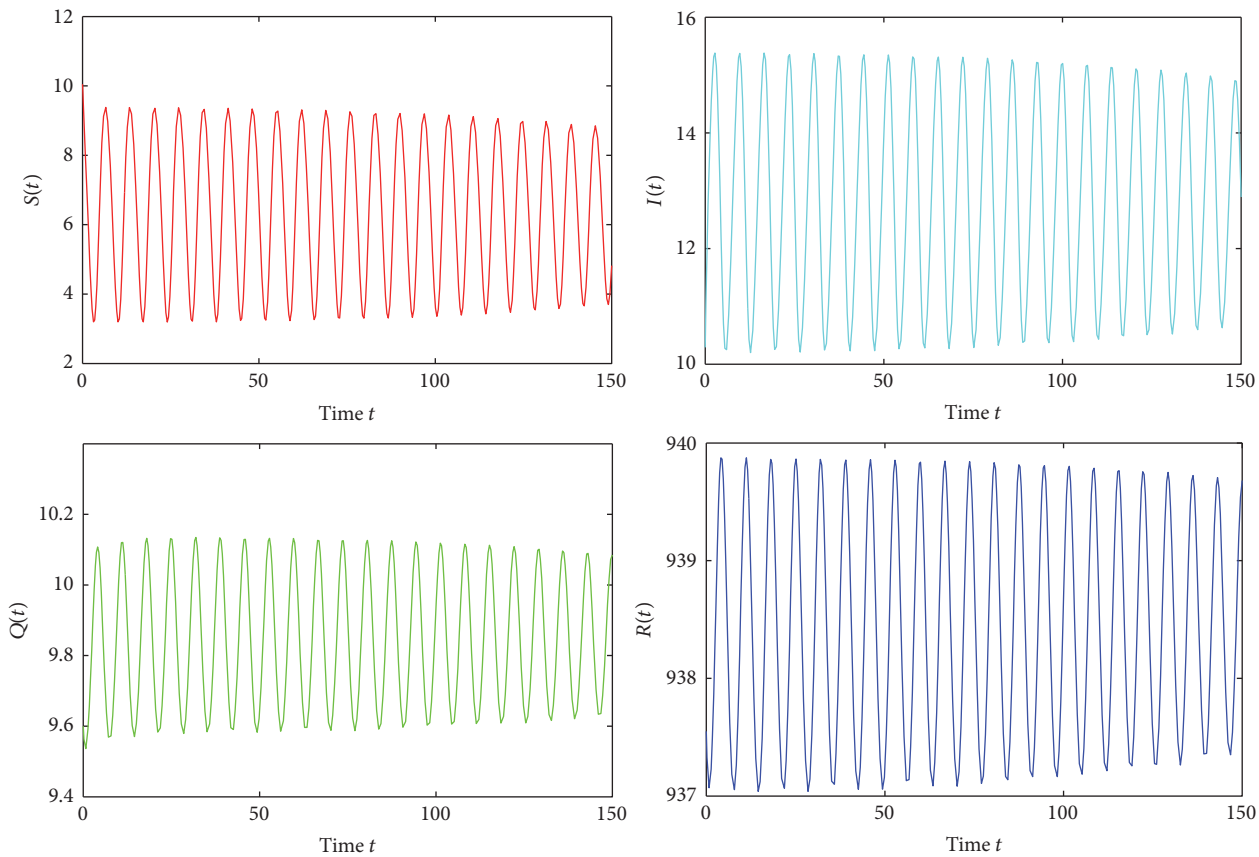

FIGURE 4: The trajectory of S, I, Q, and R when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$.

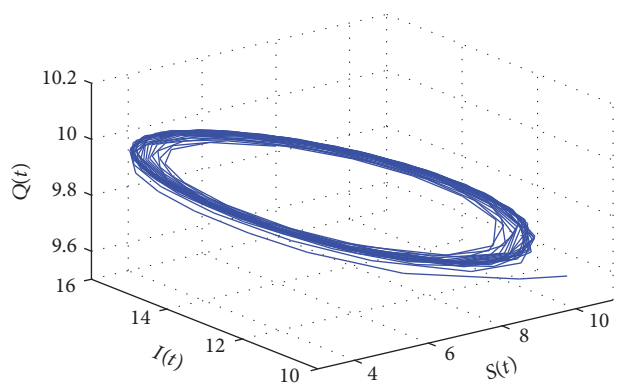

FIGURE 5: The phase plot of S, I, and Q when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$.

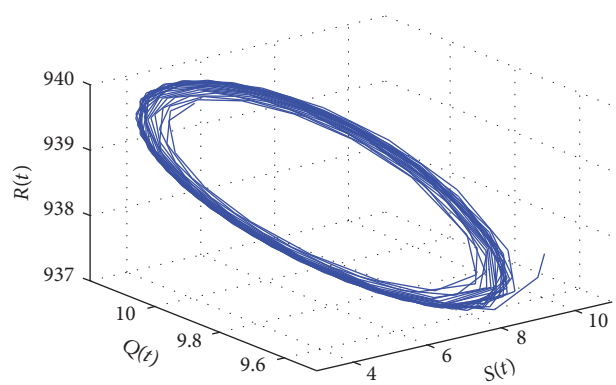

FIGURE 6: The phase plot of S, Q, and R when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$.

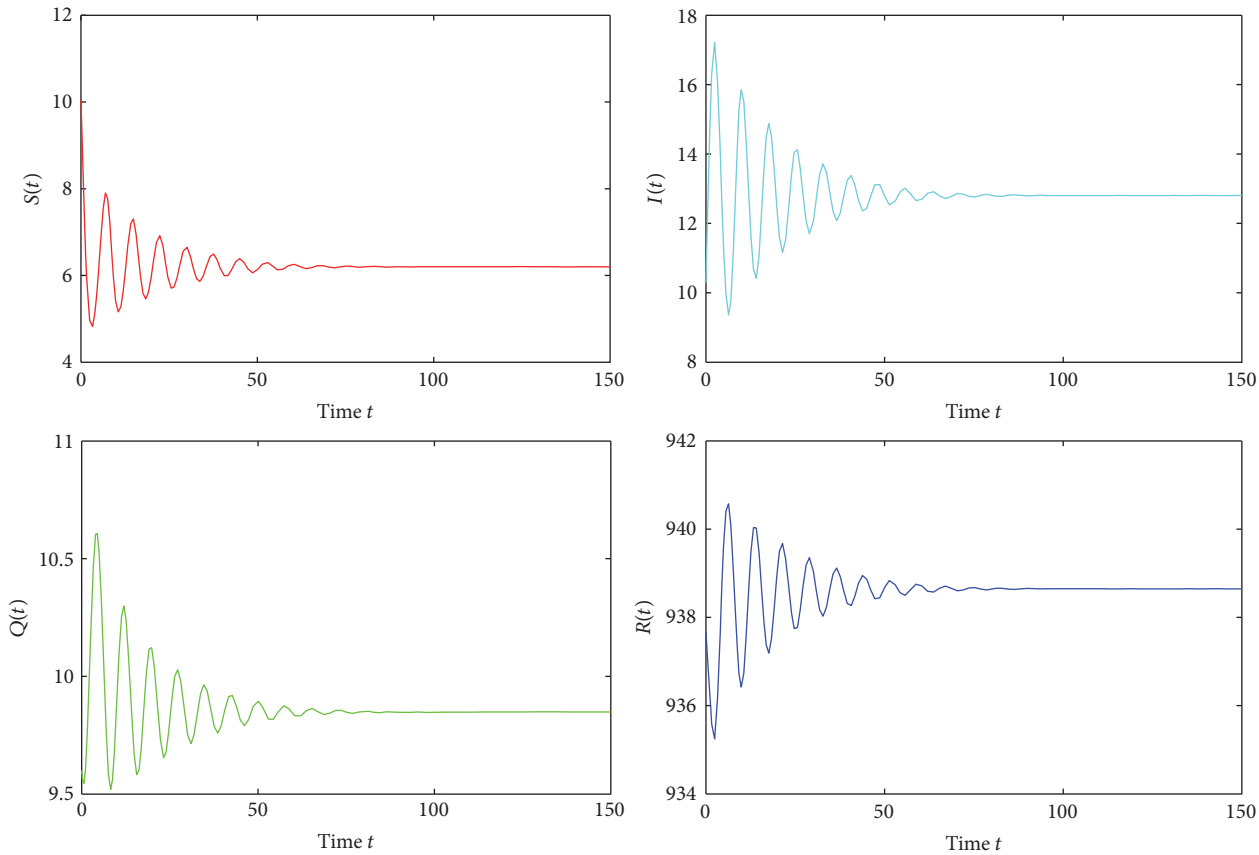

Figure 7: The trajectory of S, I, Q, and R when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.

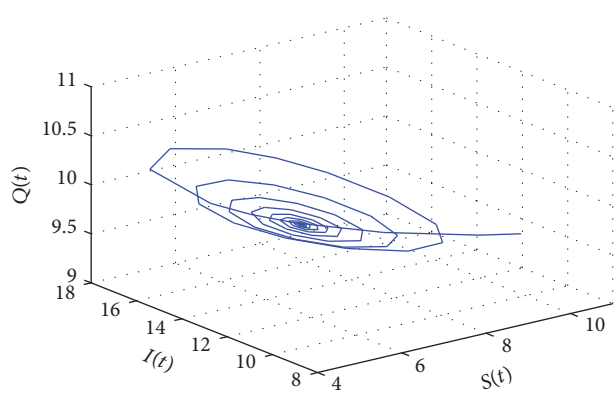

Figure 8: The phase plot of S, I, and Q when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.

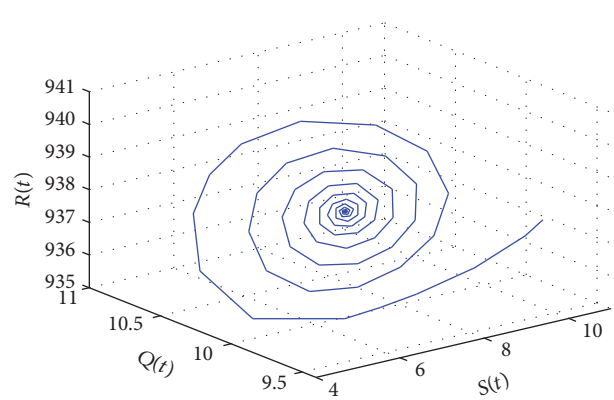

Figure 9: The phase plot of S, Q, and R when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.

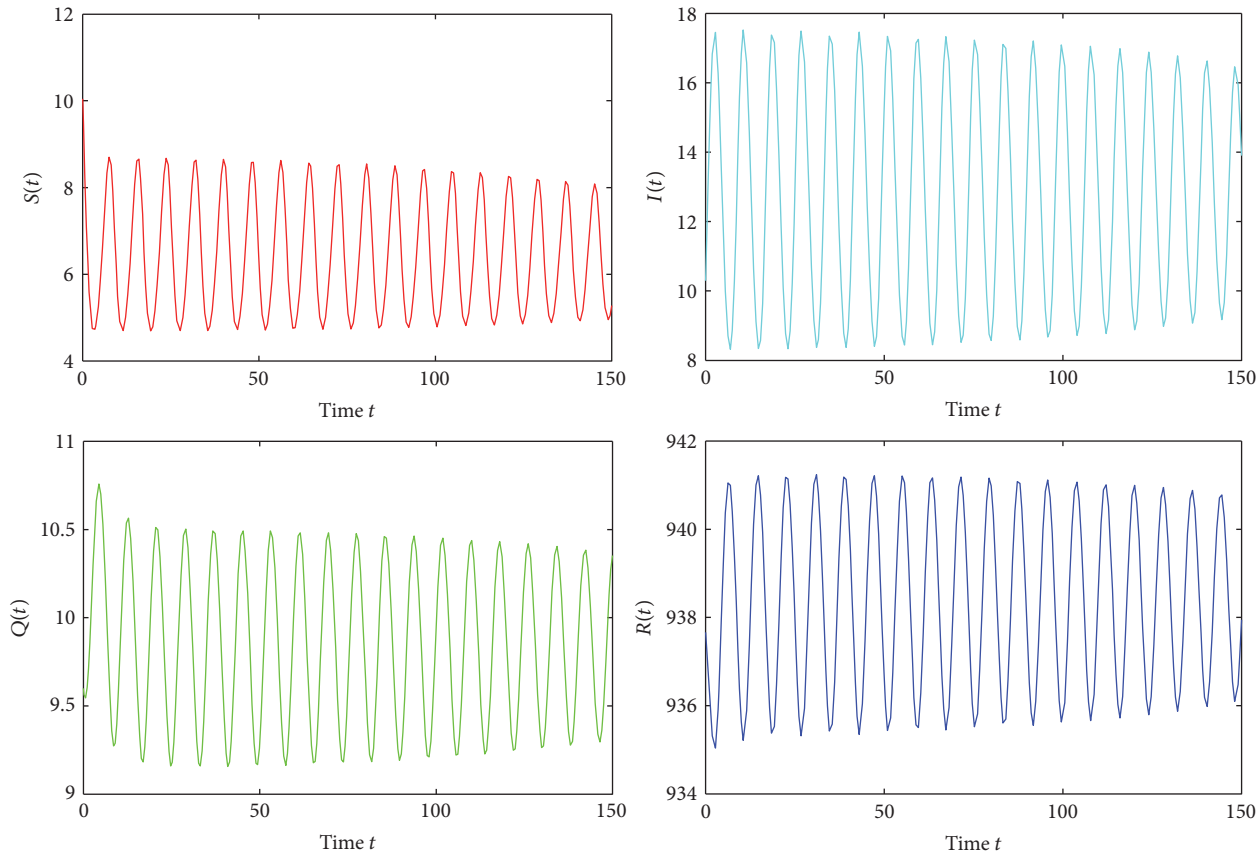

FIGURE 10: The trajectory of S, I, Q, and R when $\tau_2 = 1.8625 > 1.7852 = \tau_{20}$.

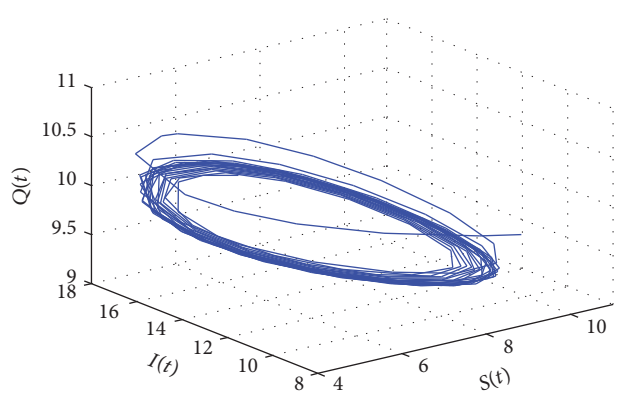

FIGURE 11: The phase plot of S, I, and Q when $\tau_2 = 1.86251.7852 = \tau_{20}$.

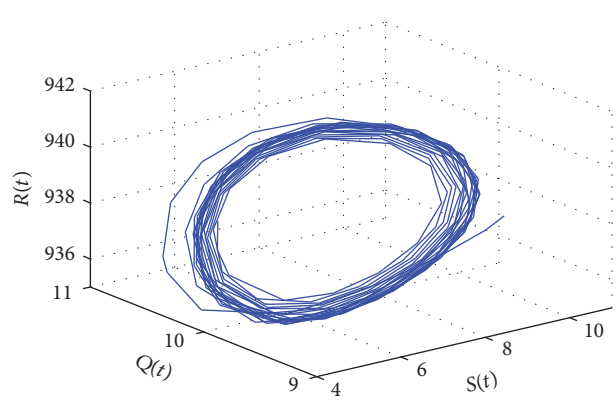

FIGURE 12: The phase plot of S, Q, and R when $\tau_2 = 1.8625 > 1.7852 = \tau_{20}$.

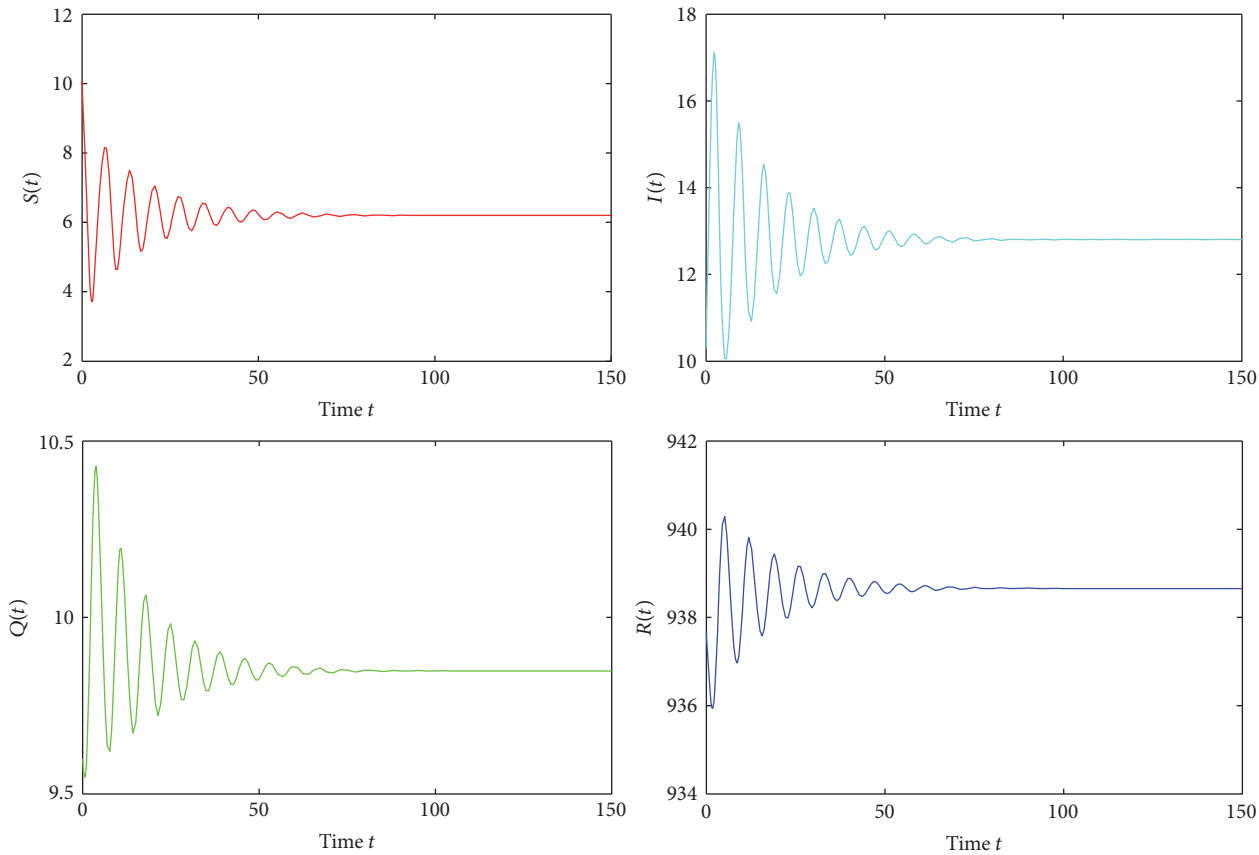

FIGURE 13: The trajectory of S, I, Q, and R when $\tau_1 = 0.785 < 0.8903 = \tau'_{10}$ and $\tau_2 = 1.05$.

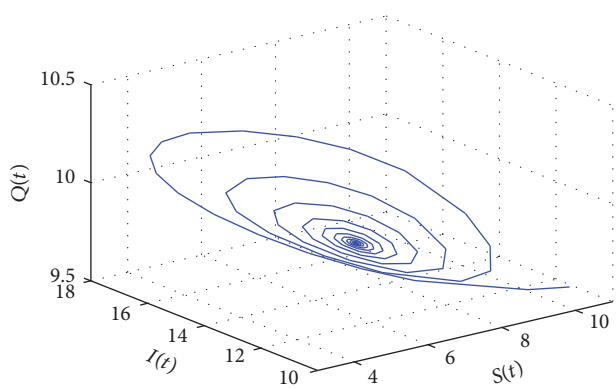

FIGURE 14: The phase plot of S, I, and Q when $\tau_1 = 0.785 < 0.8903 = \tau'_{10}$ and $\tau_2 = 1.05$.

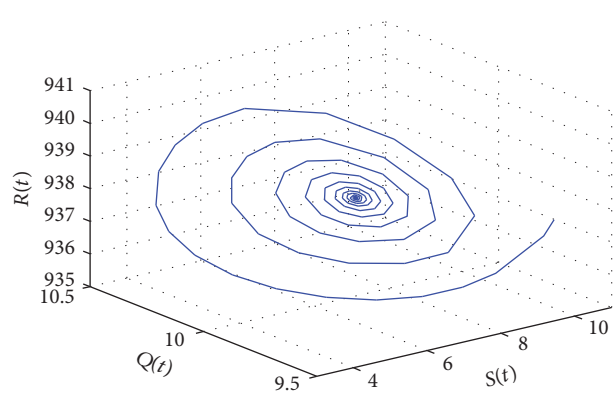

FIGURE 15: The phase plot of S, Q, and R when $\tau_1 = 0.785 < 0.8903 = \tau'_{10}$ and $\tau_2 = 1.05$.

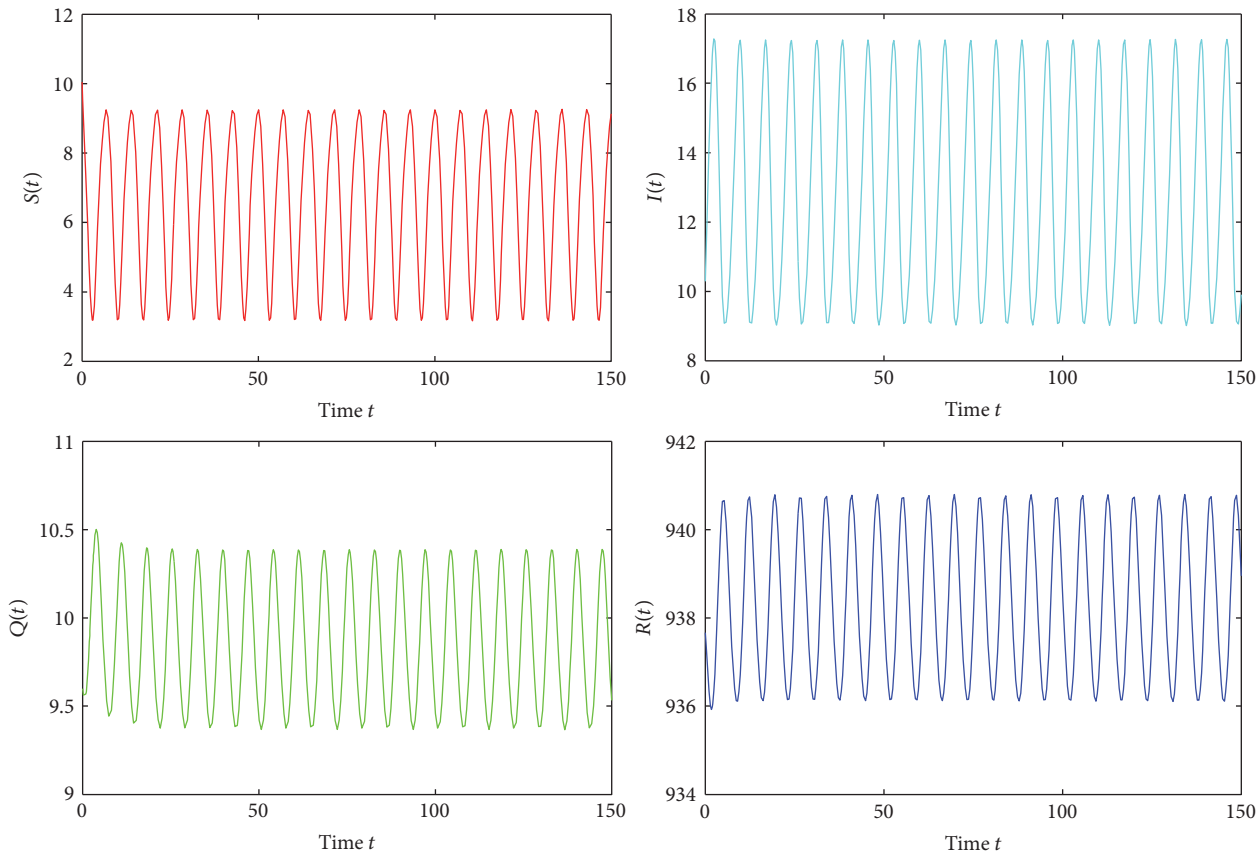

Figure 16: The trajectory of S, I, Q, and R when $\tau_1 = 0.9755 > 0.8903 = \tau'_{10}$ and $\tau_2 = 1.05$.

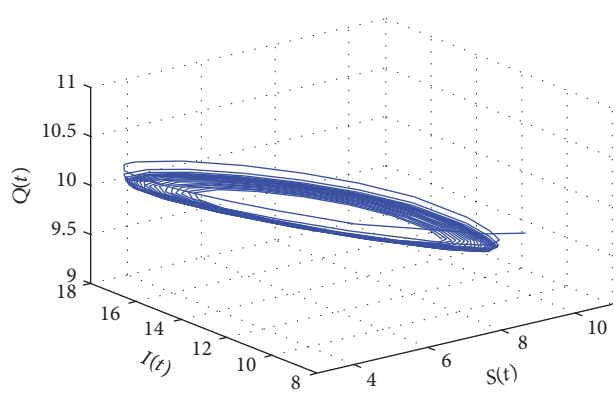

Figure 17: The phase plot of S, I, and Q when $\tau_1 = 0.9755 > 0.8903 = \tau'_{10}$ and $\tau_2 = 1.05$.

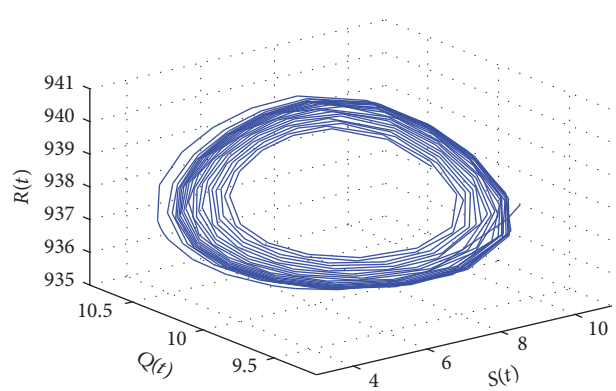

Figure 18: The phase plot of S, Q, and R when $\tau_2 = 0.9755 > 0.8903 = \tau'_{20}$ and $\tau_2 = 1.05$.

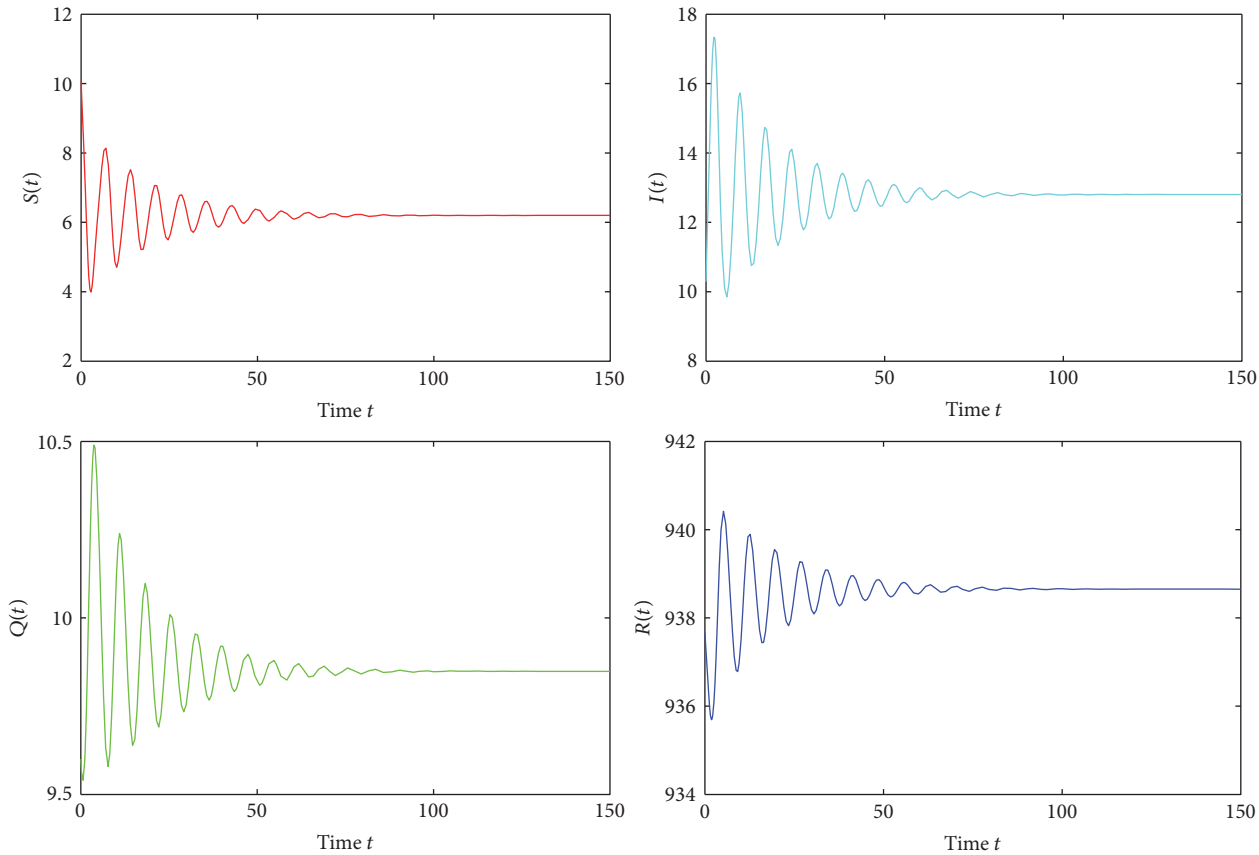

FIGURE 19: The trajectory of S, I, Q, and R when $\tau_2 = 1.235 < 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

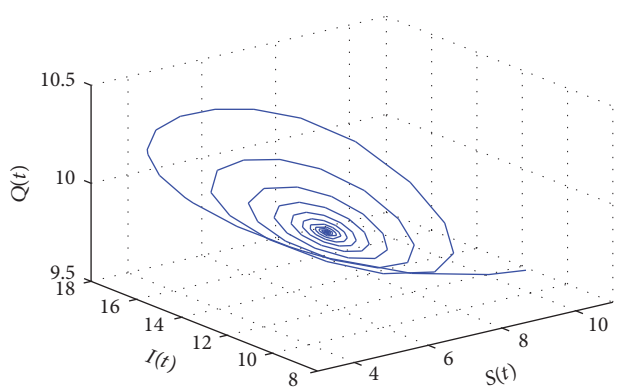

FIGURE 20: The phase plot of S, I, and Q when $\tau_2 = 1.235 < 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

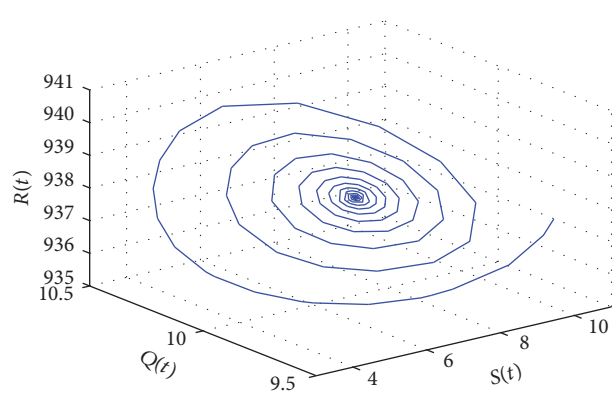

FIGURE 21: The phase plot of S, Q, and R when $\tau_2 = 1.235 < 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

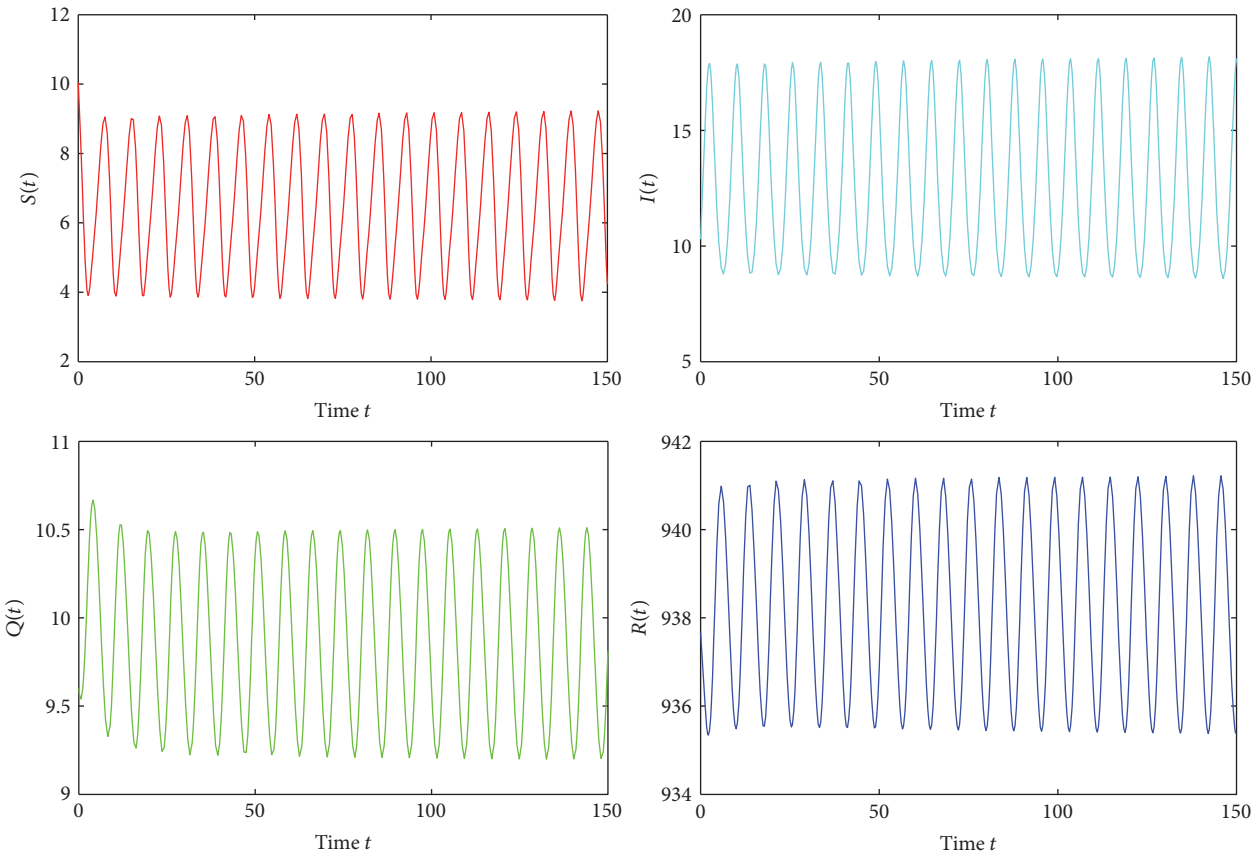

FIGURE 22: The trajectory of S, I, Q, and R when $\tau_2 = 1.52 > 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

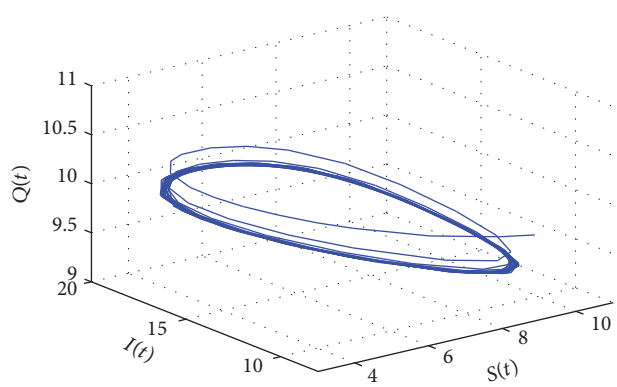

FIGURE 23: The phase plot of S, I, and Q when $\tau_2 = 1.52 > 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

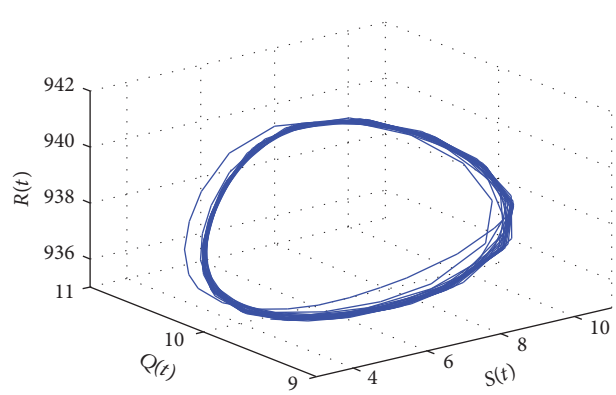

FIGURE 24: The phase plot of S, Q, and R when $\tau_2 = 1.52 > 1.3485 = \tau'_{20}$ and $\tau_1 = 0.65$.

the system in this paper is more general because it accounts for not only the time delay due to the latent period of the computer virus but also the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes. The sufficient conditions for the stability of the positive equilibrium and existence of the Hopf bifurcation for the possible combinations of two delays are obtained. When the conditions are satisfied, then there exists a critical value of the delay below which system (2) is locally asymptotically stable and above which system (2) is unstable. The direction and the stability of the bifurcating periodic solutions are determined by applying the normal theory and the center manifold theorem. Numerical simulations show that the computer viruses may be controlled by shortening the time delay due to the latent period of the computer viruses and the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes.

However, it should be pointed out that we suppose that the recovered computers have a permanent immunization period and they can no longer be infected. This is not consistent with real situation. In order to overcome this limitation and considering that the recovered computers may be infected again after a temporary immunity period, it is definitely an interesting work to investigate the following more general SIQRS model with multiple delays:

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)b - \beta S(t-\tau_1) I(t-\tau_1) - dS(t) \\
&\quad + \sigma R(t-\tau_3), \\
\frac{dI(t)}{dt} &= \beta S(t-\tau_1) I(t-\tau_1) - (\delta + d + \alpha_1) I(t) \\
&\quad - \gamma I(t-\tau_2), \\
\frac{dQ(t)}{dt} &= \delta I(t) - (d+\alpha_2) Q(t) - \varepsilon Q(t-\tau_2), \\
\frac{dR(t)}{dt} &= \gamma I(t-\tau_2) + pb + \varepsilon Q(t-\tau_2) - dR(t) \\
&\quad - \sigma R(t-\tau_3),
\end{aligned} \tag{67}$$

where σ is the transition rate from the recovered computers to the susceptible computers. τ_3 is the temporary immunity period after which a recovered computer may be infected again. We leave the analysis of the more complicated bifurcations of system (67) as the future work.

Further research directions include the possibility of linking the results obtained with the model proposed in the present paper with the results coming from the networks theory; see the recent review paper [17]. Specifically, the interest focuses on the possibility to gain a deep understanding of the impact of the network topology on the viral prevalence; see [18, 19]. In this context, the model proposed in this paper, which is a coarse-grained compartment-level model, can be considered as a network where all nodes are classified into a few compartments according to their states and whose degree distribution takes into account the rule of interactions proposed in the present paper.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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